The Theory of Parametric Control of Macroeconomic Systems and Its Applications(II)

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Abstract
This work consists of three parts and presents the recent results of development of the theory of parametric control of macroeconomic systems and some its applications for solving a number of concrete problems.
Keywords Mathematical Model, Structural Stability, Parametrical Identification, Parametric Control

Part 2. Mathematical Foundations of the Parametric Control Theory

2.1 Sufficient Conditions for the Existence of Solutions for the Problems on Synthesis and Choice of Optimal Parametric Control Laws

2.1.1 Conditions for the Existence of Solution for the Variational Calculus Problem on Synthesis of Optimal Parametric Control Law of Continuous Dynamical System

Consider continuous controllable system*

$$\dot{x}(t) = f(x(t), \mu(t), a(t)), t \in [0, T] \tag{1}$$

$$x(0) = x_0 \tag{2}$$

where t is time; $x = x(t) = (x^1(t), ..., x^m(t))$ is a vector function of system's state; $\mu = \mu(t) = (\mu^1(t), ..., \mu^q(t))$ is a vector function of control; $a = a(t) = (a^1(t), ..., a^s(t))$ is a known vector function; $s_0 = (x_0^1, ..., x_0^m)$ is an initial condition of the system, a known vector; f-a known vector function of its arguments.

The problem of synthesis of optimal economic tools values consists in finding the extremum of the following criterion

$$K = \int_0^T F(t, x(t))dt \tag{3}$$

where F is a known function, at the phase constraints

$$x(t) \in X(t), \quad t \in [0, T] \tag{4}$$

where $X(t)$ is a given set and at explicit control constraints

$$u(t) \in U(t), \quad t \in [0, T] \tag{5}$$

* All of the formulated in this part results for non-autonomous dynamical systems remain true for autonomous dynamical systems as well, when an exogenous vector function a(·) is taken as constant.

where $U(t)$ is a given set.

Define the variational calculus problem on synthesis of optimal parametric control laws for the continuous dynamical system.

Problem 2.1. At given function $a(\cdot)$ to find the control $u(\cdot)$, satisfying the condition (5), so that dynamical system (1), (2)solution meets the condition (4) and gives the maximum (the minimum) to functional (3).

To prove the solubility of the Problem 2.1 we first need the single-valued solubility of Cauchy problem (9), (10). To obtain this result we use known result from the theory of ordinary differential equations. Let be given: number $T > 0$, metrizable compact U and continuous function $\varphi : [0, T] \times R^m \times U \to R^m$ such that for any $\rho \geq 0$ exists such $\sigma \geq 0$, that the following inequality is true

$$|\varphi(t, y, \mu) - \varphi(t, y', \mu)| \leq \sigma|y - y'|\forall t \in [0, T], y, y' \in \rho B, \mu \in U \tag{6}$$

where is a unit ball in R^m and there is such a constant $\eta \geq 0$ that the following inequality is true

$$|y\varphi(t, y, \mu)| \leq \eta(1 + |y|^2)\forall t \in [0, T], y \in R^m, \mu \in U \tag{7}$$

Consider Cauchy problem

$$\dot{y}(t) = \varphi(t, y(t), \mu(t))\forall t \in [0, T] \tag{8}$$

$$y(0) = y_0 \tag{9}$$

where $y_0 \in R^m$. The following statement is true ([1], Lemma 4.1):

Lemma 2.1. Given above mentioned constraints (6), (7) for any measurable mapping $\mu : [0, T] \to U$ the problem (8), (9) has the only solution $y : [0, T] \to R^m$, satisfying the estimate

$$|y(t)| \leq \left(|y_0|^2 + 2\eta T\right)^{1/2} e^{\eta T}\forall t \in [0, T] \tag{10}$$

The following statement is true.

Theorem 2.2. Let function $a(\cdot)$ be continuous on segment $[0, T]$, U is a compact in R^q, function f is continuous, for any $\rho \geq 0$ there is such $\sigma \geq 0$, that the following inequality is true

$$|f(x, \mu, a(t)) - f(x', \mu, a(t))| \leq \sigma|x - x'|\forall t \in [0, T], x, x' \in \rho B, \mu \in U \tag{11}$$

and exists such a constant $\eta \geq 0$ that the following inequality is true

$$|xf(x, \mu, a(t))| \leq \eta(1 + |x|^2)\forall t \in [0, T], x \in R^m, \mu \in U \tag{12}$$

Then for any measurable mapping $\mu : [0,T] \to U$, Cauchy problem (2.1), (2.2) has the only solution $X : [0,T] \to R^m$ satisfying the estimate

$$|(t)| \leq \left(|x_0|^2 + 2\eta T\right)^{1/2} e^{\eta T} \forall t \in [0,T] \tag{13}$$

Proof. Let us introduce notation: $\varphi(t,x,\mu) = f\left(x,\mu,a(t)\right)$ and f functions continuity results in continuity of function φ. (6), (7) inequalities truth follows from relations (11), (12). Thus the theorem statements follow from Lemma 2.1. The theorem is proved.

For proof of solubility of the Problem 2.1 we use known result from the optimal control theory for systems, described by differential equations. Let us specify a closed subset $E \subset [0,T] \times R^m \times U$. Consider the system, described by Cauchy problem (8), (9) when Lemma 2.1 conditions are satisfied, herewith by control we mean a measurable mapping $\mu : [0,T] \to U$. Acceptable pair for the system in question is such a pair "state-control", which satisfies the relations (8), (9) and inclusion

$$(t, x(t), u(t)) \in E \tag{14}$$

Let us specify Carathéodory function Φ with non-negative values on the set $[0,t] \times (R^m \times U)$. Let us specify a functional $I = \int\limits_0^T \Phi\left(t, x(t), u(t)\right) dt$.

Problem 2.2. To find the minimum for the functional I on the acceptable pairs set of the system (8), (9).

For any point $(t,x) \in [0,t] \times R^m$ define a section $E_{t,x} = \{u \in U : (t,x,u) \in E\}$. Let us specify a set $\Gamma_{t,x} = \{\varphi(t,x,u) : (t,x,u) \in E_{t,x}\}$ and a function $g : [0,T] \times R^m \times R^m \to R$ using an equality $g(t,x,y) = \min\{\,\Phi(t,x,u)|\,(t,x,u) \in E, \varphi(t,x,u) = y\}$.

It is known the following statement about solubility of optimization problem ([1], Statement 4.2), where $\bar{R}$ implies the set of real numbers, supplemented with the values $-\infty$ and $+\infty$:

Lemma 2.3. Let, when Lemma 2.1 condition is satisfied, the set $\Gamma_{t,x}$ be convex for all $t \in [0,T].x \in R^m$, and the function $g(t,x,\cdot) : R^m \to \bar{R}$ be convex for all $(t,x) \in [0,T] \times R^m$. Then the Problem 2.2has a solution.

Let be a closure of the sum $\bigcup\limits_{t \in [0,T]} X(t)$. The following statement is true.

Lemma 2.4. Let be a compact, function F be continuouson $[0,T] \times X$. Then function Φ, specified by equality $\Phi(t,x,u) = F_0 - F(t,x)$ where F_0 is the maximum of function F on the set $[0,T] \times X$ satisfies Lemma 2.3 conditions.

Proof. Existence of the maximum for F_0 follows from the Weierstrass theorem. Non-negativity of the determined above function Φ is obvious. It is Carathéodory function because of the continuity of F. At last, the convexity of function $g(t,x,\cdot)$

for all $(t, x) \in [0, T] \times R^m$ is implemented, as function F does not depend on control u. The lemma is proved.

