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Estimating the Unmeasured Output of Nonlinear SISO System under Uncertainty

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Abstract: Nonlinear single-channel control plant with affine unmached external disturbances are considered. Their mathematical model has the so-called triangular (based on the composition of the arguments of the functions in each equation) “input-output” form, which has the following structural properties. If there are no external disturbances, then this form is controllable and observable in terms of output. If external disturbances are smooth, then the control plant model is representable in the canonical “input-output” form with respect to mixed variables (functions of state variables, external disturbances and their derivatives). In the new mixed variable basis, the system remains controllable, observable, and has the same relative degree as the original system. If the output variable is measured, then an observer of mixed variables can be constructed based on the canonical system and, using suitable feedback, ensure the desired behavior of the output variable. In this case, there is no need to detail external disturbances and design individual observers for them, which greatly simplifies the structure of the controller. This paper discusses the problem of implementing the specified synthesis procedure in the case when the output variable is not measured. Motivating examples are given and conditions are formulated under which it is possible to first restore the output variable from the measurement of other state variables invariantly with respect to the action of external disturbances. Methods for estimating external disturbances using observers with piecewise linear corrective actions, which do not require the introduction of dynamic disturbance models, constitute the methodological basis for solving this problem. Using the example of various mathematical models of single-link manipulators, the fulfillment/failure of the conditions necessary to restore an unmeasured output variable is demonstrated. The proposed approach can be extended to multichannel nonlinear control plants without loss of generality.

Keywords: nonlinear SISO system, unmatched disturbances, observability, reduced-order state observer, piecewise linear function with saturation, single-link manipulator.

1. INTRODUCTION

The problem of constructing the reduced-order state observers is considered for nonlinear minimal-phase systems with one input and one output (SISO), represented in a triangular (by the composition of the function arguments in each equation) “input-output” form. This form is convenient for synthesis and feedback when solving the tracking problem, and for the state observer for estimating unmeasured variables, when only the output (controlled) variable is measured, and all parametric uncertainties and external disturbances are match, i.e. act on the input in the last equation [12, 15, 17–19]. In this paper, we study a case little studied in theory, but common in practice, when the output (controlled) variable of the tracking system for one reason or another cannot be measured [5], the set of sensors is not complete, and the parametric and external disturbances are unmatched.

The problem of observing non-measurable (including controlled) variables is considered in a narrow setting, when the identification of unknown parameters and the construction of external disturbance generators are not provided or are not possible. Our goal is to formulate for a triangular “input-output” system the conditions necessary for constructing a physically

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realizable observer, which depend not only on the structural properties of observability, but also on the channels of action of parametric and external disturbances. The scientific novelty lies in the identification of a practically significant class of SISO systems with unmatched disturbances, where the problem of estimating unmeasured state variables is feasible without additional identification of the existing parametric and external disturbances. The reduced-order state observer of a special structure with piecewise linear control actions provides a solution to this problem [8, 9, 13], if the differential equations of the measured variables in the control plant model do not contain parametric and external disturbances. Then, the observer is constructed as a copy of these equations, where unmeasured state variables are treated as undefined bounded inputs and estimated with the observer's corrective actions. In this case, the uncertainty of differential equations of unmeasured state variables is allowed.

The proposed approach, which provides a given estimation accuracy in a finite time, is fundamentally different from the traditional methods for constructing reduced observers based on differential equations of unmeasured variables [3, 4, 6, 7, 14], which in a narrow setting can be implemented only if the entire control plant model is fully defined.

The paper is structured as follows. Design options for a state and disturbance observer with piecewise linear correcting actions for a second-order system are discussed in Section 2. The main result is presented in Section 3. The conditions are formulated necessary for constructing a physically realizable observer for estimating the controlled but unmeasured variable of the n-th order SISO system. As an example, the equations of motion of single-link manipulators with a rigid and elastic type of articulation with the shaft of a DC motor are considered.

2. MOTIVATING EXAMPLES

Methods for estimating external disturbances in terms of their act on the control plant are the methodological basis for this study. These methods provide the given accuracy of the estimating signals when certain conditions are met [8, 9, 13]. To explain them, we will consider a triangular SISO system “input-output” of the second order

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) + \eta_1(t), \\ \dot{x}_2 &= f_2(x_1, x_2) + \eta_2(t) + bu, \end{aligned} \tag{2.1}$$

$$\frac{\partial f_1(x_1, x_2)}{\partial x_2} \neq 0, b \neq 0, \tag{2.2}$$

where $x = (x_1, x_2)^{\mathrm{T}} \in X \subset R^2$ is the state vector, X is an open bounded area of change of state variables in the control process, $x_1(t)$ is controlled output, $f_i(x_1, x_2)$ are functionally and parametrically defined expressions satisfying the Lipschitz conditions, $\eta_i \in R$ are external disturbances, $u \in R$ is the control (input), which is assumed to be a known function of time, for simplicity b is assumed to be a known constant. Satisfaction of conditions (2.2) means that system (2.1) is controllable and observable with respect to the output in the absence of external disturbances.

