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On the Compatibility of Metric and Linear Order

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Abstract:
There is a classic question of introducing some reasonable order on the real plane $\mathbb{R}^2$.
We consider this problem through the prism of the relation between metric proximity and linear order proximity, introducing a way to determine the compatibility relation between metric and linearly ordered spaces, and study the basic properties of this connection.
For $\mathbb{R}$ and its metric subspaces, the order on the line is compatible with the metric according to the definition we introduce.
It turns out that for $\mathbb{R}^n$ for $n > 1$, as well as for other similar metric spaces in which there is a subspace isometric to an open ball in $\mathbb{R}^n$, it is impossible to introduce an order that is compatible with the metric.
At the same time, we also introduce a family of spaces called discrete curved lines, which are not generally isometric to the subspaces of $\mathbb{R}$, but for which it is possible to introduce a linear order compatible with the natural metric on them. For them we prove sufficient conditions for the compatibility of the metric with the order. Using them, we construct spaces with non-trivial properties in which the order is compatible with the metric. In particular, we show that although on all of $\mathbb{R}^n$ no order is compatible with the metric, in any normed space one can introduce non-trivial metric subspaces compatible with some order (in the case of infinite-dimensional normed spaces, this subspace can be constructed to be non-embeddable in any finite-dimensional subspace)

Keywords: ordered metric spaces, metric spaces, linear order.

1. INTRODUCTION

One of the natural questions one might have is how one can reasonably introduce an order on a real plane $\mathbb{R}^2$ in general given that there is a standard order on the real line $\mathbb{R}$.

A common way is to introduce lexicographic order, and there are other ways to introduce an order on a plane, but all of these methods lose some of the natural properties that order on a straight line possessed.

Lexicographic order is very inconsistent with our sense of nearness. According to the lexicographic order, it turns out that $(a, b) \leq (a, b + \epsilon) \leq (a + \delta, b)$, as arbitrarily small we do not take $\delta > 0$ and as arbitrarily large we do not take $\epsilon > 0$. This is very inconsistent with our understanding that in some ways the point (a, b) is much closer to $(a + \delta, b)$ than to $(a, b + \epsilon)$.

What gives rise to this feeling? This understanding is based on the notion of metric proximity by the natural classical Euclidean metric on $\mathbb{R}^2$.

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This gives rise to a natural way to impose the requirement to impose compatibility in the definitions of metric proximity [2–4, 6, 8] and linear proximity [1, 5, 10, 11], which we do in our article.

It turns out that there are no linear spaces compatible with the classical metric on $\mathbb{R}^n$ for $n > 1$ and similar metric spaces with linear orders according to our definition.

Further we introduce a series of non-trivial metric spaces that are compatible with respect to the order but are not isometric to subsets of the real line.

2. BASIC DEFINITIONS

Let $(X, d, \leq)$ be a set equipped with a metric d and a non-strict linear order denoted by $\leq$.

Let us introduce what it means for a metric to be compatible with an order.

Definition 2.1:
We will call the order $\leq$ on X compatible with the metric d on the left if $d(x, y) \leq d(x, z)$ for all $x, y, z \in X : x \leq y \leq z$.

Definition 2.2:
We will call the order $\leq$ on X compatible with the metric d on the right if $d(y, z) \leq d(x, z)$ for all $x, y, z \in X : x \leq y \leq z$.

Definition 2.3:
We will say that the order of X is compatible with the metric d if it is compatible on the right and left sides.

Remark 2.1:
Further, we will say both "the order is compatible with the metric" and "the metric is compatible with the order". In other words, this property connects both of these structures symmetrically. The spaces where this property holds will be called "order-compatible metric spaces."

In this article, we will focus on studying the compatibility of certain metrics with order in the sense of Definition 2.3.

Definition 2.4:
If X is a vector space, we will call the order $\leq$ on X compatible with the norm $||\cdot|| : X \to \mathbb{R}$ if it is compatible with the metric $d(x, y) = ||x - y||$ induced by it.