Let U be a closure of the sum $\bigcup_{t \in [0,T]}$. For reduction the constraints (4), (5) to the form (14) we specify the set $E \subset [0, T] \times R^m \times U$ the way to hold the relation: $E_{t_0} = \{(t, x, u) \in E | \, t = t_0\} = X(t_0) \times U(t_0), t_0 \in [0, T]$.

Lemma 2.5. Let the mappings $t \to X(t), t \to U(t)$ be continuous at each point $t \in [0, T]$ in the following sense: if the inclusions $x_k \in X(t_k), u_k \in U(t_k)$ are true, where $t_k \in [0, T], k = 1, 2, ...$ and convergence of the sequences is met $t_k \to t, x_k \to x, u_k \to u$, then the inclusions $x \in X(t), u \in U(t)$ are true.Then the set is closed.

Proof. Consider such an sequence $\{(t_k, u_k, x_k)\}$ of elements of the set , that there is a convergence $(t_k, u_k, x_k) \to (t, u, x)$. From the inclusion $(t_k, u_k, {}_k) \in E$, because of specifying the set , follow the inclusions $t_k \in [0, T], x_k \in X(t_k), u_k \in U(t_k)$, From closure of segment $[0, T]$ follows the inclusion $t \in [0, T]$. Using lemma conditions, ascertain that $x \in X(t), u \in U(t)$ Therefore, the inclusion $(t, u, x) \in E$ is true, that is the set is closed. The lemma is proved.

Let us make sure that lemma conditions are not excessively restrictive. Consider, for instance, a typical situation for scalar case, when the set $X(t) = \{x| \, a(t) \le x \le b(t)\}, t \in [0, T]$ is given, where functions and b are continuous. Let the inclusion $x_k \in X(t_k)$ be met, where $t_k \in [0, T], k = 1, 2, ...$ and there is the convergence $t_k \to t, x_k \to x$, Ipso facto, the inequalities $a(t_k) \le x_k \le b(t_k), k = 1, 2, ...$ are true. By proceeding here to the limit allowing for continuity of functions and b, we get the result $a(t) \le x \le b(t)$ what results in that $x \in X(t)$ Thereby, the mapping $t \in X(t)$ is continuous on the segment $[0, T]$.

Analogously ascertains the truth of Lemma 2.5 conditions and in more general cases, when boundaries of sets of acceptable control and condition values are continuous functions of time.

For reduction the equation (1) to the form (8) it is enough to specify $\varphi(t, x, u) = f(x, u, a(t))$ Then the set $\Gamma_{t,x}$, appearing in Lemma 2.2 conditions, defines in the following way

$$\Gamma_{t,x} = \{ f\,(x, w, a(t))| \, w \in U(t)\} \tag{15}$$

Denote by V the set of acceptable pairs "state-control" of the system (1), (2) in question at given known function , that is such vector function pairs (x, u) which satisfy the relations (1), (2), (4), (5). From Lemma 2.3 directly follows the next statement.

Theorem 2.6. Let function $(\cdot)$ be continuous on the segment $[0, t]$, U be a compact in R^q, function f be continuous in $X \times U \times A$ and for any $\rho \ge 0$ there is such $\sigma \ge 0$, that the following inequality is true

$$\left| f(x, u, a(t)) - f(x', u, a(t)) \right| \le \sigma \left| x - x' \right|, x, x' \in \rho B, u \in U \tag{16}$$

and there is such a constant $\eta \geq 0$, that the following inequality is true

$$|xf(x,u,a(t))| \leq \eta \left(1+|x|^2\right) \forall t \in [0,T], x \in R^m, u \in U \tag{17}$$

Let X be a compact, function F be continuous on $[0,T] \times X$. Let, moreover, the mappings $t \to X(t)$, $t \to U(t)$ be continuous for $t \in [0,T]$ in the following sense: if the inclusion $x_k \in X(t_k)$, $u_k \in U(t_k)$ are true, where $t_k \in [0,T], k = 1,2,...$ and there is convergence of sequences $t_k \to t$, $x_k \to x$, $u_k \to u$, then the inclusions $x \in X(t)$, $u \in U(t)$ are true. Then, in the case of non-emptiness of the set Va and convexity of the set $\Gamma_{t,x}$ for all $t \in [0,T]$, $x \in X(t)$ the Problem 2.1 has a solution in the class of measurable functions.

2.1.2 Conditions for the Existence of Solution for the Variational Calculus Problem on Choice (Among Given Finite Algorithms Set) of Optimal Parametric Control Law of Continuous Dynamical System

The continuous controllable system in question (1), (2). Control u is chosen here among the set of given control laws:

$$u_j(t) = G_j\left(v, x(t)\right), t \in (0,T), j = 1,...,r \tag{18}$$

Here G_j is known vector function of its arguments; $v = (v^1,...,v^l)$ is vector of control parameters. On control parameters lay the constraints like

$$v \in V \tag{19}$$

where V is some subset of the space R^l. Moreover, it is assumed that control parameters should be such that corresponding control law (18) satisfies the condition (5), that is the following inclusion is met

$$G_j\left(v, x(t)\right) \in U(t), t \in (0,T) \tag{20}$$

where $U(t)$ is given set. There are phase constraints on the system:

$$x(t) \in X(t), t \in (0,T) \tag{21}$$

where $X(t)$ is given set.

Consider optimality criteria

$$K_j = K_j(a,v) = \int_0^T F\left[t, x_j(t)\right] dt \tag{22}$$

where $x_j = x_j(t) = \left(x_j^1(t),...,x_j^m(t)\right)$ is solution of Cauchy problem (1), (2) at given function $(\cdot)$ and control $u = u_j(t) = \left(u_j^1(t),...,u_j^q(t)\right)$, that is for chosen j-th control law (26).

Consider the next subsidiary extremal problem:

Problem 2.3*. At given function $a(\cdot)$ for each of r control laws to find such a vector of control parameters v, that corresponding it solution $x = x_j$ of the problem (1), (2) with control law $u = u_j$, determining by formula (18), satisfies the conditions (19)-(21) and gives the maximum for the functional (22).

Define the following variational calculus problem on choice (among given finite algorithms set) of optimal parametric control law for non-autonomous continuous system.

Problem 2.3. Given known function $a(\cdot)$ among all optimal control laws in sense of the Problem 2.3* to choose the one, which corresponds to the maximal optimality criterion value (22).

Substituting into the equation (1) control value from formula (19), we get

$$\dot{x}(t) = f\left(x(t), G_j\left(v, x(t)\right), a(t)\right), t \in (0, T) \tag{23}$$

where for short we omit the index j in denoting condition function of the system, corresponding to given control law. Denote by W_a^j a set of acceptable pairs "state-control parameter" of the system in question, that is such pairs (x, v), which satisfy both the equalities (23), (2), and the inclusions (19)-(21). Thus, the Problem 2.3* comes to the maximization of functional $K = \int_0^T F\left[t, x(t)\right]dt$ on the set W_a^j.

Ascertain first a solubility of Cauchy problem (23), (2). The following theorem directly results from Lemma 2.1.

Theorem 2.7. Let the function $(\cdot)$ is continuous on the segment $[0, T]$, sets U and V are compact, functions f and G_j are continuous, for any $\rho \geq 0$ there exist such $\sigma \geq 0$, $\chi \geq 0$, that the following inequalities are true

$$|f(x, u, a(t)) - f(x', u', a(t))| \leq \sigma(|x - x'| + |u - u'|)\forall t \in [0, T], x, x' \in \rho B, u, u' \in U \tag{24}$$

$$\left|G_j\left(v, x\right) - G_j\left(v, x'\right)\right| \leq \chi\left|x - x'\right| \forall x, x' \in \rho B, v \in V \tag{25}$$

and exists such a constant $\eta \geq 0$ that the following inequalitiy is true

$$|xf(x, G_j(v, x), a(t))| \leq \eta(1 + |y|^2)\forall t \in [0, T], x \in R^m, v \in V \tag{26}$$

Then for any $v \in V$ the problem (2), (2) has the only solution $x : [0, T] \to R^m$,satisfying the estimate

$$|(t)| \leq \left(|_0|^2 + 2\eta T\right)^{1/2} e^{\eta T} \forall t \in [0, T] \tag{27}$$

Ascertain solubility of the Problem 2.3*. The following theorem is true.

