Competitive Equilibrium of a Sequence of Incomplete Markets with a Continuum of Agents

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Abstract

By introducing the Large Economical Ideas to multi-period financial market, we have constructed the multi-period economy with incomplete market and a continuum of agents. The competitive equilibrium has been proposed and the existence has been claimed. Our equilibrium definition is a development compared to that described by Radner, and our conclusion for equilibrium existence has generalized the related results obtained by Aumann and Zhang, if only the future contracts and goods are traded on security-spot markets.

Keywords perfect competition, equilibrium, incomplete markets, correspondence integral

1 Introduction

Consider a sequence of markets at successive dates, no one of which is complete in the Arrow Debreu Sense, i.e., at every date and for every commodity there will be some future dates and some events at those date for which the spot goods and future contracts contingent on those events are traded. For such economy, Radner had first proposed the concept of common expectations that require traders to associate the same future prices to same future exogenous events[1]. An equilibrium is a set of prices at the first date, a set of common price expectations for the future, and a consistent set of individual plans for agents such that, given the current prices and price expectations, each individual agents plan is optimal for him, subject to an appropriate sequence of budget constraints. Radner's common expectation is a foundation for modern incomplete market theory. But a basic assumption of such model is that the current prices and price expectations for future Contracts be not affected by a single agents action, which needs the market be perfect competition. Otherwise a change in an individuals offer to buy or sell can easily upset the prevailing prices, so that the equilibrium will never be achieved, and price system is meaningless.

As early as 1964, Aumann had suggested that the most natural mathematical model for a commodity market with such perfect competition is one in which there are a continuum of traders (like the continuum of points on a line)[2]. For decades, Aumann's large economy has always been a vigorous field of economics. A deficiency of modern incomplete market theory is the lack of introduction of Aumann's large economy ideas. As a novelty, Zhang first discussed security-spot markets with a measurable space of agents[3-4], where Aumann's ideas have been applied to security-spot market, particularly, to financial markets. But these research works are primary: the model is two-periods, and the existing discussion needs to proceed technically.

In this paper, by combining Aumann's ideas with Radner's model, we discuss the economy with a sequence of incomplete markets and a continuum of agents. First, the concept of a competitive equilibrium for such economy is proposed, which has developed Radner's definition. Then, we claim the existence of equilibrium. The sufficient conditions are all used by Aumann and Radner[5-6]. For the security-spot market with goods and future contracts, our conclusion is the generalization of related results proved by Aumann and Zhang[4,5].

2 A multiperiod model for a sequence of incomplete markets with a continuum of agents

Consider an economy extending through a finite sequence of elementary dates $1, 2, ..., T$, in an environment with a finite set S of alternative states. The set of events observable at date t will be represented by partition φ_{t} of S . It is assumed the sequence of partition, φ_{t} is monotonous, no decreasing in fineness, that is, $\varphi_{\mathrm{t+1}}$ is as fine as φ_{t}[1] . Also, take $\varphi_1 = \{S\}$.

For each date, there is a finite set of commodities, numbered $1, 2, ..., l$. Trade contract (such as future contract) at date t in event A, denoted by $\theta^{\mathrm{h}}_{\mathrm{tu}}(A, B)$, specifies the number of units of commodity h that the trader will receive from the market at date $u \geq t$ in event $B(\theta^{\mathrm{h}}_{\mathrm{tu}}(A, B)) < 0$ means the delivery to market; $u > t$ means a future trade and $u = t$ means a spot trade). For each pair of dates t and u such that $u > t$, and each commodity h, there is a given family F^h_{tu} of events, which is either empty or is a partition of S. In the latter case, φ_{u} must be as fine as F^h_{tu}. Assume that . Assume further that if F^h_{tu} is not empty and $t \leq v \leq u$, F^h_{vu} is as fine as F^h_{tu}. In other word, if at date t, one can buy a contract for receipt at date u contingent on event B, then at a later date V, one can do the same.

