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Exact Solutions of the Groundwater Filtration Equation via Dynamics

Tao Sinian

Lomonosov Moscow State University, Moscow, Russia

Abstract: The paper is devoted to the application of finite dimensional dynamics of evolutionary partial differential equations to the Boussinesq filtration equation. The Boussinesq equation is a second order nonlinear equation with three independent variables: time and two spatial coordinates. It describes a shape of the groundwater free surface as it flows through a porous medium under the influence of gravity. This paper proposes a method for constructing exact solutions of the equation, based on the method of finite dimensional dynamics. An example of the evolution of the free surface of groundwater is given.

Keywords: Boussinesq equation, filtration, groundwater, finite dimensional dynamics, completely integrable distributions, symmetry

1. INTRODUCTION

The Boussinesq equation [2]

$$\frac{\partial u}{\partial t} = k\frac{\partial}{\partial x}\left((H(x,y)+u)\frac{\partial u}{\partial x}\right) + k\frac{\partial}{\partial y}\left((H(x,y)+u)\frac{\partial u}{\partial y}\right) \tag{1.1}$$

describes a two-dimensional non-stationary filtration of groundwater in a thin layer of porous media caused by gravity.

A porous medium is a layer of permeable material (for example, sand or clay). The layer is limited below by a surface that does not allow water to pass through (for example, granite), and above by the surface of the earth. If, as a result of any physical processes (operation of artesian wells, drainage or heavy rainfall), the water level in any place of the layer changes, then under the influence of gravity, the liquid begins to move, leveling its free surface.

In equation (1.1) t is time, x, y are spatial coordinates in the fixed horizontal plane Π, and k is a constant, depending on the physical properties of water and porous media. Without loss of generality we can suppose that $k = 1$. The functions $H(x, y)$ and $u(t, x, y)$ show the distances from the plane Π to the underlying surface and to the free surface of the water at a point (x, y) respectively at a moment t.

The porous medium in this case is a thin layer of permeable material (for example, sand or clay). This layer is limited below by a surface that does not allow water to pass through (for example, granite), and above by the surface of the earth. If, as a result of any physical processes (the operation of artesian wells or precipitation), the water level in any place of the layer changes, then under the influence of gravity, the liquid begins to move, leveling its free surface.

The model is built under the following assumptions:

*Corresponding author: 86taosinian@gmail.com

– water is incompressible and has a constant density;
– the soil is homogeneous;
– the underlying surface is smooth;
– groundwater does not reach the surface of the earth anywhere;
– the pressure on the free surface of the liquid is constant.

2. EVOLUTIONARY EQUATIONS AND THEIR FLOWS

Equations (1.1) is a nonlinear evolutionary equation. The idea of the method that we implement in this work is as follows (see [14]).

Let φ be the generating function of an infinitesimal symmetry of a differential equation $\mathcal{E}$. The evolutionary equation

$$\frac{\partial u}{\partial t} = \varphi\left(x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots\right) \tag{2.2}$$

generates the flow Φ_t on the solution set of $\mathcal{E}$. Therefore, knowing some solution $v(x)$ of $\mathcal{E}$, we can construct a one-parameter family of solutions $v_t = \left(\Phi_t^{-1}\right)^* (v(x))$.

To do this we need to solve the Cauchy problem $u(0, x) = v(x)$ for equation (2.2). Then the function $u(t, x) = v_t$ is a solution of evolutionary equation (2.2). The method of (finite dimensional) dynamics is based precisely on this idea.

If for a given function φ we can find a finite type equation $\mathcal{E}$, such that this function generates a symmetry, then we can find a finite dimensional family of solutions of equation (2.2).

For equations with one spatial variable, the method of finite dimensional dynamics was proposed in [3, 11] and it was further developed in [1, 5, 8]. For equations with several spatial variables, the method was proposed in [4]. In [6] it was applied to equations with two spatial variables, and in [10] it was extended to general systems of evolutionary equations.

The practical application of the finite dimensional dynamics method involves cumbersome symbolic calculations in jet spaces. To overcome these difficulties, we used a symbolic computation system Maple (see details in [9, 13]).

This article is a direct continuation of our article [6], so we refer readers to it for basic definitions and explanations (see also [7]).

3. DYNAMICS OF THE BOUSSINESQ EQUATION

We look for dynamics of equation (1.1) in the form of a second-order overdetermined system of differential equations

$$\begin{cases} \dfrac{\partial^2 v}{\partial x^2} = P(x, y), \\ \dfrac{\partial^2 v}{\partial x \partial y} = Q(x, y), \\ \dfrac{\partial^2 v}{\partial y^2} = R(x, y), \end{cases} \tag{3.3}$$

where P, Q, R are smooth functions.