Remark 2.2:
It is easy to see that the metric d in the space X is compatible with the order $\leq$ if and only if

$$d(x, z) \geq \max\{d(y, z), d(x, y)\} \quad \forall x, y, z \in X : x \leq y \leq z.$$

Obviously, $\mathbb{R}$ is an order-compatible metric space. In addition, there are other examples of order-compatible metric spaces. A trivial example is a discrete metric space with an arbitrary linear order.

The following lemma follows from the definition of compatibility.

Lemma 2.1:
In the metric space (X, d), let the metric d be compatible with the order $\leq$. Then the induced metric on any subset of X will be compatible with the induced order.

Recall that on a linearly ordered set $(X, <)$ with strict order $<$, we can define an order topology [7] as one with a pre-base of "open rays" $\{x \mid a < x\},\ \{x \mid x < b\},\ a, b \in X.$

We might assume that the definition of compatibility between metric spaces and linear orders reduces to an order topology. We will show that this is not the case.

Remark 2.3:
Metrizable order topologies and metric spaces in which the metric is compatible with

the order do not "naturally" reduce to each other. It is not difficult to see that we can unambiguously obtain an order topology from an order and an order from an order topology. Unlike defining an order topology, an order can be compatible with different metrics, including from the point of view of the generated topology, and this metric can be compatible with different orders. For example, the discrete metric and the Euclidean metric on a straight line are compatible with the standard linear order on a straight line, but they generate different topologies. Further, a discrete metric on any set is compatible with any order on that set.

3. INCOMPATIBILITY WITH ORDER OF CLASSIC MULTIDIMENSIONAL SPACES

In this section, we will construct examples of metric spaces where it is impossible to define any linear order that is compatible with the metric. We will demonstrate for a broad range of multidimensional normed spaces that it is not possible to establish any linear order that is compatible with their norm.

Proposition 3.1:
For any metric space with at most three points, you can always create an order that is compatible with the metric.

It turns out that this is no longer always true for four points.

Theorem 3.1:
Let there be four distinct points X, Y, Z, T in the metric space $\mathcal{M}$ such that

$$min\{d(X,Z), d(Y,T)\} \geq \max\{d(Y,Z), d(X,Y), d(X,T), d(T,Z)\}.$$

Then there does not exist an order that is compatible with the metric.

Proof
Let us assume that in $\mathcal{M}$ an order-compatible metric can be introduced. Since $d(X,Z) \geq \max\{d(Y,Z), d(X,Y)\}$ by the definition of compatibility, we have $X \leq Y \leq Z$ or $Z \leq Y \leq X$. Without loss of generality, we will assume that $X \leq Y \leq Z$. Therefore, since $d(X,Z) \geq \max\{d(X,T), d(T,Z)\}$, we have that $X \leq T \leq Z$. Since $d(Y,T) \geq \max\{d(Y,X), d(T,X)\}$, we have $T \leq X \leq Y$ or $Y \leq X \leq T$. It follows that $X = T$ or $X = Y$. Hence, we have obtained a contradiction. □

This construction can be used further. To do so, we need the following definitions:

Definition 3.1:
Let the real numbers r, x and y with $r > 0$ be given. We will call $(r,x,y)_1$ - system a set of points on the plane $\{X, Y, Z, T\}$, $X = (r+x, r+y)$, $Y = (-r+x, r+y)$, $Z = (-r+x, -r+y)$, $X = (r+x, -r+y)$.

We will call $(r,x,y)_\infty$ - system a set of points on the plane $\{X, Y, Z, T\}$, $X = (r+x, y)$, $Y = (x, r+y)$, $Z = (-r+x, y)$, $T = (x, -r+y)$.

Lemma 3.1:
On the $(r,x,y)_k$-system, it is impossible to introduce an order compatible with the p-norm induced on the metric plane, for
1) $k = 1$ and $1 \leq p < \infty$;
2) $k = \infty$ and $1 < p \leq \infty$.

