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Stability Analysis of Switched Positive Persidskii Systems with Distributed and Unbounded Delays

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Abstract: Switched positive Persidskii systems with distributed and unbounded delays are studied. Right-hand sides of these systems are linear combinations of nonlinearities of a sector type. Special constructions of diagonal Lyapunov–Krasovskii functionals are proposed and conditions are derived under which the absolute stability of the considered systems can be proved with the aid of such functionals. The developed approaches are applied to the stability analysis of a mechanical system with switched nonlinear positional forces and to a problem of mobile agent deployment. Results of numerical simulations are presented confirming the theoretical conclusions.

Keywords: switched positive system, sector nonlinearity, Lyapunov–Krasovskii functional, distributed and unbounded delays, mechanical system, formation control

1. INTRODUCTION

Studying the stability of nonlinear time-delay systems is an actual problem of the contemporary control theory due to wide applications of these systems (see, e.g., [1, 2] and the bibliography therein). Moreover, it is worth noticing that, in numerous models of practical systems, state variables are restricted to be nonnegative [3, 4]. Therefore, an important class of time-delay systems is that of positive systems. Methods for stability analysis of positive systems with delay are well developed for linear systems [3,4]. In the nonlinear case, the basic approaches are the comparison method and the Lyapunov direct method [5–8]. However, under the constructing comparison systems, usually it is required that nonlinearities satisfy linear estimates (see [9]). As regards to the Lyapunov direct method, its application to time-delay systems is based on the using Lyapunov–Razumikhin functions or Lyapunov–Krasovskii functionals [1, 5]. At the same time, a general constructive technique to finding such functions and functionals is still lacking. This problem becomes even more difficult if the model under consideration needs to take into account uncertainties and switching of operation modes. As a result, stability conditions are well investigated only for special classes of nonlinear time-delay systems, for instance, for homogeneous systems, Lurie control systems, etc. [5, 10–13].

One of the interesting and important such classes is that of Persidskii systems [3]. Right-hand sides of these systems are represented as linear combinations of separable nonlinearities satisfying sector conditions. Persidskii systems are widely used for modeling automatic control systems, neural networks, opinion dynamics, digital filters, etc. [3, 14].

First, stability conditions for delay-free Persidskii systems were obtained by Barbashin via constructing a Lyapunov function in the form of linear combination of integrals from nonlinearities [15]. In [16], linear Lyapunov functions were proposed for the stability analysis

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of positive delay-free Persidskii systems. In a number of subsequent works, various canonical forms of Lyapunov functions were used to derive stability conditions for both switched and nonswitched Persidskii systems (see [3, 17, 18] and the references therein). Stability of switched positive Persidskii systems with constant delay was studied in [19] with the aid of a special construction of Lyapunov–Krasovskii functionals. In [8, 9], positive Persidskii systems with distributed delay were investigated. In recent paper [14], generalized Persidskii systems with constant delays were considered and conditions of input-to-state stability formulated in terms of LMIs were derived.

In [20], an approach to constructing Lyapunov–Krasovskii functionals for switched positive Persidskii systems with constant delay was developed. The proposed functionals depend on a positive tuning parameter. It was shown that, via an appropriate parameter choice, less conservative stability conditions can be obtained compared to known ones. The objective of the present paper is an extension of the above approach to systems with distributed and unbounded delays. In addition, we will apply our results to the stability analysis of a mechanical system with nonlinear positional forces and to a problem of formation control.

Through the paper we will use the following notation:

• R is the field of real numbers, R^n and $R^{n\times n}$ are the n-dimensional Euclidean space and the vector space of $n \times n$ matrices, respectively.

• Let $\|\cdot\|$ be the Euclidean norm of a vector.

• A matrix $C \in R^{n\times n}$ is called nonnegative if all its entries are nonnegative.

• A matrix $C \in R^{n\times n}$ is called Metzler if all its off-diagonal entries are nonnegative.

• The identity matrix is denoted by I.

• Inequalities for vectors are understood componentwise.

2. STATEMENT OF THE PROBLEM

Consider the following switched Persidskii system with distributed delay:

$$\dot{x}(t) = P_\sigma \Phi(x(t)) + Q_\sigma \int_{t-\tau}^{t} f(\xi - t)\Phi(x(\xi))d\xi. \tag{1}$$

Here $x(t) \in R^n$, $\Phi(x)$ is a separable vector function, i.e., $\Phi(x) = (\varphi_1(x_1), \ldots, \varphi_n(x_n))^\top$, where the functions $\varphi_i(x_i)$, which are said to be admissible nonlinearities, are continuous and locally Lipschitz for $|x_i| < \Delta$ ($0 < \Delta \le +\infty$) and satisfy the sector constraints $x_i\varphi_i(x_i) > 0$ for $x_i \ne 0$, $i = 1, \ldots, n$, $\tau = \text{const} > 0$, $f(\zeta)$ is nonnegative and continuous for $\zeta \in [-\tau, 0]$ scalar kernel, $\sigma = \sigma(t)$ is a piecewise constant function defining the switching law, $\sigma(t) : [0, +\infty) \mapsto \{1, \ldots, N\}$, P_s, Q_s are constant matrices, $s = 1, \ldots, N$.