Theorem 2.8. Assume that when the conditions of the theorem 2.7 are satisfied for given function $(\cdot)$ and given value of $j \in \{1, ..., r\}$ the set W_a^j is not empty, the sets V, U, X are compact, the sets $X(t), U(t)$ are compact for all $t \in [0,T]$, and function F is continuous. Then the Problem 2.3* has a solution.

Proof. From the condition (21) and continuity of function F follows that this function is bounded. Then, because of non-emptiness of the set W_a^j, upper boundary sup K of function K exists on set W_a^j of acceptable pairs of the system in question. Then there exists such an elements sequence $\{(x_k, v_k)\}$ from the set W_a^j, that convergence takes place (here $K_k = \int_0^T F[t, x_k(t)]dt$)

$$K_k \to \sup K \tag{28}$$

The inclusion $(x_k, v_k) \in W_a^j$ assumes the following relations to be true

$$\dot{x}_k(t) = f\left(x_k(t), G_j\left(v_k, x_k(t)\right), a(t)\right), t \in (0,T) \tag{29}$$

$$x_k(0) = x_0 \tag{30}$$

$$x_k(t) \in X(t), t \in (0,T) \tag{31}$$

$$v_k \in V \tag{32}$$

$$G_j\left(v_k, x_k(t)\right) \in U(t), t \in (0,T) \tag{33}$$

From the Theorem 2.7 results the estimate

$$|x_k(t)| \leq (|x_0|^2 + 2\eta T)^{\frac{1}{2}} e^{\eta T} \forall t \in [0,T] \tag{34}$$

Thus, using BolzanoCWeierstrass theorem allowing for boundedness of set V, after subsequence separation we get the convergences

$$_k(t) \to (t) \forall t \in [0,T] \tag{35}$$

$$v_k \to v \tag{36}$$

Limiting value v satisfies the inclusion (19) because of closure of set V and condition (32). Using the conditions (35), (36) and continuity of function G_j, we get the convergence

$$f\left(x_k(t), G_j\left(v_k, x_k(t)\right), a(t)\right) \to f\left(x(t), G_j\left(v, x(t)\right), a(t)\right), t \in (0,T) \tag{37}$$

Function x_k satisfies the integral relation

$$x_k(t) = x_0 + \int_0^t f\left(x_k(\tau), G_j\left(v_k, x_k(\tau)\right), a(\tau)\right) d\tau, t \in (0,T) \tag{39}$$

By proceeding here to the limit allowing for the conditions (35), (38), we get the equality

$$x(t) = x_0 + \int_0^t f\left(x(\tau), G_j\left(v, x(\tau)\right), a(\tau)\right) d\tau, t \in (0,T) \tag{40}$$

By differentiating both parts of the equality (40) with respect to t, we ascertain that the equation (1) is true. Consequently, after proceeding to the limit in equality (30) taking into account the condition (35), we ascertain that the initial condition (2) is true. Thus, the limiting pair (x, v) is acceptable.

Continuity of function F from the condition (35) results in convergence

$$F\left[t, x_k(t)\right] \rightarrow F\left[t, x(t)\right], t \in [0, T] \tag{41}$$

therefore

$$\int_0^T F\left[t, x_k(t)\right] dt \rightarrow \int_0^T F\left[t, x(t)\right] dt \tag{42}$$

From (42) and (28) follows the equality $\int\limits_0^T F\left[t, x(t)\right] dt = \sup K$. Thus, we found the pair $(x, u) \in W_a^j$ for which the upper boundary of functional K(on the set of all acceptable pairs of the system in question) is obtained. The theorem is proved.

Obviously, the Problem 2.3 comes to the search of maximum of function $\Phi_a(j) = max_{(x,v) \in W_a^j} K_j$ on finite set $\{1, ..., r\}$ at given function $(\cdot)$. The maximal value of considered functional at given function $(\cdot)$ and j-th control law is placed on the right side of previous equality. The existence of this value is proved in the Theorem 2.8. As the maximum of function on finite set is obtained always, we get the following statement.

Theorem 2.9. When Theorem 2.8 conditions are satisfied, the Problem 2.3 has a solution.

2.1.3 Conditions for the Existence of Solution for the Variational Calculus Problem on Synthesis of Optimal Parametric Control Law of Discrete Dynamical System

Consider discrete non-autonomous controllable system

$$x(t+1) = f(x(t), u(t), a(t)), \quad t = 0,\, 1, ..., n-1 \tag{43}$$

$$x(0) = x_0 \tag{44}$$

where t is time. Here $x = x(t) = \left(x^1(t), ..., x^m(t)\right)$ is vector function (of discrete argument) of systems state; $u = u(t) = \left(u^1(t), ..., u^q(t)\right)$ is control, vector function of discrete argument; $a = a(t) = \left(a^1(t), ..., a^s(t)\right)$ is known vector function of discrete argument; $x_0 = \left(x_0^1, ..., x_0^m\right)$ is initial condition of the system, known vector; f is known vector function of its arguments.

The problem of choosing optimal economic tools values consists in finding the extremum of the following criterion

$$K = \sum_{t=1}^{n} F\left[t, x(t)\right] \rightarrow \max\ (\min) \tag{45}$$

where F is known function, at phase constraints on system (43)-(44) solution like

$$x(t) \in X(t), \quad t = 1, ..., n \tag{46}$$

where X(t) is given set, and following constraints on control:

$$u(t) \in U(t),\ t = 0, 1, ..., n-1 \tag{47}$$

where $U(t)$ is given set.

Define the variational calculus problem on synthesis of optimal parametric control laws for discrete dynamical system.

Problem 2.4. Given known function $a(\cdot)$, to find control $u(\cdot)$, satisfying the condition (47), so that dynamical system solution (43), (44), corresponding it, satisfies the condition (46) and gives the maximum (minimum) for functional (45).

Denote by V_a the set of acceptable pairs "state-control" of the system in question at given known function $a(\cdot)$, that is such pairs of vector functions (x, u), which satisfy the relations (43), (44), (46), (47). Introduce notations: $X = \bigcup\limits_{t=1}^{n} X(t)$, $U = \bigcup\limits_{t=0}^{n-1} U(t)$. The following statement is true.

Theorem 2.10. Let for given function $a(\cdot)$ the set V_a be not empty; sets $X(t)$ and $U(tC1)$ are closed and bounded for all $t = 1, ..., n$; mapping f is continuous by the first two arguments on set $X \times U$, and function F is continuous by the second argument on set X. Then the Problem 2.4 has a solution.

Proof of this theorem is based on properties of functions which continuous on compact.

2.1.4 Conditions for the Existence of Solution for the Variational Calculus Problem on Choice (Among Given Finite Algorithms Set) of Optimal Parametric Control Law of Discrete Dynamical System

Consider discrete non-autonomous controllable system (43), (44). The following phase constraints are imposed on this system:

$$x(t) \in X(t), t = 1, ..., n \tag{48}$$

where X(t) is given set. In equation (43) control is chosen among the set of given control laws:

$$u_j(t) = G_j\left(v, x(t)\right), t = 1, ..., n, j = 1, ..., r \tag{49}$$

where G_j is known vector function of its arguments, $v = (v^1, ..., v^l)$ is vector of control parameters. The following constraints are imposed on control parameters

$$v \in V \tag{50}$$

where V is some subset of space R^l. Moreover, we will assume that control parameters should be such that corresponding control law (49) satisfies the condition

$$G_j\left(v, x(t)\right) \in U(t), t = 0, ..., n-1 \tag{51}$$

where $U(t)$ is given set.