A portfolio plan, which was described as a trade plan by Radner, is an array $(\theta^{\mathrm{h}}_{\mathrm{tu}}(A, B))$, one for each combination(h,t,u,A,B) such that for A in φ_{t} , B in $F^{\mathrm{h}}_{\mathrm{tu}}$, $B \subseteq A$, $t \leq u$. The security price paid at date t in event A for receipt of commodity h at date u in event B will be denoted by $p^{\mathrm{h}}_{\mathrm{tu}}(A, B)$ .When $t = u$, $p^h_{uu}(A)$ is spot price of commodity h. An array $p = \{p^h_{tu}(A, B)\}$ will be called a commodity price system.

In the situation just described, there is for each event pair (t,A), with $A \in \varphi_{\mathrm{t}}$, a market in contracts for current and future receipt, with cost to be made currently in units of account. To simplify the notation, let denote the set of all pair (t,A) such that, $t = 1, 2, ..., T$, $A \in \varphi_{\mathrm{t}}$. Endow order for $m =$ (t,A) and $n =$ (u,B) in

[1] φ is said to be as fine as partition φ' if, for every A' in φ ,either $A \subset A'$ or A' $\cap$ A = ϕ

M: $m \le n$ if and only if $t \le u$, $A \supseteq B$.

We assume short sales have low bound L, $-L \in R{+}{+}$. Then all of the portfolio plans can be denoted as $Z = \times_{\mathrm{m\in M}} Z_{\mathrm{m}}$ with Z_m denoting the vector space of all arrays of number $\theta^h_{tu}(A, B) \ge L$; $h = 1, 2, ..., l$; $u = t, t+1, ..., T$; $B \subseteq A, B \in F^h_{tu}$ for each $m = (t, A) \in M$. Thus a portfolio $\theta = (\theta_{\mathrm{m}})_{\mathrm{m\in M}}, \theta_m \in Z_m$, is a point in Z. For $m = (t, A), n = (u, B)$, if $u \ge t$ and $B \subseteq A$, $A \in \varphi_t$, $B \in F^h_{tu}$, we denote $\theta^h_{mn} = \theta^h_{tu}(A, B)$; or $\theta_{mn} = (\theta^1_{mn}, \theta^2_{mn}, \cdots \theta^l_{mn}) = 0$. The payment at m for portfolio plan $\theta = (\theta_m)_{m\in M}$, $\theta_m = (\theta_{mn})_{n\ge m} \in Z_m$,given the commodity price system $p = (p_m)_{m\in M}$, is the inner product $p_m \cdot \theta_m$.

Let $\{A, F, u\}$ be the measure space of agents. For any agent $a \in A$, we denote his preference as $\preceq_a$, which is a binary relation on $(\times_{m\in M} R^l_+) \times (\times_{m\in M} R^l_+)$. We assume the $\preceq_a$ is complete, transitive, reflexive and satisfies

a) closeness: the set $\{(x, y) \in (\times_{m\in M} R^l_+) \times (\times_{m\in M} R^l_+) | x \preceq_a y\}$ is closed in $(\times_{m\in M} R^l_+) \times (\times_{m\in M} R^l_+)$ and

b) monotony: if x,y are two points in $\times_{m\in M} R^l_+$, such that $x < y$, then $x \prec_a y$.

The set of all preference relations on $\times_{m\in M} R^l_+$ satisfying all of these assumptions is denoted by β. We endow β with the topology of closed convergence on $(\times_{m\in M} R^l_+) \times (\times_{m\in M} R^l_+)$. For each $a \in A$, we require $\varepsilon = (\preceq_a, e_a) : A \to \beta \times (\times_{m\in M} R^l_+)$ is measurable and e_a is integrable, where $e_a \in \times_{m\in M} R^l_+$ is the real endowment of agent α. We call ε large sequence of security-spot market. Each agent must select a consumption plan $x^a = (x^a_m)_{m\in M} \in \times_{m\in M} R^l_+$, and a portfolio plan $\theta^a = (\theta^a_m)_{m\in M} \in Z$ within his budget set. An pair (x^a, θ^a) is called an assign if $(x^a, \theta^a) : A \to (\times_{m\in M} R^l_+) \times Z$ is integrable (we endow Z with Bore -field).