All internal and external signals in system (2.1) are assumed to be bounded together with their derivatives in the control process, in particular:

$$\begin{aligned} &|\eta_i(t)| \leq \mathrm{H}_i, |\dot{\eta}_i(t)| \leq \overline{\mathrm{H}}_i, t \in [0, T], \\ &|f_i(x_1, x_2)| \leq F_i, \left|\tfrac{d}{dt} f_i(x_1, x_2)\right| \leq \overline{F}_i, x(t) \in X, \end{aligned} \tag{2.3}$$

where T is regulation time, $F_i, \overline{F}_i, \mathrm{H}_i, \overline{\mathrm{H}}_i$ are known constants, in this section everywhere $i = 1, 2$.

Let us consider various variants of measurements in system (2.1) and the corresponding conditions for the physical realizability of observers of unmeasured state variables and external disturbances.

Example 2.1:
If state variables $x_1(t), x_2(t)$ are measured in system (2.1), then we can obtain estimates of external disturbance s using two first-order autonomous observers:

$$\begin{aligned} \dot{z}_1 &= f_1(x_1, x_2) + v_1, \\ \dot{z}_2 &= f_2(x_1, x_2) + bu + v_2, \end{aligned} \tag{2.4}$$

where $z_i \in R$ is the state vector, $v_i \in R$ are corrective actions of the observer. Taking into account (2.1), (2.4), the system with respect to observation errors $\varepsilon_i = x_i - z_i \in R$ takes the following form

$$\dot{\varepsilon}_i = \eta_i(t) - v_i,\ i = 1,2. \tag{2.5}$$

Synthesis of observer (2.4) consists in choosing the parameters of piecewise linear corrective actions

$$v_i = m_i \mathrm{sat}(l_i \varepsilon_i) = \begin{bmatrix} m_i \mathrm{sign}(\varepsilon_i),\ |\varepsilon_i| > 1/l_i, \\ m_i l_i \varepsilon_i,\ |\varepsilon_i| \le 1/l_i, \end{bmatrix} \tag{2.6}$$

where $m_i, l_i = \mathrm{const} > 0$, so as to stabilize the observation errors and their derivatives with a given accuracy

$$|\varepsilon_i(t)| \le \delta_i, |\dot{\varepsilon}_i(t)| \le \Delta_i \Rightarrow \eta_i(t) \approx v_i(t) \pm \Delta_i, t_0 < t \le T. \tag{2.7}$$

We can obtain inequalities for the choice of observer parameters that provide (2.7) using the second Lyapunov method [8]. Taking into account the measurements in systems (2.4), (2.5), we set the initial values

$$z_i(0) = x_i(0) \Rightarrow \varepsilon_i(0) = 0 \Leftrightarrow |\varepsilon_i(0)| \le 1/l_i.$$

Then we can provide $|\varepsilon_i(t)| \le 1/l_i$ during the entire control time $t \in [0,T]$ by choosing the amplitudes of corrective actions (2.6) in the form

$$m_i > \mathrm{H}_i \Rightarrow \varepsilon_i \dot{\varepsilon}_i = \varepsilon_i(\eta_i - m_i \mathrm{sign}(\varepsilon_i)) \le |\varepsilon_i|(\mathrm{H}_i - m_i) < 0. \tag{2.8}$$

In the linear zone $|\varepsilon_i| \le 1/l_i$, system (2.5) and its derivatives take the form

$$\begin{aligned} \dot{\varepsilon}_i &= \eta_i(t) - m_i l_i \varepsilon_i, \\ \ddot{\varepsilon}_i &= \dot{\eta}_i(t) - m_i l_i \dot{\varepsilon}_i. \end{aligned}$$