Proof
The proof follows directly from checking the condition of Theorem 3.1. □

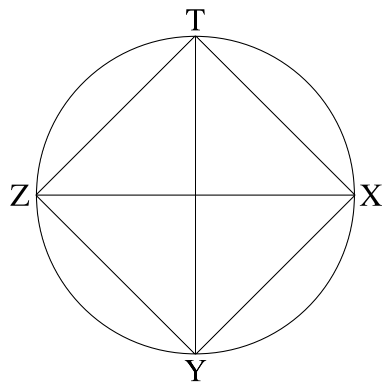

Fig. 3.1. Example of $(r, 0, 0)_\infty$ - systems $S = \{X, Y, Z, T\}$ on the plane around the origin. It is easy to see that the diameters of the resulting rhombus with vertices from S in the 2 - norm are larger than the sides, which shows that the condition of the Theorem 3.1 is true.

The definition and lemma above allow us to obtain the following useful lemma, which can be used to prove that the metric is incompatible with the order for a number of spaces:

Lemma 3.2:
On an open ball $B_r(x, y)$ in $\mathbb{R}^2$, for all $1 < p \leq \infty$, it is impossible to introduce an order that is compatible with the metric induced from the p-norm.

Proof

It is easy to see that for any $x, y \in \mathbb{R}, r > 0, 1 \leq p \leq \infty$ we have $(\frac{r}{3}, x, y)_1 \subset B_r(x, y)$ and $(\frac{r}{2}, x, y)_\infty \subset B_r(x, y)$.

□

Theorem 3.2:
There is no metric-compatible order for any metric space that contains a subset that is isometric to an open ball in the plane with p-norm, $1 \leq p \leq \infty$.

In particular, there is no metric-compatible order in the following spaces:

1. *$\mathbb{R}^n$ with the p norm for $n > 1, 1 \leq p \leq \infty$;*
2. *ℓ_p for $1 \leq p \leq \infty$;*
3. *$L_p[a, b]$ for $[a, b] \subset \mathbb{R},\ 1 \leq p \leq \infty$;*
4. *$L_p(A)$, where A is a measurable space that contains two disjoint subsets A_1 and A_2 with a finite nonzero measure, $1 \leq p \leq \infty$;*
5. *an open ball in any of the spaces listed above.*

Proof
From the statement of point (5), the statements of points (1)-(4) follow by lemma 2.1. We prove point (5) for each of these spaces. To do this, we isometrically embed an open ball from $\mathbb{R}^2$ in the corresponding space and use the Lemma 3.2.

1. Let $x \in \mathbb{R}^n$.
We can embed isometrically $B_r(0, 0)$ into $B_r(x)$ with the p - norm:
$f(y_1, y_2) = (x_1 + y_1, x_2 + y_2, x_3, \dots, x_n)$.

2. Let $x \in \ell_p$. We can isometrically embed $B_r(0, 0)$ with the p-norm into $B_r(x)$ with the ℓ_p norm:
$f(y_1, y_2) = (x_1 + y_1, x_2 + y_2, x_3, \dots)$.

3. We can embed $\mathbb{R}^2$ isometrically with the p - norm into $L_p[a, b]$:

$f(y_1, y_2) = \frac{y_1}{\sqrt[p]{\frac{b-a}{2}}} I_{[a,a+\frac{(b-a)}{2}]} + \frac{y_2}{\sqrt[p]{\frac{b-a}{2}}} I_{[a+\frac{(b-a)}{2},b]}$. We show the isometricity of f. For (y_1, y_2) and (y_1', y_2') :

$$d((y_1, y_2), (y_1', y_2')) = \sqrt[p]{|y_1 - y_1'|^p + |y_2 - y_2'|^p} =$$
$$= \sqrt[p]{\int_a^b \frac{2}{b-a}\left(|y_1 - y_1'|^p I_{[a,a+\frac{(b-a)}{2}]} + |y_2 - y_2'|^p I_{[a+\frac{(b-a)}{2},b]}\right)(z)dz} =$$
$$= d(f(y_1, y_2), f(y_1', y_2'))$$

and f is injective.

For any $B_r(x) \subset L_p[a, b]$, we can restrict f to some open ball lying in $f^{-1}(B_r(x))$, thus constructing an isometric embedding of the open ball from $\mathbb{R}^2$ into $B_r(x)$.

4. It is enough to repeat the reasoning of point 3 for $f(y_1, y_2) = \frac{y_1}{\sqrt[p]{\mu(A_1)}} I_{A_1} + \frac{y_2}{\sqrt[p]{\mu(A_2)}} I_{A_2}$.