Consider the following optimality criteria:

$$K_j = K_j(a, v) = \sum_{t=1}^{n} F\left[t, x_j(t)\right] \tag{52}$$

where $x_j = x_j(t) = \left(x_j^1(t), ..., x_j^m(t)\right)$ is problem (43), (44) solution at given function $(\cdot)$ and control $u = u_j(t) = \left(u_j^1(t), ..., u_j^q(t)\right)$, that is for chosen j-th control law.

Define the next subsidiary extremal problem:

Problem 2.5*. Given known function $(\cdot)$ for each of r control laws to find such a vector of control parameters v, that corresponding it solution $x = x_j$ of problem (43), (44) with control law $u = u_j$, determining by formula (2.49), satisfies the conditions (48), (50), (51) and gives the maximum for functional (52).

The following problem is called as variational calculus problem on choice among given finite set of algorithms of optimal parametric control laws for discrete non-autonomous system.

Problem 2.5. Given known function $(\cdot)$ among all optimal control laws in sense of the Problem 2.5* to find the one, which corresponds to the maximal optimality criterion value (52).

Ascertain first the solubility of the Problem 2.5* for fixed control law. Substituting into equation (43) the control value from formula (49), we get

$$x(t+1) = f\left(x(t), G_j\left(v, x(t)\right), a(t)\right), t = 0, 1, ..., n-1 \tag{53}$$

where for short we omit the index j in denoting condition function of the system, corresponding to j-th control law. Denote by W_a^j the set of acceptable pairs "state-control parameter" of the system in question, that is such pairs (x, v) which satisfy both the equalities (44), (53), and the inclusions (48), (49), (50). Thus, the Problem 2.5* comes to the maximization of functional $K_j =$ on the set W_a^j. Let the sets and U be determined by relations $X = \bigcup_{t=1}^{n} X(t)$, $U = \bigcup_{t=0}^{n-1} U(t)$. The following theorem is true.

Theorem 2.11. Let given known function $(\cdot)$ and j-th control law the set W_a^j is not empty, the sets V, $X(t)$ and $U(t-1)$ are closed and bounded for all $t = 1, ..., n$, function f is continuous by the first two arguments on the set $X \times U$

function G_j is continuous on the set $V \times X$ and function F is continuous by the second argument on the set . Then the Problem 2.5* has a solution.

Proof of this theorem is based on property of continuous on compact functions to reach their maximal and minimal values. As on finite set the maximum of function is obtained always, we get the following statement.

Theorem 2.12. When the Theorem 2.11 condition is satisfied, the Problem 2.5 has a solution.

2.1.5 Conditions for the Existence of Solution for the Variational Calculus Problem on Synthesis of Optimal Parametric Control Law of Discrete Stochastic Dynamical System

Consider discrete stochastic controllable system like

$$x(t+1) = f(x(t), u(t), a) + \xi(t), t = 0, 1, ..., n-1 \tag{54}$$

$$x(0) = x_0 \tag{55}$$

Here $x = x(t) \in R^m$ is function of the (54), (55) systems state, random vector function of the discrete argument (vector random process); x_0 is initial condition of the system, deterministic vector; $u = u(t) \in R^q$ is vector of controllable parameters, vector function of the discrete argument; $a \in R^s$ is vector of uncontrollable parameters, $a \in A, A \subset R^s$ is given set. $\xi = \xi(t) = (\xi^1(t), ..., \xi^m(t))$ is known vector random process, expressing noises (as such a noise can be, for instance, Gaussian noise); f is known vector function of its arguments.

Specify optimality criterion, which is subject to maximization for the present problem is of the form

$$K = E\left\{\sum_{t=1}^{n} F_t(x(t))\right\} \tag{56}$$

Here F_t are known functions, E is mathematical expectation.

There are phase constraints on the system:

$$E[x(t)] \in X(t), t = 1, ..., n \tag{57}$$

where $X(t)$ is given set.

Determined earlier constraints on control hold in the problems considered further:

$$u(t) \in U(t), t = 0, 1, ..., n-1 \tag{58}$$

where $U(t) \in R^q$ is given set.

Define the following variational problem, called as the variational calculus problem on synthesis of optimal parametric control law for discrete stochastic dynamical system.

Problem 2.6. Given known vector of uncontrollable parameters $a \in A$ to find the parametric control law u, satisfying the condition (58), so that dynamical system (54), (55) solution corresponding it satisfies the condition (57) and gives the maximum for functional (56).

Determine the set of acceptable controls U_{ad} for system under study in terms of the set of such control laws $u(t)$, satisfying the constraint (58), for which the mathematical expectation $E[x(t)]$ of corresponding solution of stochastic system satisfies the inclusion (57).

The following theorem is true.

Theorem 2.13. Let in the Problem 2.6 given $a \in A$ for any $t = 1, ..., n$ random variables $\xi(t)$ are absolutely continuous and has zero mathematical expectations, the sets $X(t)$, $U(t)$ are closed and bounded for all t, function f satisfies the Lipschitz condition, and functions F_t are continuous by Lipschitz. Functions f (for $u \in U$ and $a \in A$) and F_t by absolute value does not exceed some linear relatively $|x|$ functions. Then, if the set of acceptable controls U_{ad} is not empty, the Problem 2.6 will be soluble.

Proof. According to the Weierstrass theorem, continuous function on non-empty bounded set reaches its maximum. Thus, it is enough to show that the multivariable function $K = K(u)$, determined by (56), is continuous, and the set U_{ad} is closed and bounded. Its non-emptiness is one of the conditions of the theorem.

We will show that mathematical expectations of the magnitudes, entering into phase constraint (57) are exist. Indeed, according to equation (54), we have

$$E[x(t+1)] = E[f(x(t), u(t), a)] + E[\xi(t)]$$

The second summand of the right-hand-side of this equality has sense under the theorem conditions, and the first is calculated by formula

$$E[f(x(t), u(t), \lambda)] = \int\limits_{R^n} f(\omega, u(t), a) p_{x(t)}(\omega) d\omega$$

if the last integral absolutely converges (here by $p_{x(t)}$ is denoted probability density function of random variable $x(t)$). The latter fact is really takes place under constraints on increase of function f and existence of mathematical expectation of the magnitude $x(t)$ for any $t = 1, ..., n$ (this fact is tested using the mathematical induction method).

Existence of mathematical expectation on the right-hand-side of equality (56) results from constraints on increase of function F_t and existence of mathematical expectation of the variable $x(t)$. Let convergence of vectors $u_k \to u$, $u_k \in U_{ad}$ takes place. From Equation (54) follows the equality

$$|x_k(t+1) - x(t+1)| = |f(x_k(t), u_k(t), a) - f(x(t), u(t), a)|$$

where u_k and x are solutions of the problem (2.54), (2.55) at controls u_k and u, correspondingly. Then the following relation is true

$$|x_k(t+1) - x(t+1)| \leq L_f[|x_k(t) - x(t)| + |u_k(t) - u(t)|]$$

where L_f is the Lipschitz constant of function f. By repeating analogous reasoning and taking into account that under the condition (55), $x_k(0) = x(0)$ we will have

$$\begin{aligned} &|x_k(t+1) - x(t+1)| \\ &\leq (L_f)^2|x_k(t-1) - x(t-1)| + (L_f)^2|u_k(t-1) - u(t-1)| + L_f|u_k(t) - u(t)| \\ &\leq \sum_{s=0}^{t}(L_f)^{s+1}|u_k(t-s) - u(t-s)| \leq \varepsilon_k \end{aligned}$$

where $\varepsilon_k \leq 0$ under $k \to \infty$.