Let $J^1 = J^1(\mathbb{R}^2)$ be the space of 1-jets with canonical coordinates $x_1 = x, x_2 = y, v, p_1, p_2$. System (3.3) defines the two-dimensional distribution

$$\mathcal{P} : a \ni J^1 \mapsto \mathcal{P}(a) = \bigcap_{i=0}^{2} \ker \omega_{i,a} \subset T_a J^1 \tag{3.4}$$

on this space. Here differential 1-forms

$$\begin{aligned}\omega_0 &= dv - p_1 dx - p_2 dy,\\ \omega_1 &= dp_1 - P(x,y)dx - Q(x,y)dy,\\ \omega_2 &= dp_2 - Q(x,y)dx - R(x,y)dy.\end{aligned}$$

The distribution integrability condition has the form

$$P = \frac{\partial^2 a}{\partial x^2}, \quad Q = \frac{\partial^2 a}{\partial x \partial y}, \quad R = \frac{\partial^2 a}{\partial y^2},$$

where $a = a(x,y)$ are some smooth function. Then the general solution of system (3.3) is

$$v(x,y) = a(x,y) + C_1 x + C_2 y + C_0, \tag{3.5}$$

where C_1, C_2, C_3 are arbitrary constants.

Rewriting the right-hand side of equation (1.1) in 2-jet space coordinates and restricting it to the distribution $\mathcal{P}$ we get the function

$$\bar{\varphi} = (H_x + p_1)p_1 + (H_y + p_2)p_2 + (H + v)(a_{xx} + a_{yy}).$$

Construct the vector field

$$\bar{S} = S_{\bar{\varphi}} = \bar{\varphi}\frac{\partial}{\partial v} + \mathcal{D}_1(\bar{\varphi})\frac{\partial}{\partial p_1} + \mathcal{D}_2(\bar{\varphi})\frac{\partial}{\partial p_2}, \tag{3.6}$$

where

$$\begin{aligned}\mathcal{D}_1 &= \frac{\partial}{\partial x} + p_1\frac{\partial}{\partial v} + a_{xx}\frac{\partial}{\partial p_1} + a_{xy}\frac{\partial}{\partial p_2},\\ \mathcal{D}_2 &= \frac{\partial}{\partial y} + p_2\frac{\partial}{\partial v} + a_{xy}\frac{\partial}{\partial p_1} + a_{yy}\frac{\partial}{\partial p_2}.\end{aligned}$$

Note that $[\mathcal{D}_1, \mathcal{D}_2] = 0$. The vector field is an infinitesimal symmetry of the distribution $\mathcal{P}$ if and only if (see [6])

$$\begin{cases}\mathcal{D}_1^2(\bar{\varphi}) - \bar{S}(a_{xx}) = 0,\\ \mathcal{D}_2^2(\bar{\varphi}) - \bar{S}(a_{yy}) = 0,\\ \mathcal{D}_1\mathcal{D}_2(\bar{\varphi}) - \bar{S}(a_{xy}) = 0.\end{cases} \tag{3.7}$$

Equations (3.7) are governing. They link functions H and a. The solvability condition of system (3.7) imposes restrictions on the function H. For example, for quadratic functions H functions a should be quadratic, too.

So, to construct a solution of equation (1.1), we need:

1. find a solution a of system (3.7) for a given function H;
2. construct vector field (3.6) and find its flow Φ_t;
3. act on solutions of system (3.3) by the flow Φ_t.

4. QUADRATIC FUNCTION

Let the function H be quadratic:

$$H(x,y) = h_{20}x^2 + 2h_{11}xy + h_{02}y^2 + h_{10}x + h_{01}y + h_{00}. \tag{4.8}$$

Substituting (4.8) into (3.7), we see that system (3.7) contains equations that do not depend on the coefficients h_{ij}. Solving them, we find that the function a is also quadratic:

$$a(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2. \tag{4.9}$$

Here we discard the linear part since it do not affect equation (3.3).

The remaining equations of system (3.7) provide restrictions on the coefficients of (4.8) and (4.9). Note that for an arbitrary quadratic function H the functions $a(x,y) = -H(x,y)$ and $a(x,y) = 0$ are solutions of (3.7).

5. EXAMPLE: EVOLUTION OF THE FREE SURFACE OF GROUNDWATER

Let us find solutions of equation (1.1) when $H = 2xy$. Then we can choose

$$a(x,y) = -\frac{1}{3}x^2 - \frac{4}{3}xy - \frac{1}{3}y^2.$$

By formula (3.5) we get

$$v(x,y) = -\frac{1}{3}x^2 - \frac{4}{3}xy - \frac{1}{3}y^2 + C_1x + C_2y + C_0. \tag{5.10}$$

Then

$$\bar{\varphi} = (2y + p_1)p_1 - \frac{8}{3}xy - \frac{4}{3}v + (2x + p_2)p_2$$

and vector field (3.6) takes the form

$$\bar{S} = \left(2p_1y - \frac{8}{3}xy - \frac{4}{3}v + 2p_2x + p_1^2 + p_2^2\right)\frac{\partial}{\partial v} \\ - \left(4y - \frac{2}{3}p_2 + \frac{8}{3}p_1 + \frac{8}{3}x\right)\frac{\partial}{\partial p_1} - \left(\frac{2}{3}p_1 + 4x + \frac{8}{3}p_2 + \frac{8}{3}y\right)\frac{\partial}{\partial p_2}.$$