The state variables $\varepsilon_i(t), \dot{\varepsilon}_i(t)$ of this virtual system will converge to the given domains (2.7) in a finite time $t_0 : 0 \le t_0 << T$ if the conditions

$$\begin{aligned} & l_i \ge \mathrm{H}_i/(m_i \delta_i) \Rightarrow \varepsilon_i \dot{\varepsilon}_i = \varepsilon_i(\eta_i(t) - m_i l_i \varepsilon_i) \le \\ & \le |\varepsilon_i|(\mathrm{H}_i - m_i l_i |\varepsilon_i|) < 0 \text{ if } |\varepsilon_i| > \delta_i; \\ & l_i \ge \overline{\mathrm{H}}_i/(m_i \Delta_i) \Rightarrow \dot{\varepsilon}_i \ddot{\varepsilon}_i = \dot{\varepsilon}_i(\dot{\eta}_i(t) - m_i l_i \dot{\varepsilon}_i) \le \\ & \le |\dot{\varepsilon}_i|(\overline{\mathrm{H}}_i - m_i l_i |\dot{\varepsilon}_i|) < 0 \text{ if } |\dot{\varepsilon}_i| > \Delta_i, i = 1,2 \end{aligned} \tag{2.9}$$

are met. The simultaneous fulfillment of conditions (2.8)–(2.9)

$$m_i > \mathrm{H}_i, l_i \geq \frac{1}{m_i}\max\left\{\frac{\mathrm{H}_i}{\delta_i}, \frac{\overline{\mathrm{H}}_i}{\Delta_i}\right\}, i = 1, 2 \tag{2.10}$$

ensures the solution of the formulated estimation problem (2.7).

Example 2.2:
If in system (2.1)–(2.2) $x_1(t)$ is measured, and $\eta_1(t) \equiv 0$, then, similarly to (2.4), we can reconstruct $x_2(t)$ with a first-order observer based on the first equation of system (2.1) in the form

$$\begin{aligned} &\dot{z}_1 = v_1,\ v_1 = m_1 \mathrm{sat}(l_1 \varepsilon_1), \\ &\varepsilon_1 = x_1 - z_1, \dot{\varepsilon}_1 = f_1 - v_1. \end{aligned} \tag{2.11}$$

Then, after analyzing virtual systems

$$\begin{aligned} &|\varepsilon_1| > 1/l_1 : \dot{\varepsilon}_1 = f_1(x_1, x_2) - m_1 \mathrm{sign}(\varepsilon_1); \\ &|\varepsilon_1| \leq 1/l_1 : \dot{\varepsilon}_1 = f_1(x_1, x_2) - m_1 l_1 \varepsilon_1, \\ &\ddot{\varepsilon}_1 = \tfrac{d}{dt} f_1(x_1, x_2) - m_1 l_1 \dot{\varepsilon}_1 \end{aligned}$$

we obtain inequalities for choosing the parameters of the observer (2.11) similarly to (2.8)–(2.10)

$$m_1 > F_1, l_1 \geq \frac{1}{m_1}\max\left\{\frac{F_1}{\delta_1}, \frac{\overline{F}_1}{\Delta_1}\right\}, \tag{2.12}$$

providing $|\varepsilon_1(t)| \leq \delta_1, |\dot{\varepsilon}_1(t)| = |f_1(x_1, x_2) - v_1| \leq \Delta_1$, then

$$f_1(x_1, x_2) \approx v_1 \Rightarrow x_2(t) \approx \tilde{x}_2(t) = h_1(x_1, v_1), t_1 < t \leq T, \tag{2.13}$$

where $h_1(x_1, v_1)$ is the solution of the equation $f_1(x_1, x_2) = v_1$ with respect to x_2, which exists due to (2.2).

Thus, we applied the technique for estimating external disturbances, presented in the first case, to estimate the unmeasured state variable $x_2(t)$ using the reduced observer (2.11), constructed on the basis of the differential equation of the measured variable, which does not depend on the external disturbance. At the same time, we did not explicitly use the second equation of system (2.1), which depends on the disturbance, so identification $\eta_2(t) \neq 0$ is not needed to solve this problem.

The main limitation of this approach is that at the design stage it is necessary to obtain the estimates $F_1, \overline{F}_1$ required to tune the observer (2.12), taking into account the specific control law and the admissible range of initial values of the state variables.

Note that in this case, the standard reduced observer, which is constructed on the basis of the differential equation of an unmeasured variable, is physically unrealizable in a narrow setting, since the second equation of system (2.1) is under the influence of an external uncontrolled disturbance [6, 7].