Give $p = (p_m)_{m\in M}$, agent a's budget set is defined as
$x_a(p) = \{(x, \theta) : x = (x_m)_{m\in M} \in \times_{m\in M} R^l_+, \theta = (\theta_m)_{m\in M} \in Z, such\ that\ for\ each$ $m \in M, p_m \cdot \theta_m \le 0, x_m \le e^a_m + \sum\limits_{j\le m} \theta_{jm}\}$

The agent a's demand correspondence is then defined as

$\xi_a(p) = \{(x, \theta) \in x_a(p) | \forall (x', \theta') \in x_a(p), x \succeq_a x'\}$

Definition 2.1

An equilibrium of the economy ε is an assign (x^a, θ^a) of consumption-portfolio plan and a price system p such that

$$a \cdot e, a \in A, (x^a, \theta^a) \in \xi_a(p) \tag{1}$$

$$\int\limits_A \theta^a du = 0, \int\limits_A x^a du = \int\limits_A e^a du \tag{2}$$

where $\int\limits_A \theta^a du = 0$ means $\int\limits_A \theta^a_m du = 0$ for each $m \in M$.

This definition has obviously generalized Radner's definition for pure exchange economy, that is, the finite market participants is replaced by infinite market participants. Note that (1) implies a.e., $a \in A$, (x^a, θ^a) is the best consumption-portfolio in his budget set and (2) implies the market is clear at each data-event pair. Our definition is also a development for the equilibrium described by Debreu[7], which only considered spot-market with one-period. But compared with general incomplete market as that Geankoplos proposed[8], we only consider future contracts as securities here.

Equilibrium Existence

Our main result can be described as the following.

Theorem 3.1 If $\{A, F, u\}$ is atomless[2] and endowment satisfies $\int_A e_m^a \gg 0$[3], for any $m \in M$, then economy ε has equilibrium.

For each $m \in M$, let $\mathrm{P} = \times_{m \in M} \mathrm{P}_m$, P_m be the set of all nonnegative vectors in Z_m whose coordinates sum to 1.Denote the excess demand correspondence as $\xi(p) = \int_A \xi_a(p) du - \int_A (e_a, 0) du$[4], here $(e_a, 0) \in (\times_{m \in M}) R_+^l \times Z$. We will give some properties of $\xi(p)$ restricted on . For the reason of shortening this paper, some simple proof processes similar to that used by Debreu are omitted here[9].

Proposition 3.1 Under conditions of theorem 3.1, $\xi(p)$ is non-empty, compact, lower bounded and upper hemicontinuous at every $p \gg 0$ in P.

Proof. We only prove $\xi(p)$ is upper hemicontinuous.

We first prove $x_a(p)$ is continuous at each $p \gg 0$ in P .The graph of correspondence $x_a(p)$ is obviously closed in $\mathrm{P} \times (\times_{m \in M} R_+^l) \times Z$. Thus, $x_a(p)$ is upper hemicontinuous on P. To show that $x_a(p)$ is lower hemicontinuous at any point $p^o \gg 0$ in P, we consider a p^t sequence in P converging to $p^o (t \to \infty)$ and a point $(x^o, \theta^o) \in x_a(p^o)$.Denote $x^o = (x_m^o)_{m \in M}$, $\theta^o = (\theta_m^o)_{m \in M}$, $\theta_m^o \in Z_M$. We will discuss the problem under two cases.

i) $p_m^o \cdot \theta_m^o < 0$ for any $m \in M$.

Because of $p_m^t \to p_m^o$, We have $p_m^t \cdot \theta_m^o < 0$ for large enough t. So $(x^o, \theta^o) \in x_a(p^t)$, which satisfies the condition appearing in the definition of lower hemicontinuity.

ii) $\exists m_1, m_2 \cdots, m_g \in M$, such that $p_{m_j}^o \cdot \theta_{m_j}^o = 0, j = 1, 2, \ldots, g and\ p_m^o \cdot \theta_m^o < 0, for\ m \neq m_1, m_2 \ldots, m_g$.