According to the standard assumption [21], we will consider the case where the function $\sigma(t)$ admits only finitely many discontinuities on every bounded interval. Such switching laws will be called admissible.

Initial functions for solutions of (1) are chosen from the space $C([-\tau, 0], R^n)$ of continuous functions $\theta(\xi) : [-\tau, 0] \mapsto R^n$ with the uniform norm

$$\|\theta\|_\tau = \max_{\xi\in[-\tau,0]} \|\theta(\xi)\|.$$

For a solution $x(t)$, x_t denotes the restriction of this solution to the interval $[t - \tau, t]$, i.e., $x_t : \xi \mapsto x(t + \xi)$ for $\xi \in [-\tau, 0]$.

Assumption 2.1:
The matrices $P_1, \ldots, P_N$ are Metzler and the matrices $Q_1, \ldots, Q_N$ are nonnegative.

Remark 2.1:
Under Assumption 2.1, the system (1) is positive (see [6]).

From the continuity of $\Phi(x)$ and the sector restrictions it follows that $\Phi(0)=0$. Hence, there exists the zero solution of (1).

Definition 2.1:
The system (1) is called absolutely stable if its zero solution is asymptotically stable for any admissible switching signal and any admissible nonlinearities.

To derive the absolute stability conditions, we will use a special construction of diagonal Lyapunov–Krasovskii functional. This functional is a counterpart of that proposed in [20] for switched positive Persidskii systems with constant delays.

Furthermore, we will study the problem of absolute stability for a switched positive Persidskii system with unbounded delay.

Finally, we will provide applications of the developed approaches to the stability analysis of a mechanical system with switched nonlinear positional forces and to a problem of mobile agent deployment on a line segment.

3. ABSOLUTE STABILITY CONDITIONS FOR THE SYSTEM WITH DISTRIBUTED DELAY

Construct a Lyapunov–Krasovskii functional candidate for (1) as follows:

$$V_1(x_t)=\sum_{i=1}^{n}\alpha_i\int_0^{x_i(t)}\varphi_i^{\nu}(u)du+\sum_{i=1}^{n}\beta_i\int_{t-\tau}^{t}\int_{-\tau}^{\xi-t}f(\zeta)d\zeta\varphi_i^{\nu+1}(x_i(\xi))d\xi, \qquad (2)$$

where α_i,β_i are positive coefficients, ν is a positive rational number with odd numerator and denominator. We will look for conditions under which the absolute stability of (1) can be proven with the aid of such a functional.

Theorem 3.1:
Let Assumption 2.1 be fulfilled. If there exist a positive rational number ν with odd numerator and denominator, positive vectors λ, η and numbers ω_1,ω_2 such that

$$\nu\omega_1+\omega_2<0, \qquad (3)$$

$$\left(P_s+\int_{-\tau}^{0}f(\zeta)d\zeta Q_s\right)\lambda\le\omega_1\lambda,\quad s=1,\dots,N,$$

$$\left(P_s+\int_{-\tau}^{0}f(\zeta)d\zeta Q_r\right)^{\top}\eta\le\omega_2\eta,\quad s,r=1,\dots,N,$$

then one can choose coefficients α_i,β_i for which the functional (2) guarantees the absolute stability of the system (1).

Proof
Differentiating $V_1(x_t)$ along the solutions of (1), we obtain

$$\dot V_1=\sum_{i,j=1}^{n}\alpha_i p_{ij}^{(\sigma)}\varphi_i^{\nu}(x_i(t))\varphi_j(x_j(t))+\sum_{i,j=1}^{n}\alpha_i q_{ij}^{(\sigma)}\varphi_i^{\nu}(x_i(t))\int_{t-\tau}^{t}f(\xi-t)\varphi_j(x_j(\xi))d\xi$$

$$-\sum_{i=1}^{n}\beta_i\int_{t-\tau}^{t}f(\xi-t)\varphi_i^{\nu+1}(x_i(\xi))d\xi+F\sum_{i=1}^{n}\beta_i\varphi_i^{\nu+1}(x_i(t)),$$

where $F = \int_{-\tau}^{0} f(\zeta)d\zeta$, $p_{ij}^{(\sigma)}$ and $q_{ij}^{(\sigma)}$ are entries of the matrices P_σ and Q_σ, respectively.