Note that due to (2.13) we can reconstruct both $x_2(t)$ and $\eta_2(t) \neq 0$ using a full-order observer

$$\begin{aligned} &\dot{z}_1 = v_1,\ \dot{z}_2 = f_2(x_1, z_2) + bu + \tilde{v}_2, \\ &\tilde{v}_2 = m_2 \mathrm{sat}(l_2 \tilde{\varepsilon}_2), \tilde{\varepsilon}_2 = \tilde{x}_2 - z_2. \end{aligned} \tag{2.14}$$

Taking into account (2.1), (2.14), the system with respect to observation errors $\varepsilon_i = x_i - z_i \in R$ takes the form

$$\begin{aligned} \dot{\varepsilon}_1 &= f_1(x_1, x_2) - v_1(\varepsilon_1), \\ \dot{\varepsilon}_2 &= f_2(x_1, x_2) - f_2(x_1, z_2) + \eta_2 - \widetilde{v}_2(\widetilde{\varepsilon}_2), \end{aligned} \tag{2.15}$$

where for $t > t_1$ the inequalities

$$\begin{aligned} &|\varepsilon_2 - \widetilde{\varepsilon}_2| = |x_2 - \widetilde{x}_2| = |h_1(x_1, v_1 + \dot{\varepsilon}_1) - h_1(x_1, v_1)| \le L_{12}\Delta_1, \\ &|f_2(x_1, z_2 + \varepsilon_2) - f_2(x_1, z_2)| \le L_2 |\varepsilon_2| \end{aligned} \tag{2.16}$$

are met, L_{12}, L_2 are well-known Lipschitz constants. Then, under the conditions similar to (2.10), taking into account (2.16), the estimation $x_2(t) \approx z_2, \eta_2(t) \approx v_2(t)$ is provided in a finite time $t_2 : t_1 \le t_2 << T$.

The conditions for observability of the controlled variable, regardless of external disturbances, have the following form: if in system (2.1) $x_2(t)$ is measured, and $\eta_2(t) \equiv 0$, $\partial f_2(x_1, x_2) / \partial x_1 \ne 0$, then we can reconstruct $x_1(t)$ using a first-order observer similar to (2.11) but based on the second equation of system (2.1), and provide $f_2(x_1, x_2) \approx v_2$ $x_1(t) \approx \widetilde{x}_1(t) = h_2(v_2, x_2)$. In addition, we can obtain estimates of both $x_1(t)$ and $\eta_1(t)$ using a full-order observer, similar to (2.14).

Example 2.3:
If in system (2.1) only $x_1(t)$ is measured and $\eta_1(t) \ne 0$, then in a narrow setting, i.e. without introducing a dynamic model of $\eta_1(t)$, it is impossible to solve the problem of separately estimating the variable $x_2(t)$ and external disturbances. Similarly, if only $x_2(t)$ is measured, and $\eta_2(t) \ne 0$, then it is impossible to solve the problem of estimating separately the variable $x_1(t)$ and external disturbances in a narrow setting.

In the next section, the presented results are used to formalize the observability conditions for the output controlled variable of a single-channel arbitrary order system.

3. MAIN RESULT

3.1. General Case of Triangular SISO System

Let us consider a minimum-phase nonlinear multidimensional SISO system, represented in a triangular affine form "input-output"

$$\begin{aligned} \dot{x}_i &= f_i(x_1, x_2, ..., x_{i+1}) + \eta_i(x_1, x_2, ..., x_i, t), i = \overline{1, n-1}; \\ \dot{x}_n &= f_n(x) + \eta_n(x, t) + bu, \end{aligned} \tag{3.1}$$

where $x = (x_1, ..., x_n)^{\mathrm{T}} \in X \subset R^n$ is the state vector, $x_1(t) \in R$ is controlled output, $u \in R$ is the control, $f_i(x_1, x_2, ..., x_{i+1})$ are functionally and parametrically defined expressions satisfying the Lipschitz conditions, b is a known constant. All uncertainties of the control plant model and external disturbances are concentrated in expressions η_i, in some equations of system (3.1) these terms are absent. It is assumed that in the control process all internal and external signals are bounded similarly to (2.3). The fulfillment of the conditions

$$\frac{\partial f_i(x_1, ..., x_{i+1})}{\partial x_{i+1}} \ne 0, i = \overline{1, n-1}, x \in X, b \ne 0 \tag{3.2}$$

means that system (3.1) is observable with respect to the output in the absence of disturbances, and the output is controllable with respect to the input.

In a narrow setting, when the identification of unknown parameters and the construction of external disturbance generators are not provided or not possible, there are two options for synthesizing feedback that provides tracking of the output variable $x_1(t)$ of a reference signal $g(t)$.