For any j, we select a point $\theta'_{\mathrm{m_j}} \in \mathrm{Z_{m_j}}$ satisfying $p_{m_j}^o \cdot \theta'_{m_j} < 0$. This is possible because $-L > 0$.Thus $p_{m_j}^o \cdot \theta'_{m_j} < p_{m_j}^o \cdot \theta_{m_j}^o$, and for large enough t, the hyperplane $\{\theta_{m_j} \in Z_{m_j} | p_{m_j}^t \cdot \theta_{m_j} = 0\}$ intersects the straight line through θ'_{m_j} and $\theta_{m_j}^o$ in a unique point $\overline{\theta}_{m_j}^t$.

[2] $\{A, F, u\}$ is called atomless if, for any $B \in F$, $u(B) > 0$, there is $C \in F$ such that $0 < u(C) < u(B)$.

[3] For $x \in {}^l$, $x \gg 0$ means all of its coordinates are strictly positive.

[4] For integrals of correspondences and their properties, see Definition of Part , D. of [7].

Define θ^{t} as $\theta^t = (\theta_m^t)_{m\in M}$,

$$\theta_m^t = \begin{cases} \theta_m^o,\ m \neq m_1, m_2, \cdots, m_g \\ \overline{\theta}_m^{\mathrm{t}},\ m = m_j\ and\ \overline{\theta}_{\mathrm{m_j}}^{\mathrm{t}}\ is\ \ between\ \theta'_{m_j}\ and\ \theta_{m_j}^o \\ 0,\ others. \end{cases}$$

It is easily checked that $\theta_m^t \to \theta_m^o (t \to \infty)$ for any $m \in M$, and $p_m^t \cdot \theta_m^t \leq 0$. Let $x_m^o = (x_m^{oh}), h = 1, 2, \ldots, l,\ e_m^a = (e_m^{ah}), h = 1, 2, \ldots, l$ for each $m \in M$. For any h, if $x_m^{oh} < e_m^{ah} + \sum\limits_{j\leq m} \theta_{jm}^{oh}$, we can take x_m^{th} such that $x^{th} < e_m^{ah} + \sum\limits_{j\leq m} \theta_{jm}^{th}$, and $x_m^{th} \to x_m^{oh} (t \to \infty)$. This is possible because $x_m^{oh} < e_m^{ah} + \sum\limits_{j\leq m} \theta_{jm}^{th}$ for large enough t. If $x_m^{oh} = e_m^{ah} + \sum\limits_{j\leq m} \theta_{jm}^{oh}$, then we take $x_m^{th} = e_m^{ah} + \sum\limits_{j\leq m} \theta_{jm}^{th}$, which also implies $x_m^{th} \to x_m^{oh}$.

Therefore, we can find $x^t = (x_m^t)_{m\in M} \to x^o$ such that (x^t, θ_m^t) satisfies the condition for lower hemicontinuity of $x_a(p)$. Note that $x_a(p)$ is nonempty $((0,0) \in x_a(p))$ and compact, According to Debreu[10],$\xi(p)$ is upper hemicontinuous at each $p \in P$, $p \gg 0$. #

Proposition 3.2 Under the conditions of theorem 3.1, $\xi(p)$ satisfies Boundary Condition, that is, if $p^t \gg 0$ in converges to p^0 in ∂P, then $d(0, \xi(p^t)) \to (t \to \infty)$.

Proof. By the similar discussion to Debreu (1982, p. 729), the proposition holds if only for a.e.,$a \in A$ $d(0, \xi(p^t)) \to \infty$.

Suppose that the conclusion does not hold. Then there is a subsequence $(p^{t'})$ such that $d(0, \xi_a(p^{t'}))$ is bounded. For each t', one can select $(c^{t'}, \theta^{t'}) \in \xi_a(p^{t'})$ in such a way that sequence $(c^{t'}, \theta^{t'})$ is bounded. Therefore, one can extract from $(p^{t'}, c^{t'}, \theta^{t'})$ a sequence $(p^{t''}, c^{t''}, \theta^{t''})$ converging to (p^o, c^o, θ^o). By proposition 3.1 we have $(c^o, \theta^o) \in \xi_a(p^o)$. Since $p^o \in \partial P$, there is $m_o \in M$ such that some coordinates of $p_{m_o}^o$ is zero. Without loss of generality, we suppose the first coordinate of $p_{m_o}^o$ is zero. Then by replacing the first coordinate of $\theta_{m_o}^o$ with large one, we can get $\theta' = (\theta'_m)_{m\in M} \in Z$, satisfy in $p_m^o \cdot \theta'_m \leq 0$ and $e_u^a + \sum\limits_{j\leq u} \theta'_{ju} > e_u^a + \sum\limits_{j\leq u} \theta_{ju}^o$. Thus, we can obtain $x' = (x'_m)_{m\in M}$, such that $(x', \theta') \in x_a(p^o)$ and $x' > x^0$. This contradicts the monotony of preference $\preceq_a$.#

