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The Mean-Field Approximation for the SCARDO Model in the Case of 3-element Opinion Space: Fixed Points and Exact Solutions

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Abstract: We analyze the SCARDO model in the case of the 3-element opinion space under specific constraints on the transition table parameters that allows to link the problem at stake to the case of the 2-element opinion space that has been thoroughly studied previously. We characterize the properties of fixed points and support our findings by numerical experiments. Further, we manage to find out those settings that ensure the system almost surely reaches a specific domain in the phase space, after which its behavior can be predicted analytically.

Keywords: opinion dynamics models, SCARDO-model, mean-field approximation

1. INTRODUCTION

The recent proliferation of social media platforms as well as emergent speculations around the issues of opinion polarization [18], destructive content [14], echo-chambers [5], and filter bubbles [1] have motivated scientists to focus their attention on the analysis of these social effects by applying the framework of opinion formation models [4, 12, 15–17]. These models, initially designated to describe how individuals' opinions change following peer interactions [2], are now intensively upgraded to account for opinion formation patterns in the online domain [3].

The current paper is dedicated to the analysis of a relatively recent opinion formation model introduced in Ref. [8]. This minimal model (hereafter – the SCARDO-model) was to suggest a flexible, easy-to-validate framework that could approximate a huge variety of micro-level mechanisms of opinion formation at both quantitative and qualitative levels. Further, an extension of the model introduced in Ref. [7] allows to account for the fact that individuals' social power may depend on their socio-demographic characteristics [13].

As was reported in Ref. [8], the typical behavior of the model displays out-of-equilibrium patterns and has a hard time getting an analytical description. Nonetheless, under specific restrictions common in the field of socio-physics (the number of agents in the system is huge and they communicate via a complete graph), a mean-field approximation in the form of a system of ordinary differential equations can be derived. These equations explain opinion dynamics in terms of the populations of opinions, at the macroscopic level [15].

Previously, the mean-field approximation for the SCARDO-model in the case of the 2-element opinion space has been thoroughly investigated [6]. These settings correspond to the situation when there are two opinion factions. However, in many respects this description of real-world processes is too simplified. Indeed, in the case of political attitudes, individuals'

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opinions are not simply divided into two groups, but display rather substantial diversity, with various levels of political radicalism. At least, we should account for people with neutral views, as they usually substitute the majority of the population.

In this paper, we focus on studying the properties of the mean-field approximation system in the situation when agents' opinions belong to the 3-element opinion space. Clearly, this case is much more complicated. However, under specific constraints on the model parameters, we will investigate fixed points and, in some cases, derive precise solutions for the system.

2. OPINION DYNAMICS MODEL

In this section, we briefly present the SCARDO-model that was introduced in Ref. [8] and later investigated in Ref. [6] for the case of the two-element opinion space.

In this model with discrete time $t = 0, 1, 2, \ldots, N$, agents update their opinions following consecutive pairwise interactions that unfold on a social network $\mathcal{G}$. Agents' opinions (denoted by o) belong to a discrete opinion space with m elements:

$$Z = \{z_1, \ldots, z_m\}.$$

At each iteration, an agent i is chosen at random and then one of i's peers (j) is selected also at random. Then, agent j (influence source) influences on i (influence object). As a result, the influence object's opinion updates in accordance with a specific opinion distribution that is a function of the opinions of interacting agents. The distribution that outlines how agent i updates their opinion is represented as follows:

$$(p_{s,l,1}, \ldots, p_{s,l,m}),$$

with the element $p_{s,l,k}$ standing for the probability that i's opinion will become z_k following the interaction. The first two indices of $p_{s,l,k}$ stand for the opinions of i and j before the communication event respectively. In turn, the third index links to the index of probable opinion. As a result, the quantity $p_{s,l,k}$ is just a conditional probability:

$$p_{s,l,k} = \Pr\{o_i(t+1) = z_k \mid o_i(t) = z_s, o_j(t) = z_s\}. \tag{2.1}$$

Coupled together, these probabilities form a 3-D object

$$\mathcal{P} = [p_{s,l,k}]_{s,l,k=1}^{m},$$

which is referred to as the transition table [8]. This table can be safely expressed via its slices over the first index:

$$\mathcal{P} = [P_1, \ldots, P_m],$$

where

$$P_1 = \begin{bmatrix} p_{1,1,1} & \cdots & p_{1,1,m} \\ \cdots & \cdots & \cdots \\ p_{1,m,1} & \cdots & p_{1,m,m} \end{bmatrix}, \ldots, P_m = \begin{bmatrix} p_{m,1,1} & \cdots & p_{m,1,m} \\ \cdots & \cdots & \cdots \\ p_{m,m,1} & \cdots & p_{m,m,m} \end{bmatrix} \tag{2.2}$$

Within these shorthands, $P_1, \ldots, P_m$ are $m \times m$ matrices that encode opinion change strategies of individuals espousing opinions $z_1, \ldots, z_m$ correspondingly. All these matrices are row-stochastic:

$$p_{s,l,1} + \ldots + p_{s,l,m} = 1$$

for each s and l.

An important feature of the SCARDO-model is that it can describe both ordered and categorical opinions. The model mechanics does not depend on if the opinion alphabet is ordered or not. The presence/absence of order is important only at the stage of the transition table interpretation/estimation. Once the transition table is defined, all the necessary information on the opinion dynamics is stored therein.

3. MEAN-FIELD APPROXIMATION FOR THE SCARDO-MODEL

The following mean-field approximation was obtained in Ref. [8] for the SCARDO-model. Let $y_i(t)$ stand for the fraction of individuals espousing opinion z_s at a time moment t:

$$y_s(t) = \frac{\#\{i \mid o_i(t) = z_s\}}{N}.$$

With the assumptions $N \to \infty$ and $\mathcal{G}$ is a complete graph, the mean-field approximation can be derived in a form of a system of ordinary differential equations in the scaled time τ:

$$\begin{cases} \dfrac{dy_1}{d\tau} = \sum\limits_{s,l=1}^{m} y_s y_l p_{s,l,1} - y_1, \\ \dots \\ \dfrac{dy_3}{d\tau} = \sum\limits_{s,l=1}^{m} y_s y_l p_{s,l,m} - y_m, \end{cases} \tag{3.3}$$

where $\tau = t/N, \delta\tau = 1/N$.

The fixed points of system (3.3) are given by the fixed point system:

$$\begin{cases} \sum\limits_{s,l=1}^{m} y_s y_l p_{s,l,1} - y_1 = 0, \\ \dots \\ \sum\limits_{s,l=1}^{m} y_s y_l p_{s,l,m} - y_m = 0. \end{cases} \tag{3.4}$$

The initial point of system (3.3) is defined by

$$y_1(0) = q_1, \dots, y_m(0) = q_m, \tag{3.5}$$

where

$$q_1 \dots, q_m \geq 0, \quad q_1 + \dots + q_m = 1.$$

System (3.3) and the corresponding Cauchy problem (3.3), (3.5) feature the following properties (see Refs. [8] and [10] for proofs).

Corollary 3.1:
The function $u = y_1 + \dots + y_m$ is the first integral of (3.3).

Theorem 3.1:
The Cauchy problem (3.3), (3.5) *has a unique solution $y(\tau)$, which can be extended on the whole τ-axis. The components of $y(\tau)$ are nonnegative and sum up to one for each $\tau \in \mathbb{R}$.*

4. MAIN ASSUMPTION

Hereinafter, we will study system (3.3) in the case of the 3-element opinion alphabet ($m = 3$). This situation is much more complex than the case $m = 2$ and we have failed to obtain analytical solutions. However, we report that it is possible to get some analytical results for a specific family of transition tables:

$$P_1 = \begin{bmatrix} 1-\alpha & \alpha & 0 \\ 1-\beta & \beta & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, P_2 = \begin{bmatrix} \gamma & 1-\gamma & 0 \\ \delta & 1-\delta & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix}, P_3 = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}, \quad (4.6)$$

where $\alpha, \beta, \gamma, \delta \in [0, 1]$, and other parameters are arbitrary.