The first variant [9]: if conditions $f_i, \eta_i \in C^{n-i}, i = \overline{1, n-1}$ and $g(t) \in C^n$ are satisfied, system (3.1) is representable in the canonical basis of mixed variables (functions of state variables, external influences and their derivatives), for the estimation of which it is enough to measure $x_1(t)$ и $g(t)$, at the same time, it is possible to provide $\lim\limits_{t\to\infty} x_1(t) = g(t)$ using dynamic feedback.

The second variant [1]: if the functions $\eta_i(t), \dot{g}(t)$ are not smooth, then when measuring $x_1(t),...,x_n(t)$ and $g(t)$, it is possible to provide a given accuracy of the tracking error using static feedback. With an incomplete set of sensors, the possibility of estimating the state variables necessary for feedback synthesis depends on the observability structure of system (3.1) and the absence of disturbances $\eta_j(t) \equiv 0$ in specific j-th equations.

Let the first variant of feedback synthesis be used for system (3.1)–(3.2), but the controlled variable $x_1(t)$ is not measured. Let us formulate the conditions for its observability regardless of disturbances, based on the motivating examples from Section 2. In what follows, a narrow statement of the problem is considered by default, i.e., without additional identification of existing uncertainties.

Theorem 3.1:
If in system (3.1)–(3.2) there is i-th equation ($i = 2,...,n$) such that:

1. $\eta_i(t) \equiv 0,\ \partial f_i(x_1,...,x_{i+1}) / \partial x_1 \neq 0,\ x \in X$;

2. $x_{i+1}(t)$ *is measured (*$x_{n+1}(t) \equiv 0$*);*

3. $x_i(t)$ *is measured or observed independently of disturbances (see Example 2.2);*

4. $x_2(t),...,x_{i-1}(t)$ *either are not arguments of the function* f_i*, or are measured, or are observed regardless of disturbances,*
then for estimating $x_1(t)$ *it is possible to construct a physically realizable reduced observer.*

Proof
The conditions formulated in the lemma are dictated by the triangular form of the considered system (3.1)–(3.2). In the simplest case, when $x_i(t)$ ($i = 2,...,n$) and all available arguments of the function f_i are measured (except for $x_1(t)$), $\eta_i(t) \equiv 0$, then the estimate of $x_1(t)$ can be obtained using the corrective action of the first-order observer constructed on the basis of the i-th equation of system (3.1) similarly to (2.11)–(2.13):

$$\begin{aligned} &\dot{z}_i = v_i, v_i = m_i \mathrm{sat}(l_i \varepsilon_i), \\ &\varepsilon_i = x_i - z_i, \dot{\varepsilon}_i = f_i - v_i, \\ &f_i(x_1, ..., x_{i+1}) \approx v_i \Rightarrow \\ &x_1(t) \approx \widetilde{x}_1(t) = h_i(v_i, x_2, ..., x_{i+1}). \end{aligned} \tag{3.3}$$

Note that if $i = n$, then the first expression (3.3) will take the form $\dot{z}_n = v_n + bu$, but the following expressions (3.3) will not change.

If $x_i(t)$ and/or the available arguments $x_2(t),...,x_{i-1}(t)$ of the function f_i necessary for restoring $x_1(t)$ (3.3) are not measured, then the conditions for their observability are similar to those formulated in the lemma. For example, if in system (3.1)–(3.2) $x_2(t)$ is measured, then under conditions $\eta_j(t) \equiv 0,\ j = 2,3,...,k,\ k \leq i-1$, based on the jth equations, we can construct a physically realizable observer for estimating $x_3(t),...,x_{k+1}(t)$ similarly to (2.14)–

(2.16). If in system (3.1)–(3.2) $x_3(t)$ is measured, then under conditions $\eta_j(t)\equiv 0$, $j=3,...,k$, $k\le i-1$, based on the jth equations, we can construct a physically realizable observer for estimating $x_4(t),...,x_{k+1}(t)$, if at the same time $x_2(t)$ is measured or is not an argument of functions f_j, etc. □

If the conditions of Theorem 3.1 are satisfied and an estimate $\tilde{x}_1(t)$ can be obtained, then for the second variant of the synthesis of the tracking system, the conditions for constructing physically realizable observers for the remaining unmeasured variables with respect to the measured (observed) variables are verified similarly. For example, if $\eta_1(t)\equiv 0$, then there is the possibility of estimating $x_2(t)$; if at the same time $\eta_2(t)\equiv 0$, then and $x_3(t)$ etc.