Let

$$\alpha_i = \eta_i/\lambda_i^{\nu}, \quad i = 1, \dots, n, \tag{4}$$

$$y_i(t) = \varphi_i(x_i(t))/\lambda_i, \quad i = 1, \dots, n. \tag{5}$$

Here λ_i and η_i are components of the vectors λ and η, respectively. Then the derivative of the functional can be rewritten in the form

$$\dot V_1 = \sum_{i,j=1}^{n} \eta_i \lambda_j p_{ij}^{(\sigma)} y_i^{\nu}(t) y_j(t) + \sum_{i,j=1}^{n} \eta_i \lambda_j q_{ij}^{(\sigma)} y_i^{\nu}(t) \int_{t-\tau}^{t} f(\xi - t) y_j(\xi) d\xi$$

$$- \sum_{i=1}^{n} \beta_i \lambda_i^{\nu+1} \int_{t-\tau}^{t} f(\xi - t) y_i^{\nu+1}(\xi) d\xi + F \sum_{i=1}^{n} \beta_i \lambda_i^{\nu+1} y_i^{\nu+1}(t).$$

Using the Young inequality [1], we arrive at the estimate

$$\dot V_1 \le \frac{\nu}{\nu+1} \sum_{i=1}^{n} \eta_i y_i^{\nu+1}(t) \sum_{j=1}^{n} \left(p_{ij}^{(\sigma)} + F q_{ij}^{(\sigma)} \right) \lambda_j$$

$$+ \sum_{i=1}^{n} \lambda_i y_i^{\nu+1}(t) \left(\frac{1}{\nu+1} \sum_{j=1}^{n} \eta_j p_{ji}^{(\sigma)} + F \beta_i \lambda_i^{\nu} \right)$$

$$+ \frac{1}{\nu+1} \sum_{i=1}^{n} \lambda_i \int_{t-\tau}^{t} f(\xi - t) y_i^{\nu+1}(\xi) d\xi \sum_{j=1}^{n} \eta_j q_{ji}^{(\sigma)}$$

$$- \sum_{i=1}^{n} \beta_i \lambda_i^{\nu+1} \int_{t-\tau}^{t} f(\xi - t) y_i^{\nu+1}(\xi) d\xi.$$

Choose the coefficients β_i as follows:

$$\beta_i = \frac{1}{(\nu+1)\lambda_i^{\nu}} \left(\max_{r=1,\dots,N} \sum_{j=1}^{n} \eta_j q_{ji}^{(r)} + \delta \right), \tag{6}$$

$$i = 1, \dots, n,$$

where δ is a positive parameter. If $\delta F < -(\nu\omega_1 + \omega_2)/2$, then

$$\dot V_1 \le \frac{\nu\omega_1 + \omega_2}{2(\nu+1)} \sum_{i=1}^{n} \eta_i \lambda_i y_i^{\nu+1}(t) - \frac{\delta}{\nu+1} \sum_{i=1}^{n} \lambda_i \int_{t-\tau}^{t} f(\xi - t) y_i^{\nu+1}(\xi) d\xi.$$

Hence (see [1]), the zero solution of the system (1) is asymptotically stable. This completes the proof. □

Remark 3.1:
It is worth mentioning that, under Assumption 2.1, the conditions of Theorem 3.1 are fulfilled iff one of the following conditions is satisfied:

(a) there exists a vector $\lambda > 0$ such that

$$(P_s + FQ_s)\lambda < 0, \quad s = 1, \dots, N, \tag{7}$$

(b) there exists a vector $\eta > 0$ such that

$$(P_s + FQ_r)^\top \eta < 0, \quad s, r = 1, \dots, N.$$

Really, in the case (a) the conditions of Theorem 3.1 hold for sufficiently large values of ν, whereas in the case (b) they hold for sufficiently small values of ν. However, in some problems, e.g., estimation of attraction domain or the convergence rate of solutions, analysis of the impact of external perturbations, etc., to derive less conservative results, it is useful to have opportunity for an appropriate choice of the parameter ν (see, for instance, [20]). The presented statement of Theorem 3.1 permits us to obtain the domain of admissible values for this parameter.

4. ABSOLUTE STABILITY CONDITIONS FOR THE SYSTEM WITH UNBOUNDED DELAY

Next, consider the system

$$\dot{x}(t) = P_\sigma \Phi(x(t)) + Q_\sigma \int_0^t g(t-\xi)\Phi(x(\xi))d\xi, \tag{8}$$

where $g(\zeta)$ is nonnegative and continuous for $\zeta \geq 0$ scalar kernel and the remaining notation is the same as for (1). Thus, we will study switched Persidskii system with unbounded delay.

It should be noted that models with unbounded delays are widely used in various applications such as PID controller design, population dynamics, networked control, social science, etc. (see [1, 5, 22, 23]).