From this point, we assume that the transition table belongs to the class (4.6). It is worth noting that transition tables (4.6) are not somewhat disconnected from the real world. In fact, the restraint (4.6) just prescribes that opinion shift $z_1 \to z_3$ can only happen if the influence comes from opinion z_3, an assumption that is quite natural. Further, from (4.6) it follows that opinion shift $z_2 \to z_3$ cannot occur if the influence subject has opinion z_1 or z_2, indicating thus that agents having opinion z_2 are not sensitive to the negative and anti-conformity forms of influence.

5. AUXILIARY SYSTEM

We are going to need an auxiliary social system which is defined in the 2-element opinion alphabet by the following transition table:

$$Q_1 = \begin{bmatrix} 1-\alpha & \alpha \\ 1-\beta & \beta \end{bmatrix}, Q_2 = \begin{bmatrix} \gamma & 1-\gamma \\ \delta & 1-\delta \end{bmatrix}, \quad (5.7)$$

where $\alpha, \beta, \gamma, \delta$ are the same as in (4.6). One can easily notice that this transition table is just a chunk of transition table (4.6).

Below, this social system and the corresponding transition table (5.7) will be referred to as Auxiliary ones. In order to avoid confusion, the initial 3-element opinion space system will be denoted as the main. The properties of Auxiliary system have been thoroughly studied in Ref. [6]. In particular, the exact solution of (3.3) has been found, and fixed points as well as their stability properties have been systematically characterized. For now, we will harness these findings for our purposes.

6. FIXED POINTS ON THE LINE $y_3 = 0$

Let us turn to the analysis of system (3.3) in the case of the main system. We will start with the characterization of fixed points that are located on the line $y_3 = 0$. Because of (4.6), the third equation of (3.3) turns out to have the following form:

$$\frac{dy_3}{d\tau} = y_3 \times \Big[(y_1 p_{1,3,3} + y_2 p_{2,3,3} + y_1 p_{3,1,3} + y_2 p_{3,2,3} + y_3 p_{3,3,3}) - 1\Big]. \quad (6.8)$$

With $y_1 + y_2 + y_3 = 1$ we can rewrite (6.8) as follows:

$$\frac{dy_3}{d\tau} = y_3 \times \Big[Ay_1 + By_3 + C\Big], \quad (6.9)$$

where

$$\begin{aligned} A &= p_{1,3,3} - p_{2,3,3} + p_{3,1,3} - p_{3,2,3}, \\ B &= p_{3,3,3} - p_{2,3,3} - p_{3,2,3}, \\ C &= p_{2,3,3} + p_{3,2,3} - 1. \end{aligned}$$

Let us now assume that $y_3 = 0$. In this case, $\frac{dy_3}{d\tau} = 0$ and the first equation of (3.3) (for the derivative of y_1) turns out to:

$$\frac{dy_1}{d\tau} = (1-\alpha)y_1^2 + (1-\beta)y_1y_2 + \gamma y_2y_1 + \delta y_2^2 - y_1.$$

Let us denote $y = y_1$ and make use of $y_1 + y_2 = 1$:

$$\frac{dy}{d\tau} = (1-\alpha)y^2 + (1-\beta)y(1-y) + \gamma(1-y)y + \delta(1-y)^2 - y =$$
$$(1-\alpha)y^2 + (1-\beta)y - (1-\beta)y^2 + \gamma y - \gamma y^2 + \delta - 2\delta y + \delta y^2 - y =$$
$$= (\delta + \beta - \alpha - \gamma)y^2 + (\gamma - \beta - 2\delta)y + \delta.$$

Now it is time to recall that this expression is exactly similar to the one obtained for Auxiliary system (see Ref. [6], p. 107, formula (4.7)). Further, from $y_3 = 0$ it follows that $y_1 + y_2 = 1$. With both of these facts, we obtain that the fixed points of (3.3) that are located on the line $y_3 = 0$ can be found by solving the fixed point equation (3.4) for the Auxiliary system. Its solutions $[y_1^* \quad y_2^*]^T$ will define the fixed points of the main system as follows: $[y_1^* \quad y_2^* \quad 0]^T$.

7. SIMULATION EXPERIMENTS

Let us now pinpoint the above findings with simulation experiments. We will consider two basic scenarios.