By denoting the maximal of the Lipschitz constants of functions F_t by L_F for $t = 1, ..., n$ we get the estimate

$$|F_t[x_k(t)] - F_t[x(t)]| \leq L_F \varepsilon_k$$

After calculating mathematical expectations of both of parts of this inequality, we get inequality

$$E\{|F_t[x_k(t)] - F_t[x(t)]|\} \leq L_F \varepsilon_k$$

This implies that $E\{|F_t[x_k(t)] - F_t[x(t)]|\} \to 0$ and convergence of the sequence in question $E\{|F_t[x_k(t)] - F_t[x(t)]|\}$. Hence, function $K = K(u)$ is continuous because of (56).

Boundedness of the set U_{ad} results from boundedness of the set $U(t)$. Closure of the set U_{ad} results from continuity of the mapping $U_{ad} \to X$, determined by defining of the set U_{ad} and compactness of the set X(theorem about closure of complete preimageof compact at continuous mapping). Now, existence of solution for the problem under study follows from the Weierstrass theorem.

The theorem is proved.

2.1.6 Conditions for the Existence of Solution for the Variational Calculus Problem on Choice (Among Given Finite Algorithms Set) of Optimal Parametric Control Law of Discrete Stochastic Dynamical System

Controllable discrete dynamical system with given additive noise, described by equations (54), (55) with phase constraints (57) is considered again in the following parametric control problem for discrete dynamical system. In this problem control is chosen among the set of given control laws:

$$u_j(t) = G_j(v, x(t)), t = 0, ..., n-1, j = 1, ..., r \tag{59}$$

where G_j is known vector function of its arguments, $v = (v^1, ..., v^l)$ is parameters vector of control law G_j.

On adjustable coefficients v impose the constraints

$$v \in V \tag{60}$$

where V is compact in space R^l. Moreover, it is assumed that parameters of control law should be such that corresponding control law (59) satisfies the condition (58), that is the following inclusion would be met

$$E\left[G_j(v, x_j^v(t))\right] \in U(t) \tag{61}$$

Here x_j^v is solution of problem (54), (55) at chosen v coefficient values, uncontrollable parameter a and j-th parametric control law.

Consider optimality criteria

$$K_j = K_j(v, a) = E\left\{\sum_{t=1}^{n} F_t(x_j^v(t))\right\} \tag{62}$$

Define the following variational problem, called as the variational calculus problem on choice of optimal parametric control law for discrete stochastic dynamical system.

Problem 2.7. Given known vector of uncontrollable parameter $a \in A$ to find each of r control laws to find such a vector of adjustable coefficients v, so that corresponding it solution $x = x_j$ of problem (54), (55) with control law $u = u_j$, determined by formula (59), satisfies the conditions (60), (61) and gives the maximum for functional (62) with consequent choice of the best of found optimal control laws, i.e. the one, to which corresponds the maximal value of optimality criterion.

Now we get sufficient conditions for the existence of solution of the problem 2.7.

Denote by x_j^v solution of the system (54), (55) for chosen j-th parametric control law (59), its adjustable coefficient v and parameter α:

$$x_j^v(t+1) = f\left(x_j^v(t), G_j\left(v, x_j^v(t)\right), a\right) + \xi(t), t = 0, 1, ..., n-1 \tag{63}$$

$$x_j^v(0) = x(0) \tag{64}$$

For considered problem denote the set of acceptable values of adjustable coefficients as the set V_{ad}^j, consisting of such values $v \in V$ satisfying the condition (60), for which corresponding solution of the problem (63), (64) satisfies the inclusions

$$E\left[G_j(v, x_j^v(t))\right] \in U(t), t = 0, 1, ..., n-1 \tag{65}$$

$$E\left[x_j^v(t)\right] \in X(t), t = 1, ..., n-1 \tag{66}$$

We will call the Problem 2.7 as nontrivial, if the set V_{ad}^j, corresponding it, is not empty and contains some open set for each $j = 1, .., r$.

Theorem2.14. Let in the Problem 2.7 $a \in A$, the sets $U(t)$, $X(t)$, V are compact, functions f, G_j, F_t satisfy the Lipschitz condition. These functions satisfy constraints on increase as well: functions $|f(x, G_j(v, x), a)|$, $|F_t(x)|$ do not exceed linear relatively $|x|$ functions uniformly by $v \in V$. Random variable $\xi(t)$ is absolutely continuous and has zero mathematical expectation. Then in the case of non-emptiness of the sets V_{ad}^j the Problem 2.7 has a solution.

Proof. It is enough to ascertain that all of the functions (62) K_j are continuous, and all of the sets V_{ad}^j are closed and bounded, where $j = 1, ..., r$. The existence of all used below mathematical expectations is proved in the same way as when proving the Theorem 2.13.

Allowing for mathematical expectation additivity, we find the values of

$$K_j = K_j(v) = \sum_{t=1}^{n} E\left\{F_t[x_j^v(t)]\right\}$$

whence it follows inequality for $v, w \in V_{ad}^j$

$$\begin{aligned}|K_j(v) - K_j(w)| &= |\sum_{t=1}^{n} E\left\{F_t[x_j^v(t)]\right\} - \sum_{t=1}^{n} E\left\{F_t[x_j^w(t)]\right\}| \\ &\leq \sum_{t=1}^{n} |E\left\{F_t[x_j^v(t)]\right\} - E\left\{F_t[x_j^w(t)]\right\}|\end{aligned}$$

From relations (63), (64) follows that

$$\begin{aligned}|x_j^v(t+1) - x_j^w(t+1)| &= \left|f\left(x_j^v(t), G_j\left(v, x_j^v(t)\right), a\right) - f\left(x_j^w(t), G_j\left(w, x_j^w(t)\right), a\right)\right| \\ &\leq L_f\left[|x_j^v(t) - x_j^w(t)| + |G_j\left(v, x_j^v(t)\right) - G_j\left(w, x_j^w(t)\right)|\right]\end{aligned}$$

where L_f is the Lipschitz constant of function f. After denoting by L_a the maximal one of the Lipschitz constants of function G_j, we will get inequality

$$|x_j^v(t+1) - x_j^w(t+1)| \leq L_f(1 + L_a)|x_j^v(t) - x_j^w(t)| + L_f L_G|v - w|, t = 0, 1, ..., n-1$$

Taking into account equality $|x_j^v(0) - x_j^w(0)| = 0$, we get the estimate

$$|x_j^v(t+1) - x_j^w(t+1)| \leq L_f L_G \sum_{l=0}^{t} \left[L_f(1 + L_G)\right]^l |v - w| \leq \beta|v - w| \forall v, w \in V_{ad}^j$$

where

$$\beta = L_f L_G \sum_{l=0}^{t} [L_f(1 + L_G)]^l$$

After denoting by L_F the maximal one of the Lipschitz constants of function F_t, we will have

$$\left|F[x_j^v(t)] - F[x_j^w(t)]\right| \le L_f \left|x_j^v(t) - x_j^w(t)\right| \le L_F \beta |v - w| \forall v, w \in V_{ad}^j$$

Thus, in the case of sufficient smallness of difference of adjustable coefficients v and w the values of $x_j^v(t)$ and $x_j^w(t)$ (and the same $F[x_j^v(t)]$ and $F[x_j^w(t)]$) will be arbitrary close to each other. Determine converged sequence $w = v_k \to v$. Then, after finding mathematical expectations of LHS and RHS of the last inequality, we will get the following inequality:

$$E\left[\left|F[x_j^v(t)] - F[x_j^w(t)]\right|\right] \le L_F \beta |v - w|$$

Hence results the following convergence

$$E\left\{F_t\left[x_j^{v_k}(t)\right]\right\} \to E\left\{F_t\left[x_j^{w_k}(t)\right]\right\}$$

from which follows continuity of function K_j^v.