□

4. CONSTRUCTIONS OF ORDER-COMPATIBLE METRIC SPACES

In the previous section, we provided numerous examples of spaces that are not compatible with any linear order. These spaces were primarily created by naturally embedding certain subdomains of $\mathbb{R}^n$ into them, meaning they were ”locally multidimensional” and ”closely resembled” a multidimensional real space.

At the same time, only degenerate finite metric spaces (consisting of no more than 3 points), discrete metric spaces, and $\mathbb{R}$ fit the conditions for compatibility with some order.

Let us construct an example of a more nontrivial space with such properties obtained by a certain deformation of a countable subset of points on a line.

To begin with, we recall the definition of a uniformly discrete set [9].

Definition 4.1:
If (M, d) is a metric space, and $A \subset M$, then the packing radius r of the set A is defined as:

$$r = \frac{1}{2} \inf_{x,y \in A} d(x, y).$$

Definition 4.2:
A set is called uniformly discrete if it has a nonzero covering radius.

Now we introduce the definition of a discrete curved line, which will be a model example of a set for many constructions in this section.

Definition 4.3:
Let (M, d) be a metric space. Let D be a countable uniformly discrete subset of $\mathbb{R}$ with covering radius r. Denote by X the image of an isometric embedding of D into M.

Let $u_x \in M$ be chosen for any $x \in X$ such that there exists $C > 0$ such that for all $x \in X$ it is true that $d(x, u_x) \leq C$. Let $\varepsilon = \inf\{C | \forall x \in X\ d(x, u_x) \leq \delta\}$.

We refer to a set $\{u_x\}_{x \in X}$ with a metric induced from M and an order induced from X (meaning that $u_{x_1} \leq u_{x_2}$ if $x_1 < x_2$) as a discrete curved line. Alternatively, if we want to highlight the parameters, we can call it a (r, ε)-discrete curved line.

Let’s try to consider the motivation for such a definition.

It comes down to two simple concepts:

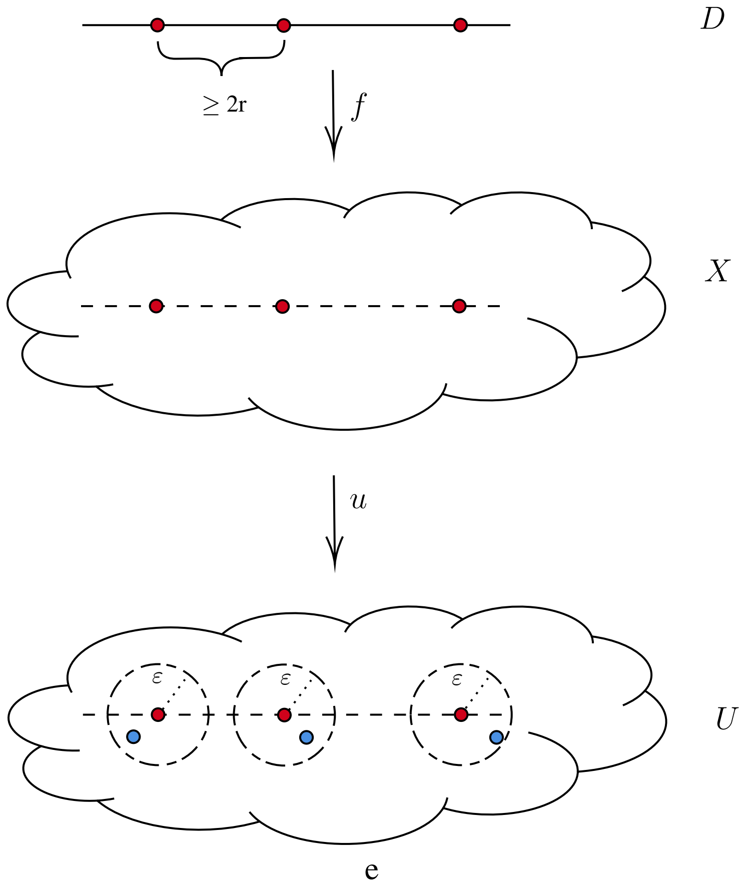

Fig. 4.2. The concept of a discrete curved line: We start by selecting a set of points in $D \subset \mathbb{R}$. These points are uniformly distributed and have a covering radius of r, meaning that no two points are closer than $2r$. We then embed this set into a new metric space, creating a new space called X. Next, we define a subset of U by shifting the elements of X by no more than ε. This allows us to create a curved line structure within the space U.