7.1. Scenario 1

In this scenario, we will specify the transition table as follows:

$$P_1 = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix}, P_3 = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

On this occasion, the transition table for Auxiliary system is given by:

$$Q_1 = \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix}, Q_2 = \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix}. \tag{7.10}$$

It is known from Ref. [6] that in this case, the only fixed point for Auxiliary system is $y_1^* = \frac{5}{7}, y_2^* = \frac{2}{7}$. From our previous derivations, it follows that $\begin{bmatrix} \frac{5}{7} & \frac{2}{7} & 0 \end{bmatrix}^T$ should be a fixed point of the main system.

7.2. Scenario 2

In this scenario, we consider the following transition table:

$$P_1 = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.5 & 0.5 & 0 \\ 0.24 & 0.24 & 0.52 \end{bmatrix}, P_2 = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.5 & 0.5 & 0 \\ 0.24 & 0.24 & 0.52 \end{bmatrix}, P_3 = \begin{bmatrix} 0.24 & 0.24 & 0.52 \\ 0.24 & 0.24 & 0.52 \\ 0 & 0 & 1 \end{bmatrix}.$$

The transition table of the corresponding Auxiliary system is given by (7.10) again. In this vein, one should expect $\begin{bmatrix} \frac{5}{7} & \frac{2}{7} & 0 \end{bmatrix}^T$ to be a fixed point of the main system.

7.3. Stability analysis for Scenarios 1 and 2

From Ref. [6], we know that $y_3 = 0$ ensures $\frac{dy_1}{d\tau} > 0$ for every $y_1 = y_1^* - \varepsilon$ and $\frac{dy_1}{d\tau} < 0$ for every $y_1 = y_1^* + \varepsilon$, where y_1^* is the first component of Auxiliary system's fixed point $[y_1^* \quad y_2^*]$ and $\varepsilon > 0$. As such, the properties of the fixed point $[y_1^* \quad y_2^* \quad 0]$ of the main system are only defined by the sign of the derivative $\frac{dy_3}{d\tau}$.

In the case of Scenario 1, we end up with the following system:

$$\begin{cases} \frac{dy_1}{d\tau} = -0.3 \cdot y_1 \cdot y_3 - 0.5 \cdot y_3 \cdot y_3 - 0.7 \cdot y_1 + 0.5, \\ \frac{dy_3}{d\tau} = y_3 \cdot y_3 - y_3. \end{cases} \tag{7.11}$$

The second equation in (7.11) has the negative coefficient before the linear term, which means that the equilibrium point $\begin{bmatrix} \frac{5}{7} & \frac{2}{7} & 0 \end{bmatrix}^T$ is a stable node.

In Scenario 2, the system (3.3) turns out to be:

$$\begin{cases} \frac{dy_1}{d\tau} = -0.3 \cdot y_1 \cdot y_3 + 0.02 \cdot y_3 \cdot y_3 - 0.7 \cdot y_1 - 0.52 \cdot y_3 + 0.5, \\ \frac{dy_3}{d\tau} = -0.04 \cdot y_3 \cdot y_3 + 0.04 \cdot y_3. \end{cases} \tag{7.12}$$

The second equation in (7.12) has the positive coefficient before the linear term. Because of this, the equilibrium point $\begin{bmatrix} \frac{5}{7} & \frac{2}{7} & 0 \end{bmatrix}^T$ is a saddle. That is, the difference between Scenario 1 and Scenario 2 lies in the stability of fixed point $\begin{bmatrix} \frac{5}{7} & \frac{2}{7} & 0 \end{bmatrix}^T$. It is worth noting that in the case of Scenario 2, the phase portrait of the system contains one more fixed point $[0 \quad 0 \quad 1]^T$, which is stable.

7.4. Results of simulations

In fig. 7.1, we present the results of the simulations conducted. We see that in the case of Scenario 1, the system steadily converges to the fixed point $\begin{bmatrix} \frac{5}{7} & \frac{2}{7} & 0 \end{bmatrix}^T$, just as predicted by our theoretical derivations. However, in Scenario 2, the system does not converge to

$\begin{bmatrix}\frac{5}{7} & \frac{2}{7} & 0\end{bmatrix}^T$, which is unstable on this occasion. Instead, the system first reaches a specific line located near the red and blue curves (these two lines are defined by the fixed point equation (3.4) for Auxiliary system and their intersections outline fixed points) and then tends to drift along this line in the direction of the stable fixed point $[0 \quad 0 \quad 1]^T$.