As $V_{ad}^j \subset V$, then all of the sets V_{ad}^j are bounded. Closure of the sets V_{ad}^j results from closure of the sets $U(t)$, $X(t+1)$, of proved above continuity of mappings $v \to E[x_j^v(t+1)]$, $v \to E[G_j(v, x_j^v(t))]$ and determining the set V_{ad}^j as complete preimageof pointed sets at continuous mappings. The theorem statement follows from the Weierstrass theorem about reaching continuous on compact function of its upper boundary.

2.2 Sufficient Conditions for Continuous Dependence of Optimal Criteria Values of the Parametric Control Problems on Uncontrollable Functions

In this section, within the framework of developing the 4th component of the parametric control theory, there are derived sufficient conditions of continuous dependence on uncontrollable functions $a(\cdot)$ of optimal criteria values for all of considered above parametric control problems for non-autonomous deterministic dynamical systems and sufficient conditions of continuous dependence on uncontrollable parameters of optimal criteria values for considered above parametric control problems for autonomous stochastic dynamical systems. All of the results defined for non-autonomous systems remain true for autonomous dynamical systems as well, where exogenous vector function $a(\cdot)$ is taken as constant.

The following theorem determines sufficient conditions of continuous dependence of optimal criterion values for the Problem 2.4.

Theorem 2.15. Assume that when conditions of the Theorem 2.10 are met, in neighborhood (in Euclidean topology) of function $a(\cdot)$ function f is continuous by the third argument and satisfies the Lipschitz condition by the first argument on uniformly by the second and third arguments.Then optimal criterion value for the Problem 2.4 continuously depends on uncontrollable function at the point $a(\cdot)$.

Proof. Let the following convergence occurs

$$a_k \to aBR^{sn} \tag{67}$$

According to the Theorem 2.10 the Problem 2.4 given the values of uncontrollable functions and ak has a solutions, which we denote, correspondingly, by u and u_k. Denote by $x[b, w]$ solution of condition equation (43), (44), corresponding to uncontrollable function b and control w. Ipso facto the function $x[b, w]$ satisfies relations

$$x\left[b,w\right](t+1) = f\left(x\left[b,w\right](t), w(t), b(t)\right), t = 0, 1, ..., n-1 \tag{68}$$

$$x\left[b,w\right](0) = x_0 \tag{69}$$

Then the following inequalities are true

$$0 \geq K\left(x\left[a,u_k\right]\right) - K\left(x\left[a,u\right]\right), K\left(x\left[a_k,u_k\right]\right) - K\left(x\left[a_k,u\right]\right) \geq 0$$

where by $k(Y)$ is denoted the value of criterion K at given value of system state . Consequently, we get the following relations:

$$\begin{aligned} &0 \leq K\left(x\left[a,u\right]\right) - K\left(x\left[a,u_k\right]\right) \leq K\left\{\left(x\left[a,u\right]\right) - K\left(x\left[a_k,u\right]\right)\right\} + \\ &\left\{\left(x\left[a_k,u\right]\right) - \left(x\left[_k,u_k\right]\right)\right\} + \left\{K(x\left[a_k,u_k\right]) - K\left(x\left[a,u_k\right]\right)\right\} \\ &\leq 2\sup_{w\in U}\left|K(x\left[a,w\right]) - K\left(x\left[a_k,w\right]\right)\right| \end{aligned} \tag{70}$$

From the conditions (68) we derive inequalities

$$\begin{aligned} &\left|x\left[a_k,w\right](t+1) - x\left[a,w\right](t+1)\right| \\ &= \left|f\left(x\left[a_k,w\right](t), w(t), a_k(t)\right) - f\left(x\left[a,w\right](t), w(t), (t)\right)\right| \\ &\leq \left|f\left(x\left[a_k,w\right](t), w(t), a_k(t)\right) - f\left(x\left[a_k,w\right](t), w(t), (t)\right)\right| \\ &+ \left|f\left(x\left[a_k,w\right](t), w(t), a(t)\right) - f\left(x\left[a,w\right](t), w(t), a(t)\right)\right| \\ &\leq \sup_{\in X, \varphi\in U}\left|f\left(y,\varphi,a_k(t)\right) - f\left(y,\varphi,a(t)\right)\right| + L\left|x\left[a_k,w\right](t) - x\left[a,w\right](t)\right|, \\ &t = 0, 1, ..., n-1, \end{aligned}$$

where L is the Lipschitz constant of function f by the first argument, which does not depend on w. Consequently, we get the following relation

$$\left|\psi_k(t+1)\right| \leq \eta_k + L\left|\psi_k(t)\right|, t = 0, 1, ..., n-1$$

where

$$\psi_k(t) = x\,[a_k, w]\,(t) - x\,[a, w]\,(t), t = 0, 1, ..., n-1,$$
$$\eta_k = \max_{t=0,1,\dots,n-1} \sup_{\in X, \varphi \in U} |f\,(y, \varphi, a_k(t)) - f\,(y, \varphi, a(t))|$$

Then we determine relations

$$|\psi_k(t+1)| \le \eta_k + L|\psi_k(t)| \le \eta_k + L(\eta_k + L|\psi_k(t-1)|)$$
$$= (1+L)\eta_k + L^2|\psi_k(t-1)| \le (1+L)\eta_k + L^2(\eta_k + L|\psi_k(t-2)|)$$
$$= (1+L+L^2)\eta_k + L^3|\psi_k(t-2)| \le ... \le \sum_{r=0}^{t} L^r \eta_k + L^{t+1}|\psi_k(0)|, t = 0, 1, ..., n-1$$

From initial condition (69) follows that $|\psi_k(0)| = 0$. Then the following estimate is true

$$|\psi_k(t+1)| \le \eta_k \sum_{r=0}^{t} L^r, t = 0, 1, ..., n-1$$

from which follows that

$$|x[a_k, w](t+1) - x[a, w](t+1)| \le \eta_k \sum_{r=0}^{t} L^r, t = 0, 1, ..., n-1$$

Note that right-hand-side of this inequality does not depend on w.

From the condition (67) follows the convergence $\eta_k \to 0$ under continuity of function f by the third argument, as well as closure and boundedness of the sets X and U. Then we derive from the last inequality that

$$x\,[a_k, w]\,(t) \to x\,[a, w]\,(t) \ uniformly\ by\ w, t = 1, ..., n-1.$$

Hence under continuity of function F follows that

$$F\,[t, x\,[a_k, w]\,(t)] \to F\,[t, x\,[a, w]\,(t)] \ \ uniformly\ by\ w, t = 1, ..., n-1.a$$

therefore,

$$K(x[a_k, w]) \to K(x[a, w]) \ uniformly\ by\ w, t = 1, ..., n-1.a$$

whence the convergence

$$\sup_{w \in U} |K(x\,[a, w]) - K\,(x\,[a_k, w])| \to 0 \tag{71}$$

follows.

Taking the limits in inequality (70) and taking into account the conditions (58), we will have

$$K\,(x\,[a, u_k]) \to K\,(x\,[a, u]) \tag{72}$$

The following estimate is true.

$$|K(x[a_k,u_k]) - (Kx[a,u])| \leq |K(x[a_k,u_k]) - (Kx[a,u_k])| + |(Kx[a,u_k]) - (Kx[a,u])|$$

Taking the limits and taking into account the conditions (71), (72), we will determine the convergence

$$K(x[a_k,u_k]) \rightarrow K(x[a,u])$$

Hence under defining controls u and u_k results that the maximal value of criterion K, corresponding to uncontrollable function a_k, converges to its maximum, corresponding to the extreme value of uncontrollable function . The theorem is proved.