Proposition 3.3 Under conditions of theorem 3.1, Walras Law holds, that is, for any $p \in P$, $p \gg 0$ and $(x, \theta) \in \int\limits_A \xi_a(p) du$, we have $(p_m \cdot \theta_m)_{m\in M} = 0$ and $x_m = \int\limits_A e_m^a du + \sum\limits_{j\leq m} \int\limits_A \theta_{jm}^a du$.

Proof. For any $a \in A$, $(x, \theta) \in \xi_a(p)$, by the monotony of preference $\preceq_a$, it is easily proven that $(p_m \cdot \theta_m^a)_{m\in M} = 0$ and $x_m^a = e_m^a + \sum\limits_{j\leq m} \theta_{jm}^a$. By integrating the two sides of $(p_m \cdot \theta_m^a)_{m\in M} = 0$ and $x_m^a = e_m^a + \sum\limits_{j\leq m} \theta_{jm}^a$, the conclusion holds. #

Proposition 3.4 Under conditions of Theorem 3.1, is convex-valued.

Noticing the fact that $\xi_a(p)$ is convex-valued for any $a \in A$, the proposition is the direct corollary of theorem 3 of Part I, D.II of [9].

Proof of theorem 3.1. For the real number $b > 0$, let $b \cdot P_m = \{b \cdot p : p \in p_m\}$. Notice that if we replace P by $\mathrm{P}' : \mathrm{P}' = \times_{m\in M}(b_m \cdot P_m)$, all of the conclusions of proposition 3.1-proposition 3.4 are true. Assume the number of dimensions of P_m is v_m, and $\sum\limits_{m\in M} \nu_m = \nu$. Let $P'_m = \frac{\nu_m}{\nu} \cdot P_m$, $\mathrm{P}' = \times_{m\in M} P'_m$, and

$$E = \{p \in \mathrm{P}' | p \gg 0, there\ is (x, \theta) \in \xi(p), such\ that \sum_{n\geq m,n,m\in M} \sum_{h=1}^{l} \theta^h_{mn} \leq 0\}.$$

Since $\xi(p)$ is bounded below, when p varies in E, the θ with $(x, \theta) \in \xi(p)$ remains bounded. So does χ. So, by the Boundary Conditions, there cant be in E a sequence p^t converging to p^0 in $\partial P'$. Consequently, the distance from $p \in E$ to $\partial P'$ is bounded below by a strictly positive real number. Thus, there is a closed convex cone C with vertex 0 in $\times_{m\in M} R_m$, such that $E \subset \mathrm{int}C$ and $C\backslash 0 \subset \mathrm{int}(\times_{m\in M} R^+_m)$, where R_m denotes the vector space of all arrays of real number $g^h_{mn} \in R$ for any $n = (u, B) \geq m$,$B \in F^h_{mn}$ and $h = 1, 2, ..., l$, and R^+_m is the positive cone of R_m.

Let $\xi'(p) = \{\theta \in Z : thereis(x, \theta) \in \xi(p)\}$. If we restrict $\xi'(p)$ on P', it is easily proved that $\xi'(p)$ is convex-valued, bounded below and satisfies Walras Law. We further claim $\xi'(p)$ is upper hemicontinuous at each $p \gg 0, p \in \mathrm{P}'$.