Let us look at this trajectory in more detail. One can notice that each point on this line is almost an equilibrium as it lies near the red and blue curves at once. Nevertheless, the phase velocity on this trajectory is directed towards $[0 \quad 0 \quad 1]^T$. To get a deeper understanding of this effect, let us consider a specific family of transition tables that satisfy the following constraints:

$$\begin{aligned} p_{3,3,3} - p_{2,3,3} - p_{3,2,3} &= 0, \\ p_{1,3,3} - p_{2,3,3} + p_{3,1,3} - p_{3,2,3} &= 0, \\ p_{2,3,3} + p_{3,2,3} - 1 &= 0. \end{aligned} \tag{7.13}$$

In fact, the transition table from Scenario 1 almost fits (7.13): whereas for the second equation, we get the perfect matching, the first and third ones have a residual of 0.04. With a transition table that meets criteria (7.13), one can ensure that $\frac{dy_3}{d\tau} = 0$ holds for all y_1 and y_2, and the infinite number of fixed points exists. These fixed points form a line which is extremely close to the trajectory of the system in Scenario 2. Each of these fixed points can be easily computed by reducing the problem at hand to Auxiliary system. To this end, one should pose $0 \leq y_3 = c < 1$ and find a fixed point $[y_1^* \quad y_2^*]$ from Auxiliary system subject to $y_1 + y_2 = 1 - c$.

8. FURTHER PROPERTIES

The results presented above center around the line $y_3 = 0$, which is an extremely narrow domain of the phase space. However, after imposing additional restrictions on the transition table components, we can guarantee that with probability 1, the system will always reach this line in a finite time interval. Further, after reaching the line $y_3 = 0$, the system will not leave it, and its behavior to be described by the solution of the corresponding Auxiliary system.

Let us now look at equation (6.9) in more detail. Because $y_3 \geq 0$, we can rewrite inequality $\frac{dy_3}{d\tau} < 0$ as

$$\begin{aligned} y_3 \times (p_{3,3,3} - p_{2,3,3} - p_{3,2,3}) + \\ + y_1 \times (p_{1,3,3} - p_{2,3,3} + p_{3,1,3} - p_{3,2,3}) + (p_{2,3,3} + p_{3,2,3} - 1) < 0, \\ y_3 > 0. \end{aligned} \tag{8.14}$$

The first inequality in (8.14) is ensured if the following constraints hold (recall that $y_1 \geq 0, y_3 \geq 0, y_1 + y_3 \leq 1$):

$$\begin{aligned} p_{3,3,3} - p_{2,3,3} - p_{3,2,3} &\leq 0, \\ p_{1,3,3} - p_{2,3,3} + p_{3,1,3} - p_{3,2,3} &\leq 0, \\ p_{2,3,3} + p_{3,2,3} - 1 &< 0, \end{aligned} \tag{8.15}$$

Note that inequalities (8.15) are imposed upon unspecified components of (4.6) and thus do not contradict it.