Determine now sufficient conditions of continuous dependence on uncontrollable functions of optimal criterion values for the variational calculus problem 2.5 on choice (among given finite algorithms set) of optimal parametric control laws based on discrete non-autonomous dynamical system.

Determine first corresponding result for the Problem 2.5*.

Theorem 2.16. Assume that when conditions of the Theorem 2.11 are met in neighborhood of point function f is continuous by the third argument and satisfies the Lipschitz condition by the first two arguments on $X \times U$ uniformly by the third argument, and function G_j satisfies the Lipschitz condition by the second argument on uniformly by the first argument.Then optimal criterion value for the Problem 2.5* continuously depends on uncontrollable function at point .

Proof. Let the convergence occurs

$$a_k \rightarrow aBR^{sn} \tag{73}$$

According to the Theorem 2.11 the Problem 2.5* at values of uncontrollable functions and ak has a solution, which we denote, correspondingly, by v and v_k. Denote by $x[b,w]$ solution of state equations (53), (44), corresponding to uncontrollable function $b(\cdot)$ and controllable parameter w. Ipso facto function $x[b,w]$ meets relations

$$x[b,w](t+1) = f\left(x[b,w](t), G_j(v, x[b,w](t)), b(t)\right), t = 0,1,...,n-1 \tag{74}$$

$$x[b,w](0) = x_0 \tag{75}$$

Analogously with inequality (70) the following relation is determined

$$0 \leq K(x[a,v]) - K(x[a,v_k]) \leq 2 \sup_{w \in V} |K(x[a,w]) - K(x[a_k,w])| \tag{76}$$

After denoting

$$a = x\left[a, w\right], a_k = x\left[a_k, w\right]$$

from equalities (74) we derive inequalities

$$\begin{aligned}
&\left|x_k(t+1) - x(t+1)\right| \\
&= \left|f\left(x_k(t), G_j\left(w, x_k(t)\right), a_k(t)\right) - f\left(x(t), G_j\left(w, x(t)\right), a(t)\right)\right| \\
&\leq \left|f\left(x_k(t), G_j\left(w, x_k(t)\right), a_k(t)\right) - f\left(x_k(t), G_j\left(w, x_k(t)\right), a(t)\right)\right| \\
&+ \left|f\left(x_k(t), G_j\left(w, x_k(t)\right), a(t)\right) - f\left(x(t), G_j\left(w, x(t)\right), a(t)\right)\right| \\
&\leq \sup_{y \in X, \varphi \in U} \left|f\left(y, \varphi, a_k(t)\right) - f\left(y, \varphi, a(t)\right)\right| \\
&+ L\left[\left|x_k(t) - x(t)\right| + \left|G_j\left(w, x_k(t)\right) - G_j\left(w, x(t)\right)\right|\right] \\
&\leq \sup_{y \in X, \varphi \in U} \left|f\left(y, \varphi, a_k(t)\right) - f\left(y, \varphi, a(t)\right)\right| \\
&+ L(1+M)\left|x_k(t) - x(t)\right|, t = 0, 1, ..., n-1,
\end{aligned}$$

where L is the Lipschitz constant of function f by the first two arguments, and is the Lipschitz constant of function G_j by the second argument. Consequently, we get inequality

$$\left|\psi_k(t+1)\right| \leq \eta_k + N\left|\psi_k(t)\right|, t = 0, 1, ..., n-1$$

where

$$N = L(1+M)$$

$$\psi_k(t) = x_k(t) - x(t), t = 0, 1, ..., n-1,$$

$$\eta_k = \max_{t=0,1,...,n-1} \sup_{y \in X, \varphi \in U} \left|f\left(y, \varphi, a_k(t)\right) - f\left(y, \varphi, a(t)\right)\right|$$

Using technique, described above in proof of the Theorem 2.15, we determine the estimate

$$\left|\psi_k(t+1)\right| \leq \eta_k \sum_{r=0}^{n} N^r, t = 0, 1, ..., n-1$$

and the convergence $\eta_k \to 0$. Thereby, $\left|\psi_k(t)\right| \to 0, t = 1, ..., n$ therefore,

$$x\left[a_k, w\right](t) \to x\left[a, w\right](t) \ uniformly\ by\ w, t = 1, ..., n-1.$$

Hence under continuity of function F follows that

$$F\left[t, x\left[a_k, w\right](t)\right] \to F\left[t, x\left[a, w\right](t)\right] \ uniformly\ by\ w, t = 1, ..., n-1.$$

therefore,

$$K\left(x\left[a_k, w\right]\right) \to K\left(x\left[a, w\right]\right) \ uniformly\ by\ w.$$

Then under finite dimensionality of the space of controls and boundedness of the set U we get

$$\sup_{w\in V} |K(x[a,w]) - K(x[a_k,w])| \to 0 \tag{77}$$

Taking the limits in inequality (76) and taking into account the condition (77), we will have

$$K(x[a,v_k]) \to K(x[a,v]) \tag{78}$$

The following estimate is true

$$|K(x[a_k,v_k]) - K(x[a,v])| \leq |K(x[a_k,v_k]) - K(x[a,v_k])| + |K(x[a,v_k]) - K(x[a,v])|$$

Taking the limits here and taking into account the conditions (77), (78), we determine the convergence

$$K(x[a_k,v_k]) \to K(x[a,v])$$

Hence under definition of control parameters v and v_k follows that the maximal value of criterion K, corresponding to uncontrollable function a_k, converges to its maximum, corresponding to extreme value of uncontrollable function. The theorem is proved.

As the maximum of two (therefore, any finite number as well) of continuous functions is continuous, then from the Theorem 2.16 follows similar result for the Problem 2.5.

Theorem 2.17. Assume that when conditions of the Theorem 2.11 are met in neighborhood of point , function f is continuous by the third argument and satisfies the Lipschitz condition by the first two arguments on $X \times U$ uniformly by the third argument, and function G_j satisfies the Lipschitz condition by the second argument on uniformly by the first argument.Then optimal criterion value for the Problem 2.5 continuously depends on uncontrollable function at point a.

The purpose of the next study is to determine sufficient conditions of continuous dependence on uncontrollable functions of optimal criterion values for the Problem 2.1 based on continuous non-autonomous dynamical system.

Theorem 2.18. Assume that when conditions of the Theorem 2.6 are met in neighborhoods of point , function f is continuous by the second argument and satisfies the Lipschitz condition by the first and third arguments on $X \times A$ uniformly by the second argument.Then optimal criterion value for the Problem 2.1 continuously depends on uncontrollable function at point .