The translation along this vector field is

$$\Phi_t : \begin{cases} x \mapsto x, \\ y \mapsto y, \\ v \mapsto [-(1/3(x^2 + y^2 + 4xy))\exp((4/3)t) \\ \quad - (1/3(x - y+))(x - y - (3/2)p_1 + (3/2)p_2)\exp(-(2/3)t) \\ \quad - (3/8)(y + x + (1/2)p_2 + (1/2)p_1)^2\exp(-(16/3)t) \\ \quad - (1/12)(x - y - (3/2)p_1 + (3/2)p_2)^2\exp(-(8/3)t) \\ \quad + (y + x + (1/2)p_2 + (1/2)p_1)(y + x)\exp(-2t) + (1/8)x^2 \\ \quad - (1/3((21/8)p_1 - (15/8)p_2 + (9/4)y))x + (1/8)y^2 - \\ (1/3((21/8)p_2 - (15/8)p_1))y + (9/32)p_2^2 \\ \quad + (9/32)p_1^2 - (3/16)p_2p_1 + v]\exp((-4/3)t), \\ p_1 \mapsto -(2/3(x + 2y))\exp(2t)\exp((10/3)t)\exp(-(16/3)t) \\ \quad + (1/6(6y + 6x + 3p_2 + 3p_1))\exp(-(10/3)t) \\ \quad - (1/3)\exp(-2t)(x - y - (3/2)p_1 + (3/2)p_2), \\ p_2 \mapsto -(4/3)x - (2/3)y + ((y + x + (1/2)(p_2 + p_1)))\exp(-(10/3)t) \\ \quad + (1/6)\exp(-2t)(-2y + 2x + 3p_2 - 3p_1). \end{cases}$$

Applying the transformation Φ_t^{-1} to (5.10) we get the following 3-parameter solutions family of Boussinesq equation (1.1):

$$
\begin{aligned}
u(t,x,y) = & -\frac{1}{3}x^2 - \frac{4}{3}xy + \frac{1}{2}x\exp\left(-\frac{10}{3}t\right)C_1 + \frac{1}{2}\exp(-2t)C_3x \\
& - \frac{1}{2}\exp(-2t)C_1x + \frac{1}{2}x\exp\left(-\frac{10}{3}t\right)C_3 - \frac{1}{3}y^2 \\
& + \frac{1}{2}\exp(-2t)C_1y - \frac{1}{2}\exp(-2t)C_3y + \frac{1}{2}\exp\left(-\frac{10}{3}t\right)yC_3 \\
& + \frac{1}{2}\exp\left(-\frac{10}{3}t\right)yC_1 + \exp\left(-\frac{4}{3}t\right)C_2 - \frac{3}{16}\exp(-4t)C_3^2 \\
& + \frac{9}{32}\exp\left(-\frac{4}{3}t\right)C_3^2 - \frac{3}{32}\exp\left(-\frac{20}{3}t\right)C_1^2 - \frac{3}{16}\exp(-4t)C_1^2 \\
& + \frac{9}{32}\exp\left(-\frac{4}{3}t\right)C_1^2 - \frac{3}{32}\exp\left(-\frac{20}{3}t\right)C_3^2 + \frac{3}{8}\exp(-4t)C_3C_1 \\
& - \frac{3}{16}\exp\left(-\frac{4}{3}t\right)C_3C_1 - \frac{3}{16})\exp\left(-\frac{20}{3}t\right)C_3C_1.
\end{aligned}
\tag{5.11}
$$

A direct check shows that this is indeed a solution of equation (1.1) when $H(x,y) = 2xy$. Graphs of function (5.11) whit $C_1 = 1, C_2 = C_3 = 0$ for various time values are presented in Fig. 5.1.

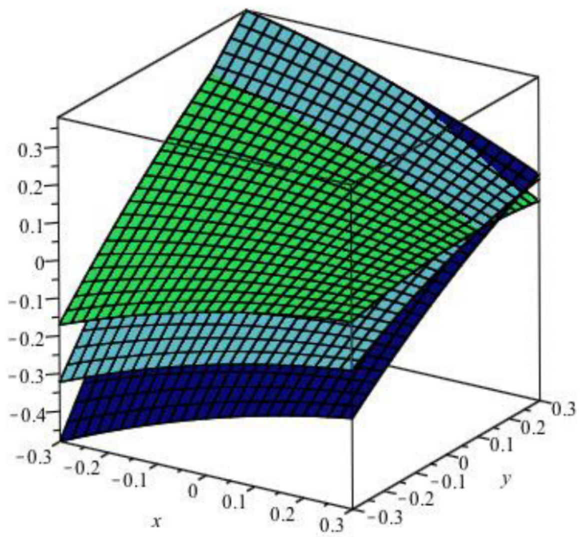

Fig. 5.1. Evolution of the free surface of groundwater at $t = 0, 0.1, 0.3$.

Another example of the use of differential geometric methods for constructing exact solutions of nonlinear partial differential equations is proposed in [12].

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