Every solution $x(t)$ of (8) is defined by an initial time instant $t_0 \geq 0$ and an initial function $\theta(\xi) \in C([0, t_0], R^n)$, where $C([0, t_0], R^n)$ is the space of continuous functions $\theta(\xi) : [0, t_0] \mapsto R^n$ with the uniform norm

$$\|\theta\|_{[0,t_0]} = \max_{\xi \in [0,t_0]} \|\theta(\xi)\|.$$

Let x_t be the restriction of a solution $x(t)$ of (8) to the interval $[0, t]$, i.e., $x_t : \xi \mapsto x(t+\xi)$ for $\xi \in [-t, 0]$.

Remark 4.1:
It is known [7] that, under Assumption 2.1, the system (8) is positive.

Definition 4.1:
The system (8) is called absolutely stable if its zero solution is asymptotically stable for any admissible switching signal and any admissible nonlinearities.

To obtain absolute stability conditions for (8) we impose additional constraints on the kernel $g(\zeta)$.

Assumption 4.1:
Let $\int_0^{+\infty} g(\zeta)d\zeta < +\infty$.

Assumption 4.2:
Let $\int_0^{+\infty} G(\zeta)d\zeta < +\infty$, *where* $G(\zeta) = \int_\zeta^{+\infty} g(\xi)d\xi$.

Remark 4.2:
In particular, Assumptions 4.1 and 4.2 are satisfied for the function $g(\zeta) = \exp(-c\zeta)$, *where* $c = \text{const} > 0$. *It is worth mentioning that PID-controllers with exponential kernels are widely used in the modern control theory (see [1]).*

We will choose a Lyapunov–Krasovskii functional candidate for (8) in the form

$$V_2(x_t) = \sum_{i=1}^{n} \alpha_i \int_0^{x_i(t)} \varphi_i^{\nu}(u)du + \sum_{i=1}^{n} \beta_i \int_0^t G(t-\xi)\varphi_i^{\nu+1}(x_i(\xi))d\xi, \tag{9}$$

where α_i, β_i are positive coefficients, ν is a positive rational number with odd numerator and denominator.

Theorem 4.1:
Let Assumptions 2.1, 4.1, 4.2 be fulfilled. If there exist a positive rational number ν with odd numerator and denominator, positive vectors λ, η and numbers ω_1, ω_2 such that

$$(P_s + G(0)Q_s)\,\lambda \leq \omega_1 \lambda, \ \ s = 1, \ldots, N, \tag{10}$$

$$(P_s + G(0)Q_r)^{\top}\,\eta \leq \omega_2 \eta, \ \ s, r = 1, \ldots, N, \tag{11}$$

and the inequality (3) hold, then one can choose coefficients α_i, β_i for which the functional (9) guarantees the absolute stability of the system (8).

Proof
Consider the derivative of the functional (9) along the solutions of (8). We obtain

$$\dot{V}_2 = \sum_{i,j=1}^{n} \alpha_i p_{ij}^{(\sigma)} \varphi_i^{\nu}(x_i(t))\varphi_j(x_j(t))$$

$$+ \sum_{i,j=1}^{n} \alpha_i q_{ij}^{(\sigma)} \varphi_i^{\nu}(x_i(t)) \int_0^t g(t-\xi)\varphi_j(x_j(\xi))d\xi$$

$$- \sum_{i=1}^{n} \beta_i \int_0^t g(t-\xi)\varphi_i^{\nu+1}(x_i(\xi))d\xi + G(0)\sum_{i=1}^{n} \beta_i \varphi_i^{\nu+1}(x_i(t)).$$

Define the coefficients α_i and β_i by the formulae (4) and (6), where λ_i and η_i are components of the positive vectors λ and η satisfying the inequalities (10) and (11), respectively.

Similarly to the proof of Theorem 3.1, using the substitution (5) and the Young inequality, we arrive at the estimate

$$\dot{V}_2 \leq -c_1 \sum_{i=1}^{n} y_i^{\nu+1}(t) - c_2 \sum_{i=1}^{n} \int_0^t g(t-\xi) y_i^{\nu+1}(\xi)d\xi,$$

where $c_1 > 0$, $c_2 > 0$. Hence (see [5]), the zero solution of the system (8) is asymptotically stable. This completes the proof. □

It is worth noticing that, regarding Theorem 4.1, a remark similar to Remark 3.1 can be formulated.

5. APPLICATIONS

Consider two applications of the proposed approaches.