1. We select a specific subset of points from a straight line (embedded in another metric space). These points are chosen such that they are a sufficient distance apart from each other, for example, using sets like $\mathbb{N}$ or $\mathbb{Z}$, where all points are at least 1 unit apart.

2. We perform a small shift on each point, relative to the distance between them.

Since the shift is minimal, we can expect that after the shift, for any three ordered points $x \leq y \leq z$, the distances $d'(x, z)$, $d'(y, y')$, and $d'(y', z')$ will remain larger than the original distances $d(x, y)$, $d(y, z)$, and $d(x, z)$, respectively. This is because all pairwise distances change slightly compared to the significant difference between $d(x, z)$ and the distances $d(x, y)$ and $d(y, z)$.

A discrete curved straight line allows us to make any desired shifts relative to the original location of points from the perspective of the geometry of the space where the embedding occurs.

Now we will prove the general theorem on the necessary and sufficient conditions for the compatibility of the metric and order for discrete curved lines.

Theorem 4.1:
(Sufficient condition for the compatibility of the metric with the order for a discrete curved line)

Let U be an (r, ε)-discrete curved line in the metric space (M, d). If $\varepsilon \leq \frac{r}{2}$, then the metric on U is compatible with the order on U.

Proof
According to the definition 4.3 for U, there exists X, which is a uniformly discrete subset of $\mathbb{R}$ with a covering radius of r. The set U is defined as $\{u_x\}_{x\in X}$, where $d(x, u_x) < \varepsilon$.

Consider three points: $u_{x_1} \leq u_{x_2} \leq u_{x_3}$ (which implies $x_1 \leq x_2 \leq x_3$)

Then, $d(u_{x_1}, u_{x_2})$ is less than or equal to the sum of three terms: $d(x_1, x_2)$, $d(u_{x_1}, x_1)$, and $d(u_{x_2}, x_2)$.

This can be shown using the triangle inequality and the definition of a uniformly discrete line. Thus, we have $d(u_{x_1}, u_{x_2}) \leq d(x_1, x_2) + d(u_{x_1}, x_1) + d(u_{x_2}, x_2) \leq 2r + 2\varepsilon$.

It is not difficult to see that on X (since X is isometric to a subset of the real line with covering radius r) $d(x_1, x_3) = d(x_1, x_2) + d(x_2, x_3) \geq 4r$.

Similar to the reasoning above, we have: $d(x_1, x_3) \leq d(u_{x_1}, u_{x_3}) + d(u_{x_1}, x_1) + d(u_{x_3}, x_3) \leq d(u_{x_1}, u_{x_3}) + 2\varepsilon$ from which it follows that $d(u_{x_1}, u_{x_3}) \geq d(x_1, x_3) - 2\varepsilon \geq 4r - 2\varepsilon$.

It is easy to see that if $\varepsilon \leq \frac{r}{2}$ then $d(u_{x_1}, u_{x_2}) \leq 2r + 2\varepsilon \leq 4r - 2\varepsilon \leq d(u_{x_1}, u_{x_3})$.

Similarly, for the case where $d(u_{x_2}, u_{x_3}) \leq d(u_{x_1}, u_{x_3})$, the metric is compatible with the order.

□

As we can see, for a small enough value of ε, a discrete curved line's metric is compatible with its order. This is an example of a space with a non-trivial metric that is compatible with a certain order.

On the other hand, if ε is increased relative to r, the order compatibility property may be violated.

Example 4.1:
(Left shift of a point of the stretched natural number line)

Consider the set of natural numbers scaled by $2r$ as a subset of $\mathbb{R}$ with the standard metric. Let $X = 2r\mathbb{N}$. Move the point $4r$ to the left by an amount $\varepsilon > 2r$. Let $U = \{u_x\}_{x\in X}$, where

$$u_x = \begin{cases} 4r - \varepsilon, & \textit{if } x = 4r; \\ x, & \textit{otherwise.} \end{cases}$$

Thus, we obtain a (r, ε)-discrete curved line, where the order is not compatible with the metric.