3.2. Application to Single Link Manipulator

As an example, let us consider two systems of the form (3.1)–(3.2), which describe the equations of motion of a single-link manipulator [16] with a rigid joint type with a DC motor shaft

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \bar{c}_{21}\sin x_1 - \bar{c}_{22}x_2 + \bar{c}_{23}x_3 + \eta(t), \\ \dot{x}_3 &= -\bar{c}_{32}x_2 - \bar{c}_{33}x_3 + \bar{b}u \end{aligned} \tag{3.4}$$

and with an elastic joint type

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \bar{a}_{21}(x_3 - x_1) - \bar{a}_{22}\sin x_1 + \eta(t), \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= -a_{43}(x_3 - x_1) - a_{44}x_4 + a_{45}x_5, \\ \dot{x}_5 &= -\bar{a}_{54}x_4 - \bar{a}_{55}x_5 + \bar{b}u, \end{aligned} \tag{3.5}$$

where $x_1(t)$ is the angular position of the pendulum (controlled output), an overline means that this parameter is not exactly known, $\eta(t)\neq 0$.

As we can see, the conditions of the Theorem 3.1 are not satisfied in system (3.4). If the controlled variable $x_1(t)$ is not measured, then in a narrow formulation it is impossible to obtain its estimate from measurements of $x_2(t), x_3(t)$, this is possible only in the absence (or identification) of uncertainties $\bar{c}_{21},\bar{c}_{22},\bar{c}_{23},\eta$. Thus, in system (3.4), the first variant of the tracking system synthesis is realized only when measuring x_1.

For the second variant of the synthesis of the tracking system, the minimum set of sensors is measurements $x_1(t), x_3(t)$, since, based on the first equation of system (3.4), it is possible to construct a reduced observer for estimating $x_2(t)$, and it is not necessary to identify the existing uncertainties to solve the observation problem.

In system (3.5), the conditions of the lemma are satisfied for $i=4$ and measurements $x_3(t), x_4(t), x_5(t)$. Then the corrective action of the first-order observer constructed on the basis of the fourth equation of system (21) will give the estimate $x_1(t)$. Sensors $x_3(t), x_5(t)$ are the minimum set for estimating $x_1(t)$. In this case, estimates of both $x_1(t)$ and $x_4(t)$ can be obtained using a second-order observer constructed on the basis of the third and fourth equations of system (3.5) [10, 11].

Sensors $x_3(t), x_5(t)$ are the minimum set for the second variant of the synthesis of the tracking system. In this case, the third-order observer is constructed on the basis of the unperturbed subsystem

$$\begin{aligned} \dot{x}_3 &= x_4, \\ \dot{x}_4 &= -a_{43}(x_3 - x_1) - a_{44}x_4 + a_{45}x_5, \\ \dot{x}_1 &= x_2 \end{aligned} \tag{3.6}$$

in the form of

$$\begin{aligned} \dot{z}_3 &= z_4 + v_3, \\ \dot{z}_4 &= -a_{43}(x_3 - z_1) + a_{45}x_5 + v_4, \\ \dot{z}_1 &= v_1 \end{aligned}$$

allows one to obtain the estimates $x_4(t), x_1(t), x_2(t)$.

If $x_1(t), x_3(t)$ are measured, then the observer constructed on the basis of subsystem (3.6) in the form of

$$\begin{aligned} \dot{z}_3 &= z_4 + v_3, \\ \dot{z}_4 &= -a_{43}(x_3 - x_1) - a_{44}z_4 + v_4, \\ \dot{z}_1 &= v_1 \end{aligned}$$

allows one to obtain the estimates $x_4(t), x_5(t), x_2(t)$.

An example of constructing a reduced-order observer for a multilink manipulator whose movements are described by matrix equations of the type (3.4) is presented in the work [11]. Simulation modeling of closed systems (3.4), (3.5) with different sets of sensors and with the corresponding reduced observers confirmed their performance [2, 9, 10].

4. CONCLUSION

The control plant model is often presented in the canonical “input-output” form in order to make it convenient to perform mathematical analysis and synthesis of the tracking system. However, the canonical form completely depersonalizes the control plant model and hides its features. This paper shows that if the output variable is not measured in tracking systems operating under parametric and external disturbances, then it is advisable to use the triangular shape “input-output” to solve the problem of its observation. At the same time, the use of the technique for estimating external disturbances for estimating unmeasured state variables makes it possible to construct robust reduced-order observers.

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