5.1. Stability Analysis of a Mechanical System

Let motions of a mechanical system be defined by the equations

$$\ddot{q}(t) + hA\dot{q}(t) + B_\sigma \Phi(q(t)) + C_\sigma \int_0^t g(t-\xi)\Phi(q(\xi))d\xi = 0. \tag{12}$$

Here $q(t), \dot{q}(t) \in R^n$ are vectors of generalized coordinates and generalized velocities, respectively, $\Phi(q) = (\varphi_1(q_1), \dots, \varphi_n(q_n))^\top$, where the functions $\varphi_i(q_i)$ are continuous for $|q_i| < \Delta$ $(0 < \Delta \leq +\infty)$ and satisfy the sector constraints $q_i\varphi_i(q_i) > 0$ for $q_i \neq 0$, $i = 1, \dots, n$, $g(\zeta)$ is nonnegative and continuous for $\zeta \geq 0$ scalar kernel, $\sigma = \sigma(t)$ is an admissible switching law, $\sigma(t) : [0, +\infty) \mapsto \{1, \dots, N\}$, A, B_s, C_s are constant matrices, $s = 1, \dots, N$, h is a positive parameter. It is worth noting that the term $C_\sigma \int_0^t g(t-\xi)\Phi(q(\xi))d\xi$ can be treated as integral part of a PID-controller [5,22].

Every solution $q(t)$ of (12) is defined by an initial time instant $t_0 \geq 0$ and an initial function $\theta(\xi) \in C^1([0, t_0], R^n)$, where $C^1([0, t_0], R^n)$ is the space of continuously differentiable functions $\theta(\xi) : [0, t_0] \mapsto R^n$ with the uniform norm

$$\|\theta\|_{[0,t_0]} = \max_{\xi \in [0,t_0]} \left(\|\theta(\xi)\| + \|\dot{\theta}(\xi)\| \right).$$

The system (12) admits the equilibrium position

$$q = \dot{q} = 0. \tag{13}$$

To derive stability conditions for (13), we will use a special approach based on the decomposition method. This approach was first proposed in [24] for stability analysis of linear delay-free mechanical systems and it was subsequently extended to some classes of linear and nonlinear mechanical systems with constant delay [12,25].

Consider the following auxiliary subsystems:

$$\dot{x}(t) = -A^{-1} \left(B_\sigma \Phi(x(t)) + C_\sigma \int_0^t g(t-\xi)\Phi(x(\xi))d\xi \right), \tag{14}$$

$$\dot{y}(t) = -Ay(t). \tag{15}$$

Assumption 5.1:
The matrices

$$P_s = -A^{-1}B_s, \quad s = 1, \dots, N, \tag{16}$$

are Metzler and the matrices

$$Q_s = -A^{-1}C_s, \quad s = 1, \dots, N, \tag{17}$$

are nonnegative.

Assumption 5.2:
The system (15) is asymptotically stable.

Assumption 5.3:
Vector function $\Phi(q)$ satisfies the Lipschitz condition for $\|q\| < \Delta$ with a constant $L > 0$.

Theorem 5.1:
Let the matrices P_s, Q_s be defined by the formulae (16), (17), respectively, and Assumptions 4.1, 4.2, 5.1–5.3 be fulfilled. If one of the following conditions is satisfied:

(a) there exists a vector $\lambda > 0$ such that

$$(P_s + G(0)Q_s)\lambda < 0, \quad s = 1, \dots, N,$$

(b) there exists a vector $\eta > 0$ such that

$$(P_s + G(0)Q_r)^\top \eta < 0, \quad s, r = 1, \dots, N,$$

then one can find a number $\bar{h} > 0$ such that the equilibrium position (13) is asymptotically stable for $h \geq \bar{h}$, for any admissible switching law and any admissible nonlinearity $\Phi(q)$.

Proof
Define new variables as follows:

$$x(t) = q(t) + \frac{1}{h}A^{-1}\dot{q}(t), \quad y(t) = \dot{q}(t). \tag{18}$$

Then

$$\begin{aligned} h\dot{x}(t) &= P_\sigma \Phi(x(t)) + Q_\sigma \int_0^t g(t-\xi)\Phi(x(\xi))d\xi \\ &+ P_\sigma \left(\Phi\left(x(t) - \frac{1}{h}A^{-1}y(t) \right) - \Phi(x(t)) \right) \\ &+ Q_\sigma \int_0^t g(t-\xi) \left(\Phi\left(x(\xi) - \frac{1}{h}A^{-1}y(\xi) \right) - \Phi(x(\xi)) \right) d\xi, \\ \dot{y}(t) &= -hAy(t) + P_\sigma \Phi\left(x(t) - \frac{1}{h}A^{-1}y(t) \right) \\ &+ Q_\sigma \int_0^t g(t-\xi)\Phi\left(x(\xi) - \frac{1}{h}A^{-1}y(\xi) \right) d\xi. \end{aligned} \tag{19}$$

It is worth mentioning that (19) can be considered as a complex system describing the interaction of subsystems (14) and (15).

Under the conditions of Theorem 5.1, there exist a positive rational number ν with odd numerator and denominator and positive numbers α_i, β_i, $i = 1, \dots, n$, such that

$$W(t, x_t) \leq -c_1 \|\Phi(x(t))\|^{\nu+1} - c_2 \int_0^t g(t-\xi)\|\Phi(x(\xi))\|^{\nu+1} d\xi,$$

where $W(t, x_t)$ is the derivative of the functional (9) along the solutions of the subsystem (14) and c_1, c_2 are positive coefficients.