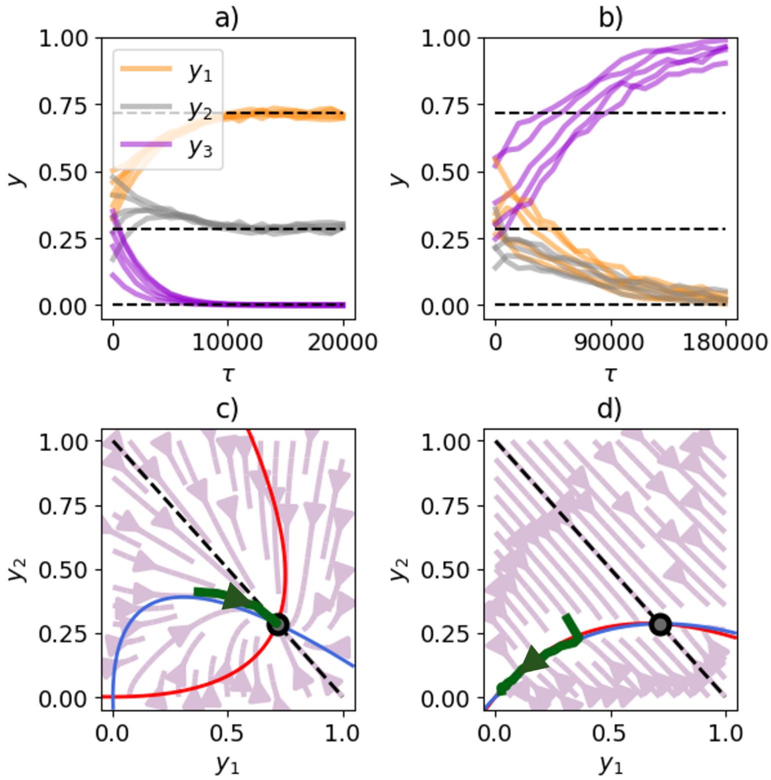

Fig. 7.1. On panels (a) and (b), 5 simulation experiments for Scenarios 1 and 2 are depicted. Panels (c) and (d) demonstrate the phase portraits for these Scenarios. The gray dots on panels (c) and (d) plot the fixed point $\begin{bmatrix}\frac{5}{7} & \frac{2}{7} & 0\end{bmatrix}^T$. The same point is defined on panels (a) and (b) by the dashed lines. The blue and red lines are defined by the equations of (3.4). Their intersections mark fixed points. The green curves stand for the trajectories obtained in simulations (one trajectory per Scenario).

If (8.15) is true, then equation $\frac{dy_3}{d\tau} = 0$ has the only root ($y_3 = 0$), and $\frac{dy_3}{d\tau} < 0$, excepting for the only fixed point $[y_1^* \quad y_2^* \quad 0]^T$ (we will characterize this fixed point later).

For such systems, the following results can be obtained.

Theorem 8.1:
Let us consider the Cauchy problem (3.3), (3.5) *for* $m = 3$. *Let assume that restrictions* (4.6) *and* (8.15) *hold. Then, the system will almost surely reach the line* $y_3 = 0$ *in a finite amount of time.*

Theorem 8.2:
Let assume that the conditions of Theorem 8.1 hold. Let assume that the system has just ($t = T$) *reached the line* $y_3 = 0$ *at the point* $[\hat{q}_1 \quad \hat{q}_2 \quad 0]^T$. *Let us redefine the time as follows:* $\hat{\tau} = \tau - T$, *and initialize Auxiliary system with the initial point* $[\hat{q}_1 \quad \hat{q}_2]^T$.

Then the solution of the main system $y(\tau)$ *is given by*

$$y(\tau) = [y_1(\tau) \quad y_2(\tau) \quad 0]^T,$$

where

$$y_1(\tau) = \hat{y}_1(\hat{\tau}), \quad y_2(\tau) = \hat{y}_2(\hat{\tau}),$$

and $[\hat{y}_1(\hat{\tau}) \quad \hat{y}_2(\hat{\tau})]^T$ *is the solution of Auxiliary system.*

Proof

First of all, let us show that with probability 1, the line $y_3 = 0$ can be reached from any initial point in finite time. Let us denote $p_{1,j,k}$ as $a_{j,k}$, $p_{2,j,k}$ as $b_{j,k}$, and $p_{3,j,k}$ as $c_{j,k}$.