Generally, when we create various examples of discrete curved lines, we observe that if we increase the parameter ε compared to r, there will be more instances where the metric is incompatible with the order.

It would be natural to assume that for large ε, the metric is guaranteed to cease to be compatible with the order. However, we will demonstrate a counterexample.

Example 4.2:
(The set of natural numbers scaled and shifted upwards.)

Let $U = 2r\mathbb{N}e_1 + \varepsilon e_2 = \{(2rn, \varepsilon)\}_{n\in\mathbb{N}} \subset \mathbb{R}^2$. It is not difficult to see that this set is (r, ε)-discrete curved line. However, the order is compatible with the metric, no matter how small or large ε we choose. This is because all metric properties and order properties remain unchanged under this isometric shift.

Thus, we see the following picture for the (r, ε)-discrete curved line:

1) For a small ε relative to r (it is enough if $\varepsilon \leq \frac{r}{2}$ according to Theorem 4.1), the metric is compatible with the order.

2) If we exceed a certain limit (following the example of 4.1, it is enough to have $\varepsilon > 2r$), then we observe discrete curved lines where the metric is not compatible with the order. However, for arbitrarily large ε there are also cases where the metric is compatible with the order, as demonstrated by the example 4.2.

This leads us to the question: what is the highest possible value of ε that ensures that the order is compatible with the metric?

Problem 4.1:
What is the minimum $m(r)$ such that any (r, ε)-discrete curved line with $\varepsilon \leq m(r)$ is guaranteed to have a compatible metric with the order? According to Theorem 4.1 and the 4.1 example, it is true that $m(r)$ is between $\frac{r}{2}$ and $2r$.

Of course, the exact conditions on ε in relation to r for making the metrics compatible with the order may change depending on the specific metric spaces we consider, their classes, or the methods we use to implement the isometric embedding and select new points.

One might think that all types of order-compatible discrete curved lines, in terms of their metric and general properties, are exactly the same as an ordinary lines. This is because they are created by simply shifting certain points on a line.

At the same time, we can select discrete curved lines that can significantly differ from a straight line in terms of certain properties. This allows us to explore non-trivial examples of spaces with a metric that is compatible with an order.

Lemma 4.1:
For any infinite-dimensional normed space V over $\mathbb{R}$, there exists a countable subset of U compatible with some order that does not does not belong to any finite-dimensional subspace, meaning that $\dim span(U) = \infty$.

Proof
Let us consider a specific subset of the basis vectors: $\{e_1, \dots\}$. Consider an $(r, \frac{r}{2})$-discrete curved line U in the vector space V. Let U be a set of points that can be represented as $u_n = 2nre_1 + \frac{r}{2}e_n$, where e_n is the n-th basis vector.

It is easy to see that U is an $(r, \frac{r}{2})$-discrete curved line with a specific structure. It is obtained by shifting up along the nth coordinate of the embedding of natural numbers, which are scaled by $2r$. This scaled set is then embedded on the first coordinate axis.

By the Theorem 4.1, we know that the metric in this curved line is compatible with the order and vectors in U are linearly independent, since for each unique u_i in U, there is a corresponding unique e_i in the basis expansion.

□

Remark 4.1:
Theorem 4.3 shows that there are nontrivial examples of metric spaces that are not isometric to any subset of the real line, but in which the order is compatible with the metric.

Theorem 4.1 gives sufficient conditions for the discrete curved line to be compatible with the order. At the same time, it is possible to find examples of spaces where the metric is compatible with the order through specific directed deformations of individual points. Sometimes we can use general conditions for the entire line, but in general it is preferred to have a more precise method for evaluating the allowed changes at a specific point.