It is known (see [26]) that Assumption 5.2 implies the existence of a Lyapunov function $V_3(y)$ with the following properties:

(i) $V_3(y)$ is continuously differentiable for $y \in R^n$;

(ii) $V_3(y)$ is positive definite;

(iii) $V_3(y)$ is a homogeneous of the order $\nu + 1$ with respect to the standard dilation function;

(iv) the derivative of $V_3(y)$ along the solutions of the subsystem (15) is negative definite.

Construct a Lyapunov–Krasovskii functional candidate for (19) in the form

$$\widetilde{V}(x_t, y_t) = h \sum_{i=1}^n \alpha_i \int_0^{x_i(t)} \varphi_i^\nu(u)du + \sum_{i=1}^n \beta_i \int_0^t G(t-\xi)\varphi_i^{\nu+1}(x_i(\xi))d\xi$$

$$+ V_3(y(t)) + \int_0^t G(t-\xi)\|y(\xi)\|^{\nu+1} d\xi.$$

Differentiating this functional along the solutions of (19) and using the Lipschitz condition for $\Phi(q)$, we obtain

$$\dot{\tilde{V}} \leq -c_1\|\Phi(x(t))\|^{\nu+1} - c_2\int_0^t g(t-\xi)\|\Phi(x(\xi))\|^{\nu+1}d\xi$$

$$-(hc_3 - G(0))\|y(t)\|^{\nu+1} - \int_0^t g(t-\xi)\|y(\xi)\|^{\nu+1}d\xi$$

$$+\frac{c_4}{h}\|\Phi(x(t))\|^{\nu}\left(\|y(t)\| + \int_0^t g(t-\xi)\|y(\xi)\|d\xi\right)$$

$$+c_5\|y(t)\|^{\nu}\Bigg(\|\Phi(x(t))\| + \int_0^t g(t-\xi)\|\Phi(x(\xi))\|d\xi$$

$$+\frac{1}{h}\|y(t)\| + \frac{1}{h}\int_0^t g(t-\xi)\|y(\xi)\|d\xi\Bigg),$$

where c_3, c_4, c_5 are positive constants.

With the aid of the Young inequality, it is easy to prove the existence of a number $\bar{h} > 0$ such that

$$\dot{\tilde{V}} \leq -\frac{1}{2}\Bigg(c_1\|\Phi(x(t))\|^{\nu+1} + hc_3\|y(t)\|^{\nu+1}$$

$$+\int_0^t g(t-\xi)\left(c_2\|\Phi(x(\xi))\|^{\nu+1} + \|y(\xi)\|^{\nu+1}\right)d\xi\Bigg)$$

for $h \geq \bar{h}$. This implies the asymptotic stability of the zero solution of (19), see [5]. Taking into account the properties of the transformation (18), we obtain that, for such values of h, the equilibrium position (13) of the system (12) is asymptotically stable, as well. This completes the proof. □

5.2. A Problem of Mobile Agent Deployment

Let a group of n mobile agents on a line be given. The agents are interpreted as numbered points with coordinates $z_i(t) \in R$, $i = 1, \ldots, n$, and the agent dynamics is modeled by the first-order integrators

$$\dot{z}_i(t) = u_i, \qquad i = 1, \ldots, n,$$

where u_i is a control input.

Assume that a segment $[a, b]$ of the line is given. Consider the problem of synthesis of a decentralized control protocol that provides a prescribed agent deployment on the segment.

In [27], this problem was solved for the case of the equidistant agent distribution under the condition that each agent receives information about the distances between itself and its nearest left and right neighbors. It should be noted that neighbors are understood in terms of agent numbers.

In [28, 29], the result of [27] was extended to the case where each agent receives information about the distances between itself and several its left neighbors and several its right neighbors (not necessarily nearest neighbors). Furthermore, the effect of communication delay and switching of network topology (replacing chosen neighbors by the other ones) was investigated [28,29]. Linear control protocols were designed for which neither constant delay values nor switching disturb the agent convergence to the equidistant distribution. In addition, the problem of nonlinearly-uniform (uniform with respect to a given nonlinear function) agent deployment was studied [29]. It is worth mentioning that such a problem is important for coverage control of mobile sensors, where a cost function is introduced to evaluate how well a given curve or domain is covered by the sensor network, and for synchronization of processes relatively to certain functions of phase coordinates [30, 31]. In [29], nonlinear control protocols were proposed and robustness of these protocols with respect to constant communication delays and network topology switching was proved.

However, it should be noted that, in various formation control models, instead of discrete delays, distributed ones are used (see [1,32]). In particular, an example of such a model is the traffic flow dynamics [32]. Therefore, in the present subsection, we will consider the problem of nonuniform agent deployment under distributed delay in the communication channels.