Suppose p^t in P' converge to $p^o \in \mathrm{P}', p^o \gg 0$ and $\theta^t \in \xi'(p^t)$ converge to θ^o, which is a portfolio plan. By the definition of $\xi'(p)$, there is $(x^t, \theta^t) \in \xi(p^t)$. Let U be a compact neighborhood of p^0 contained in the relative interior of P'. Then when t large enough, $(x^t, \theta^t) \in \xi(p^t)$ is uniformly bounded. Thus we can take a subsequence $(x^{t'}, \theta^{t'}) \to (x^o, \theta^o)$ with $x^o \in \times_{m\in M} R^+_m$. By proposition 3.1, we have $(x^o, \theta^o) \in \xi(p^o)$, which yields $\theta^o \in \xi'(p^o)$. This shows that $\xi'(p)$ is upper hemicontinuous at any $p^o \in \mathrm{P}', p^o \gg 0$.

According to Debreu[9], there is a point $p^* \in C \cap \mathrm{P}'$ such that $\xi'(p^*) \cap C^o \neq \phi$. Let $\theta^* \in \xi'(p^*) \cap C^o$. By Walras' Law, the point $\left(\frac{1}{v}, \frac{1}{v}, \cdots, \frac{1}{v}\right)$ in P' belongs to E, hence to C. Therefore, $\sum\limits_{n\leq m,n,m\in M} \sum\limits_{h=1}^{l} \theta^{*h}_{mn} \leq 0$. Consequently,$p^* \in E$, hence $p^* \in \mathrm{int}C$. Moreover, by another application of the Walras' Law, we has $p^* \cdot \theta^* = 0$. This equality together with $p^* \in \mathrm{int}C$ and $\theta^* \in C^o$ implies $\theta^* \in 0$. Then (x^*, θ^*) is a equilibrium for price P^*.#

References

[1] Aumann, R. J. (1964), "Space cannot be cut: Why self-identity naturally includes neighbourhood", *Integrated Psychological Behaviour*, Vol.32, pp.39-50.

[2] Aumann, R. J. (1966), "Existence of competitive equilibria in market with a continuum of traders", *Econometrica*, Vol.34, pp.1-17.

[3] Yun, X. (2007), "Optimal portfolio on optimal control", *Advances in System Science and Applications*, Vol.7, No.1, pp.66-71.

[4] Arrow, K., Debreu, G. (1954), "Existence of an equilibrium for a competitive economy", *Econometrica*, Vol.22, pp.265-290.

[5] Debreu, G. (1982), "Existence of competitive equilibrium", *Handbook of Mathematical Economics*, Vol. II, Chapter 15, pp.697-743, North Holland, New York.http://www.calphysics.org/articles/chown2007.html

[6] Radner, R. (1972), "Existence of equilibrium of plans, prices, and price expectations in a sequence of market", *Econometrica*, Vol.40, No.2, pp.289-302.

[7] Hildenbrand, W. (1974), *Core and equilibria of a large economy, Princeton University Press.*

[8] Geankoplos, J. (1990), "An introduction to general equilibrium with incomplete Asset Market", *Journal of Mathematical Economics*, Vol.19, pp.1-38.

[9] Jifeng, Z. and Xuemou, Wu. (2002), "Pansystems or principles to society-economy systems", *Advances in System Science and Applications*, Vol.2, No.1, pp.24-31.

[10] Hart, O. D. (1975), "On the optimality of equilibrium when the market structure is incomplete", *Journal of Economic Theory*, Vol.11, pp.418-443.

[11] Bingan, J. etc.(2007), "Equilibrium of manufactures' R&D decision-making in defense procurement", *Advances in System Science and Applications*, Vol.7, No.1, pp.117-120.

[12] Guosheng Zhang (1998), *On large finacial market, Mathematics in Economy (in Chinese)*, Vol.15, No.3, pp.1-6.

[13] Guosheng Zhang (2000), "Spot-Financial Market with large characteristics", *Systems Engineering-Theory & paractice (in Chinese)*, Vol.20, No.3, pp.39-45.

[14] Guosheng Zhang (2007), "Competitive equilibrium of Large Security-Spot Market with incomplete asset structure", *Journal of Systems Science and Complexity*, Vol.20, No.3, pp.386-396.

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