Theorem 4.2:
(Pointwise sufficient condition for compatibility of the metric with an order for a discrete curved line) *Let us consider a discrete curved line U that is obtained from a set X. We can assign numbers to the elements of X according to their order on the real number line using integers. If there is an upper or lower bound, we can use positive or negative integers accordingly. Let $d(u_{x_i}, x_i) = \varepsilon_i$. Then, U can be considered a order-compatible metric space if the following condition holds for all i:*

$$\varepsilon_i \leq \min_{j<i,\ k>i} \max\left(2r(k-i) - 2\varepsilon_j - \varepsilon_k, 2r(i-j) - 2\varepsilon_k - \varepsilon_j\right).$$

Proof
Let us take an arbitrary point x_i and consider arbitrary points x_j, x_k, $x_j \leq x_i \leq x_k$.

Then $d(u_{x_i}, u_{x_j}) \leq d(x_i, x_j) + d(u_{x_i}, x_i) + d(u_{x_j}, x_j) \leq 2(i-j)r + \varepsilon_i + \varepsilon_j$ (since on the straight line between x_i and x_j there is exactly $j - i$ times $2r$).

Similarly $d(u_{x_j}, u_{x_k}) \leq 2(k-i)r + \varepsilon_i + \varepsilon_k$.

Similar to the above $d(x_j, x_k) \leq d(u_{x_j}, u_{x_k}) + d(u_{x_j}, x_j) + d(u_{x_k}, x_k) \leq d(u_{x_j}, u_{x_k}) + \varepsilon_j + \varepsilon_k$ which implies that $d(u_{x_j}, u_{x_k}) \geq d(x_j, x_k) - \varepsilon_j - \varepsilon_k \geq (k-j)2r - \varepsilon_j - \varepsilon_k$.

Then if $(k-j)2r - \varepsilon_j - \varepsilon_k \geq \max(2(i-j)r + \varepsilon_i + \varepsilon_j, 2(k-i)r + \varepsilon_i + \varepsilon_k)$, then if $\varepsilon_i \leq \max(2r(k-i) - 2\varepsilon_j - \varepsilon_k, 2r(i-j) - 2\varepsilon_k - \varepsilon_j)$ our metric will be compatible with the order for the triple $u_{x_j}, u_{x_i}, u_{x_k}$. If we select an arbitrary value of i and then choose any two other values, j and k, where j is less than i and k is greater than i, and keep in mind that ε_i is bounded from above, we obtain the conditions of the lemma.

□

All the examples of order-compatible discrete curved lines in normed spaces had the property that when we projected them onto the coordinate axis passing through their starting points X, the order of the points was the same as it was on the original curved line itself. In other words, the points were simply ordered based on their coordinate along the original straight line.

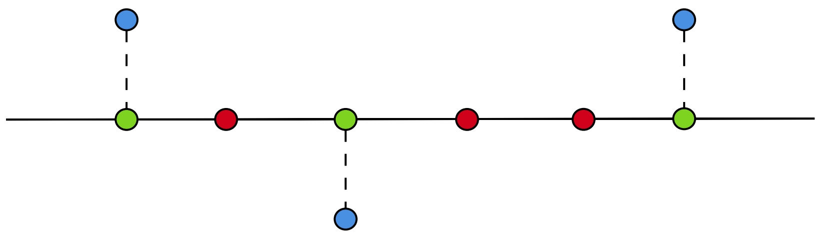

Fig. 4.3. An example of a discrete curved line where the metric is compatible with the order. Since we make a small shift, the projection preserves the order on the original set.

Let us show that this is not always the case.

Example 4.3:
Consider the set of natural numbers $X = \mathbb{N}e_1$ embedded in the plane with the Euclidean metric. Then the set $U = \{u_x\}_{x \in X}$, where

$$u_x = \begin{cases} (0.5, 1.5), & \text{if } x = (2, 0); \\ x, & \text{otherwise} \end{cases}$$

will be a discrete curved line with the metric compatible with the order, but the projection of U on the x axis (on which X lies) has an order that does not coincide with the order on this axis as a straight line.

Proof
To check compatibility with the order, note that the only modified point is $u_{(2,0)}$. It is easy to see that if we take the points $u_{x_1} = (1, 0)$, $u_{x_2} = (0.5, 1.5)$, $u_{x_3} = (3, 0)$, the compatibility is satisfied (checked by calculating distances). The remaining cases follow from Theorem 4.1 and the fact that $\varepsilon_i = 0$ for all $i \neq 2$.

□

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