Let a continuous, locally Lipschitz and strictly increasing for $z \in (-\infty, +\infty)$ scalar function $\varrho(z)$ be given. Our objective is to design a decentralized control protocol providing the convergence of agents to the positions $\bar{z}_i$ for which the corresponding points $\varrho(\bar{z}_i)$ are uniformly distributed on the segment $[\varrho(a), \varrho(b)]$. Hence,

$$\varrho(\bar{z}_i) = \varrho(a) + \frac{i}{n+1}(\varrho(b) - \varrho(a)), \quad i = 1, \ldots, n.$$

Denote $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n)^\top$. In what follows, the points a and b will be interpreted as static agents, i.e., $z_0(t) = a$, $z_{n+1}(t) = b$ for $t \in [0, +\infty)$.

We will consider the scenario where each agent receives information from some its neighbors, and connections between agents can be switched on and off at any time instant. Let $\sigma(t) : [0, +\infty) \mapsto \{1, \ldots, N\}$ be an admissible switching law defining the operating order of communication topologies, $\Xi_{il}^{(\sigma(t))}$ and $\Xi_{ir}^{(\sigma(t))}$ be the sets of indices of left and right neighbors, respectively, from which the ith agent receives information at the instant t, and $\Xi_{i}^{(\sigma(t))} = \Xi_{il}^{(\sigma(t))} \cup \Xi_{ir}^{(\sigma(t))}$, $i = 1, \ldots, n$.

Assumption 5.4:
Let $\Xi_{il}^{(s)} \neq \emptyset$ and $\Xi_{ir}^{(s)} \neq \emptyset$ for $i = 1, \ldots, n$, $s = 1, \ldots, N$.

Assumption 5.5:
Each ith agent at each time instant knows the values $\varrho(z_i(t)) - c\int_{t-\tau}^{t} f(\xi - t)\varrho(x_j(\xi))d\xi$ for $j \in \Xi_i^{(\sigma(t))}$, where $\tau = \mathrm{const} > 0$, $f(\zeta)$ is a nonnegative and continuous for $\zeta \in [-\tau, 0]$ kernel, $\int_0^\tau f(\zeta)d\zeta > 0$, $c = 1/\int_0^\tau f(\zeta)d\zeta$.

Assumption 5.6:
Each agent at each time instant knows how many agents are located between itself and agents from which the signals are received.

Define the coefficients $a_{ij}^{(s)}$ by the formulae

$$a_{ij}^{(s)} = \frac{\delta_i^{(s)}}{(i-j)M_{il}^{(s)}} \ \text{ for } \ j \in \Xi_{il}^{(s)},$$

$$a_{ij}^{(s)} = \frac{\delta_i^{(s)}}{(j-i)M_{ir}^{(s)}} \ \text{ for } \ j \in \Xi_{ir}^{(s)},$$

where $M_{il}^{(s)}$ and $M_{ir}^{(s)}$ are the numbers of elements of the sets $\Xi_{il}^{(s)}$ and $\Xi_{ir}^{(s)}$, respectively, and

$$\delta_i^{(s)} = \left(\frac{1}{M_{il}^{(s)}} \sum_{j \in \Xi_{il}^{(s)}} \frac{1}{i-j} + \frac{1}{M_{ir}^{(s)}} \sum_{j \in \Xi_{ir}^{(s)}} \frac{1}{j-i} \right)^{-1}, \quad i = 1, \ldots, n, \quad s = 1, \ldots, N.$$

Construct a control protocol as follows:

$$u_i = \sum_{j\in\Xi_i^{(\sigma(t))}} a_{ij}^{(\sigma(t))} \left(c\int_{t-\tau}^{t} f(\xi - t)\varrho(z_j(\xi))d\xi - \varrho(z_i(t)) \right), \quad i = 1,\ldots,n.$$

Then the corresponding closed-loop system takes the form

$$\dot{z}_i(t) = \sum_{j\in\Xi_i^{(\sigma(t))}} a_{ij}^{(\sigma(t))} \left(c\int_{t-\tau}^{t} f(\xi - t)\varrho(z_j(\xi))d\xi - \varrho(z_i(t)) \right), \quad i = 1,\ldots,n. \tag{20}$$

Theorem 5.2:
Let Assumptions 5.4–5.6 be fulfilled. Then the system (20) admits the equilibrium position $\bar{z}$ that is globally asymptotically stable for any admissible function $\varrho(z)$ and any admissible switching law.