The probability that the number of agents with opinion z_3 will increase during the next time step can be safely presented as follows:

$$\begin{aligned}&\Pr(o_i = z_1, o_j = z_1) \cdot 0 + \Pr(o_i = z_1, o_j = z_2) \cdot 0 + \Pr(o_i = z_1, o_j = z_3) \cdot a_{3,3} + \\ &+\Pr(o_i = z_2, o_j = z_1) \cdot 0 + \Pr(o_i = z_2, o_j = z_2) \cdot 0 + \Pr(o_i = z_1, o_j = z_3) \cdot b_{3,3}.\end{aligned}$$

In turn, the probability that the number of agents with opinion z_3 will decrease during the next step is given by the following expression:

$$\begin{aligned}&\Pr(o_i = z_3, o_j = z_1) \cdot c_{1,1} + \Pr(o_i = z_3, o_j = z_1) \cdot c_{1,2} + \Pr(o_i = z_3, o_j = z_2) \cdot c_{2,1} + \\ &+\Pr(o_i = z_3, o_j = z_2) \cdot c_{2,2} + \Pr(o_i = z_3, o_j = z_3) \cdot c_{3,1} + \Pr(o_i = z_3, o_j = z_3) \cdot c_{3,2}.\end{aligned}$$

The expected change in the number of agents with opinion z_3 during one step ΔY_3 is equal to:

$$\begin{aligned}\mathbb{E}(\Delta Y_3) = +1 \cdot [\Pr(o_i = z_1, o_j = z_1) \cdot 0 + \Pr(o_i = z_1, o_j = z_2) \cdot 0 + \\ +\Pr(o_i = z_1, o_j = z_3) \cdot a_{3,3} + \Pr(o_i = z_2, o_j = z_1) \cdot 0 + \Pr(o_i = z_2, o_j = z_2) \cdot 0 + \\ +\Pr(o_i = z_1, o_j = z_3) \cdot b_{3,3}] + (-1) \cdot [\Pr(o_i = z_3, o_j = z_1) \cdot c_{1,1} + \Pr(o_i = z_3, o_j = z_1) \cdot c_{1,2} + \\ +\Pr(o_i = z_3, o_j = z_2) \cdot c_{2,1} + \Pr(o_i = z_3, o_j = z_2) \cdot c_{2,2} + \Pr(o_i = z_3, o_j = z_3) \cdot c_{3,1} + \\ +\Pr(o_i = z_3, o_j = z_3) \cdot c_{3,2}] = \\ = \frac{y_3}{(y_1 + y_2 + y_3)^2}(y_1 \cdot a_{3,3} + y_2 \cdot b_{3,3} - y_1(1 - c_{1,3}) - y_2(1 - c_{2,3}) - y_3 \cdot (1 - c_{3,3})).\end{aligned}$$

Let us rewrite this as follows:

$$\begin{aligned}\mathbb{E}(\Delta Y_3) = \frac{y_3}{(y_1 + y_2 + y_3)^2}(y_1 \cdot (p_{3,1,3} + p_{1,3,3} - 1) + \\ +(1 - y_1 - y_3) \cdot (p_{2,3,3} + p_{3,2,3} - 1)) + y_3 \cdot (p_{3,3,3} - 1) = \\ = \frac{y_3}{(y_1 + y_2 + y_3)^2}(y_3 \cdot (p_{3,3,3} - p_{2,3,3} - p_{3,2,3}) + \\ +y_1 \cdot (p_{1,3,3} - p_{2,3,3} + p_{3,1,3} - p_{3,2,3}) + (p_{2,3,3} + p_{3,2,3} - 1)).\end{aligned}$$

Now from (8.15) we conclude that $\mathbb{E}(\Delta Y_3) \leq 0$ for every $y_1, y_2,$ and y_3. Moreover, $\mathbb{E}(\Delta Y_3)$ is only equal to zero when $y_3 = 0$.

Having $(y_1 + y_2 + y_3) = N$ is a finite natural number and

$$y_3 \cdot (p_{3,3,3} - p_{2,3,3} - p_{3,2,3}) + y_1 \cdot (p_{1,3,3} - p_{2,3,3} + p_{3,1,3} - p_{3,2,3}) + (p_{2,3,3} + p_{3,2,3} - 1) \leq D,$$

where

$$D = (p_{3,3,3} - p_{2,3,3} - p_{3,2,3}) + (p_{1,3,3} - p_{2,3,3} + p_{3,1,3} - p_{3,2,3}) + (p_{2,3,3} + p_{3,2,3} - 1))$$

is a finite negative number for every fixed transition table, so we end up with

$$|\mathbb{E}(\Delta Y_3)| \geq \frac{|D|}{N^2},$$

which means that for every initial condition, with probability 1, the line $y_3 = 0$ will be reached in a finite time T.