Proof
It is easy to verify that $\tilde{z}$ is an equilibrium position for (20). Hence, the system (20) can be rewritten in the form

$$\dot{z}_i(t) = \varrho(\bar{z}_i) - \varrho(z_i(t)) + c\sum_{j\in\Xi_i^{(\sigma(t))},\,1\le j\le n} a_{ij}^{(\sigma(t))} \int_{t-\tau}^{t} f(\xi - t)\left(\varrho(z_j(\xi)) - \varrho(\bar{z}_j)\right)d\xi,\ \ i = 1,\ldots,n.$$

Let $x_i(t) = z_i(t) - \bar{z}_i$, $i = 1,\ldots,n$. Then

$$\dot{x}_i(t) = -\varphi_i(x_i(t)) + c\sum_{j\in\Xi_i^{(\sigma(t))},\,1\le j\le n} a_{ij}^{(\sigma(t))} \int_{t-\tau}^{t} f(\xi - t)\varphi(x_j(\xi))d\xi,\ \ i = 1,\ldots,n. \tag{21}$$

Here $\varphi_i(x_i) = \varrho(x_i + \bar{z}_i) - \varrho(\bar{z}_i)$. Thus, we arrive at the system of the form (1) where $P_s = -I$ and the components $q_{ij}^{(s)}$ of the matrices Q_s are defined as follows: $q_{ij}^{(s)} = ca_{ij}^{(s)}$ for $j \in \Xi_i^{(s)}$, and $q_{ij}^{(s)} = 0$ for $j \notin \Xi_i^{(s)}$, $i, j = 1,\ldots,n$, $s = 1,\ldots,N$.

It is known (see the proof of Theorem 1 in [29]), that, for such matrices P_s and Q_s, the corresponding inequality system (7) admits a positive solution. According to Remark 3.1 we obtain that, if ν is sufficiently large, then there exists a functional of the form (2) guaranteeing the asymptotic stability of the zero solution of (21). It is worth noticing that in this case the asymptotic stability is global [5]. This implies the global asymptotic stability of the equilibrium position $\bar{z}$ of the system (20). The proof is completed. □

6. NUMERICAL EXAMPLE

For simulation, consider a group consisting of ten agents. Let $[a, b] = [1, 6]$, $\varrho(z) = \arctan z$, $\tau = 1$, $f(\zeta) = e^{-\zeta}$, $\sigma(t) = 1$ for $t \in [10k, 10k + 5)$ and $\sigma(t) = 2$ for $t \in [10k + 5, 10(k + 1))$, $k = 0, 1, 2, \ldots$. Assume that initial functions for agents are constant on the interval $[-1, 0]$. Their values and sets of agent neighbors are presented in Table 6.1.

In Fig. 6.1, the dependence of agent coordinates on time is given. The simulation results demonstrate the convergence of agents to the prescribed distribution.

Table 6.1. Network topologies and agents initial positions

Agent number	Initial position	$\Xi_i^{(1)}$	$\Xi_i^{(2)}$
1	0.41	$\{0, 4\}$	$\{0, 5\}$
2	-0.39	$\{0, 5\}$	$\{0, 6\}$
3	0.4	$\{0, 6\}$	$\{0, 7\}$
4	-0.3	$\{1, 7\}$	$\{0, 8\}$
5	-0.22	$\{2, 8\}$	$\{1, 9\}$
6	0.8	$\{3, 9\}$	$\{2, 10\}$
7	0.3	$\{4, 10\}$	$\{3, 11\}$
8	-0.5	$\{5, 11\}$	$\{4, 11\}$
9	-0.6	$\{6, 11\}$	$\{5, 11\}$
10	1.7	$\{7, 11\}$	$\{6, 11\}$

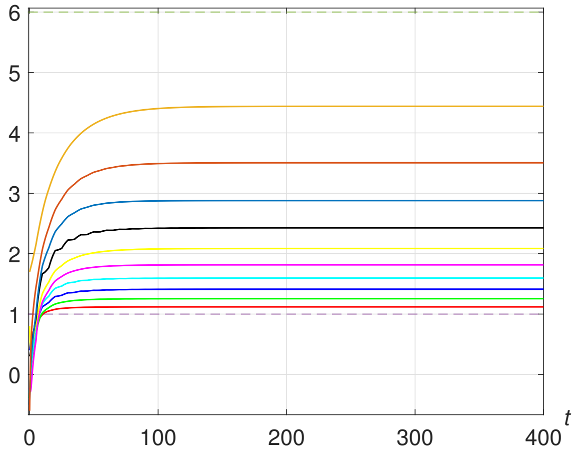

Fig. 6.1. The agent time history.

7. CONCLUSION

In the present contribution, original constructions of diagonal Lyapunov–Krasovskii functionals for switched positive Persidskii systems with distributed and unbounded delays are proposed. With the aid of these functionals, new absolute stability conditions for considered systems are derived. The obtained results are used for the stability analysis of a mechanical system with switched nonlinear positional forces and for the design of decentralized protocols ensuring a prescribed mobile agent deployment on a line segment. It is worth noticing that the proofs of Theorems 3.1 and 4.1 provide us constructive algorithms for finding parameters of the functionals. An interesting direction for further research is an application of the developed approaches in stability analysis of generalized Lotka–Volterra models of population dynamics.

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