After reaching the line $y_3 = 0$, the system ends changing the third phase component: the probability of increasing the number of opinion z_3's followers is zero due to $p_{1,1,3} = p_{1,2,3} = p_{2,1,3} = p_{2,2,3} = 0$, and $\frac{dy_3}{d\tau} = 0$. As such, we can safely throw away the third dimension and analyze the behavior of the system along the first two phase components y_1 and y_2. In other words, we can switch ourselves to consider Auxiliary system.

□

Remark 8.1:
Theorem 8.1's conditions ensure that the state $[0 \quad 0 \quad 1]^T$ *cannot be a point of no return for the agent system (not to be confused with the mean-field equation system), because from* (8.15) *it holds that*

$$p_{3,3,3} < p_{2,3,3} + p_{3,2,3} < 1.$$

In this vein, even if the agent system reaches the state $[0 \quad 0 \quad 1]^T$, *there is always a nonzero probability of leaving it in the next time moment.*

9. DISCUSSION

In this paper, we studied the mean-field approximation for the SCARDO-model in the case of the 3-element opinion alphabet. These settings are more realistic and meaningful and, at the same time, much more complicated than the 2-element opinion alphabet, for which in Ref. [6], the precise analytical description has been derived. Unfortunately, for the 3-element opinion space, we failed to get analytical solutions for the mean-field system without imposing additional constraints on the model parameters.

The main idea of our approach was to reduce the 3-element opinion alphabet system to the 2-dimensional phase space and then harness the results obtained in Ref. [6]. To this end, we focused on a specific family of transition tables that affords such a treatment. For such transition tables, we managed to jump to the 2-element opinion alphabet. After that, using a combination of analytical and simulation methods, we investigated the properties of fixed points located on the line $y_3 = 0$. After that, after implementing additional restrictions on the transition table, we demonstrated that the system will reach the line $y_3 = 0$ in a finite amount of time and then its behavior can be easily predicted. The reason is that after reaching the line $y_3 = 0$, the system will remain on it. In this vein, we can make use of the analytical results for the case $m = 2$ from Ref. [6].

As we already discussed in Section 4, the constraints 4.6 that were imposed on the transition table are not too rigid and still allow handling many scenarios of opinion formation. It is worth noting that instead of (4.6), we could use different specifications of the transition table that are just the symmetric displacements of (4.6):

$$P_1 = \begin{bmatrix} 1-\alpha & \alpha & 0 \\ 1-\beta & \beta & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, P_2 = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}, P_3 = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ 0 & \delta & 1-\delta \\ 0 & \gamma & 1-\gamma \end{bmatrix}$$

or

$$P_1 = \begin{bmatrix} 1-\alpha & \alpha & 0 \\ 1-\beta & \beta & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, P_2 = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}, P_3 = P_3 = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ 0 & \delta & 1-\delta \\ 0 & \gamma & 1-\gamma \end{bmatrix}.$$

For these transition tables, all the derivations would be virtually the same.

Next, in Section 8, we introduced a new set of restrictions on the transition table. In short, these restrictions take away the competitive edge from the third opinion camp and ensure that this opinion will not have supporters in the long run. We should say that, like (4.6), these restrictions are not somewhat exotic. For example, the transition table estimated on the empirical data from Ref. [9]:

$$P_1 = \begin{bmatrix} 0.96 & 0.04 & 0 \\ 0.942 & 0.057 & 0.001 \\ 0.907 & 0.091 & 0.002 \end{bmatrix}, P_2 = \begin{bmatrix} 0.039 & 0.952 & 0.008 \\ 0.021 & 0.969 & 0.01 \\ 0.02 & 0.944 & 0.036 \end{bmatrix}, P_3 = \begin{bmatrix} 0.001 & 0.082 & 0.917 \\ 0.001 & 0.07 & 0.929 \\ 0.001 & 0.054 & 0.945 \end{bmatrix},$$

meets inequalities (8.15).

We believe that these results would be useful in performing research on opinion dynamics with the SCARDO-model, as they provide analytical insights into model behavior for some configurations of the transition table.

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