Proof. Let the convergence occurs

$$a_k \to a \ (C[0,T])^s \tag{79}$$

According the Theorem 2.6 the Problem 2.1at values of uncontrollable functions and ak has solutions, which we denote, correspondingly, by u and u_k. Denote by $x[b,w]$ solution of state equations (1), (2), corresponding to uncontrollable function b and control w. Ipso facto function $x[b,w]$ meets relations

$$\dot{x}\left[b,w\right](t) = f\left(x[b,w]\,, w(t), b(t)\right), t \in (0,T) \tag{80}$$

$$x\left[b,w\right](0) = x_0. \tag{81}$$

Then, by repeating reasoning from proof of the Theorem 2.13, analogously with relation (70) we determine inequality

$$0 \leq K\left(x\left[a,u\right]\right) - K\left(x\left[a,u_k\right]\right) \leq 2\sup_{w \in U}\left|K(x\left[a,w\right]) - K\left(x\left[a_k,w\right]\right)\right| \tag{82}$$

After denoting

$$x = x\left[a,w\right], x_k = x\left[a_k,w\right]$$

from the conditions (80) we derive inequalities

$$\dot{x}_k(t) - \dot{x}(t) = f\left(x_k(t), w(t), a_k(t)\right) - f\left(x(t), w(t), a(t)\right), t \in (0,T)$$

By integrating by t and taking into account equalities (81), we get

$$\begin{aligned} |x_k(t) - x(t)| &\leq \textstyle\int_0^t \left|f\left(x_k(\tau), w(\tau), a_k(\tau)\right) - f\left(x(\tau), w(\tau), (\tau)\right)\right| d\tau \\ &\leq L \textstyle\int_0^t |x_k(\tau) - x(\tau)|\, d\tau + L \int_0^t |a_k(\tau) - (\tau)|\, d\tau \\ &\leq L \textstyle\int_0^t |x_k(\tau) - x(\tau)|\, d\tau + LT\|a_k - \|_\Theta, t \in (0,T) \end{aligned}$$

where L is the Lipschitz constant of function f, not depending on w. Using the Gronwall lemma, we will have

$$|x_k(t) - x(t)| \leq \|a_k - a\|_\Theta, t \in (0,T)$$

where positive constant depends only on L and on .

Using the condition (79), we get that

$$x_k(t) \to x(t), t \in (0,T)$$

and therefore,

$$x\left[a_k,w\right](t) \to x\left[a,w\right](t)\ uniformly\ by\ w, t \in (0,T)$$

Hence under continuity of function F follows the convergence

$$K\left(x\left[a_k,w\right]\right) \to K\left(x\left[a,w\right]\right)\ \ uniformly\ by\ w$$

Consequently, we get

$$\sup_{w\in U} |K(x[a,w]) - K(x[a_k,w])| \to 0 \tag{83}$$

Taking the limits in inequality (82) and taking into account the conditions (83), we will have

$$K(x[a,u_k]) \to K(x[a,u]) \tag{84}$$

The following estimate is true

$$\begin{gathered} |K(x[a_k,u_k]) - K(x[a,u])| \\ \le |K(x[a_k,u_k]) - K(x[a,u_k])| + |K(x[a,u_k]) - K(x[a,u])| \end{gathered}$$

Taking the limits here and taking into account the conditions (83), (84), we will get the convergence

$$K(x[a_k,u_k]) \to K(x[a,u])$$

Hence under definition of controls u and u_k it follows that the maximal value of criterion K, corresponding to uncontrollable function a_k, converges to its maximum, corresponding to extreme value of uncontrollable function. The theorem is proved.

Determine sufficient conditions of continuous dependence on uncontrollable functions of optimal criterion values for the variational calculus problem on choice (among given finite algorithms set) of optimal parametric control laws based on continuous non-autonomous dynamical system.

Theorem 2.19. Assume that when conditions of the Theorem 2.8 are met in neighborhood of point , function f satisfies the Lipschitz condition on the set $X \times U \times A$, and function satisfies the Lipschitz condition by the second argument on uniformly by the first argument.Then optimal criterion value for the Problem 2.3* continuously depends on uncontrollable function at point .

Proof. Let the convergence occurs

$$a_k \to aB(C[0,T])^s \tag{85}$$

According the Theorem 2.8 the Problem 2.3* at values of uncontrollable functions and a_k has a solutions, which we denote, correspondingly, by v and v_k. Denote by $x[b,w]$ solution of state equations (23), (2), corresponding to uncontrollable function b and control w. Ipso facto function $x[b,w]$ meets relations

$$\dot{x}[b,w](t) = f\left(x[b,w](t), G_j(w,x(t)), b(t)\right), t \in (0,T) \tag{86}$$

$$x[b,w](0) = x_0 \tag{87}$$

Then, by repeating reasoning from proof of the Theorem 2.15, analogously with relation (70) we determine inequality

$$0 \le K\left(x\left[a,u\right]\right) - K\left(x\left[a,u_k\right]\right) \le 2\sup_{w\in V}\left|K(x\left[a,w\right]) - K\left(x\left[a_k,w\right]\right)\right| \tag{88}$$

After denoting

$$x = x\left[a,w\right], x_k = x\left[a_k,w\right]$$

from the conditions (86) we derive inequalities

$$\dot{x}_k(t) - \dot{x}(t) = f\left(x_k(t), G_j\left(w, x_k(t)\right), a_k(t)\right) - f\left(x(t), G_j\left(w, x(t)\right), a(t)\right), t \in (0,T)$$

By integrating by t and taking into account equalities (87), we get

$$\begin{gathered}\left|x_k(t) - x(t)\right| \le \int_0^t \left|f\left(x_k(\tau), G_j\left(w, x_k(\tau)\right), a_k(\tau)\right) - f\left(x(\tau), G_j\left(w, x(\tau)\right), a(\tau)\right)\right| \\ d\tau \le L\int_0^t \left[\left|x_k(\tau) - x(\tau)\right| + \left|G_j\left(w, x_k(\tau)\right) - G_j\left(w, x(\tau)\right)\right| + \left|a_k(\tau) - a(\tau)\right|\right] d\tau \\ \le L(1+)\int_0^t \left|x_k(\tau) - x(\tau)\right| d\tau + LT\left\|a_k - a\right\|_{\Theta}, t \in (0,T),\end{gathered}$$

where L is the Lipschitz constant of function f, and is the Lipschitz constant of function by the second argument. Using the Gronwall lemma, we will have

$$\left|x_k(t) - x(t)\right| \le \left\|a_k - a\right\|_{\Theta}, t \in (0,T)$$

where positive constant depends only on L and on .

Using the condition (85), we get that

$$x\left[a_k,w\right](t) \to x\left[a,w\right](t)\ uniformly\ by\ w, t \in (0,T)$$

Hence under continuity of function F follows that

$$F\left[t, x\left[a_k,w\right](t)\right] \to F\left[t, x\left[a,w\right](t)\right]\ uniformly\ by\ w$$

therefore,

$$K\left(x\left[a_k,w\right]\right) \to K\left(x\left[a,w\right]\right)\ uniformly\ by\ w$$

Consequently, we get

$$\sup_{w\in V}\left|K(x\left[a,w\right]) - K\left(x\left[a_k,w\right]\right)\right| \to 0 \tag{89}$$

Taking the limits in inequality (88) and taking into account the condition (89), we will have

$$K\left(x\left[a,u_k\right]\right) \to K\left(x\left[a,u\right]\right) \tag{90}$$

The following estimate is true

$$
\begin{gathered}
\left|K(x\left[a_k,u_k\right]) - K\left(x\left[a,u\right]\right)\right| \\
\le \left|K(x\left[a_k,u_k\right]) - K\left(x\left[a,u_k\right]\right)\right| + \left|K(x\left[a,u_k\right]) - K\left(x\left[a,u\right]\right)\right|
\end{gathered}
$$

Taking the limits here and taking into account the conditions (89), (90) we will get the convergence

$$
K\left(x\left[a_k,u_k\right]\right) \to K\left(x\left[a,u\right]\right)
$$

Hence under definition of controls u and u_k follows that the maximal value of criterion K, corresponding to uncontrollable function a_k, converges to its maximum, corresponding to extreme value . The theorem is proved.

Using the Theorem 2.19 analogously with the Theorem 2.17 (in this case system continuity is not a matter of principle), we come to the following statement.

Theorem 2.20. When for all $j = 1, ..., r$ conditions of the Theorem 2.19 are met, optimal criterion value for the Problem 2.5 continuously depends on uncontrollable function at point .

Present statements of the theorems about continuous dependence on uncontrollable parameter of optimal criterion values for the parametric control problems of stochastic dynamical systems. Proves of these theorems are similar to those presented above.

Theorem 2.21. Let given any $a \in A$ conditions of the Theorem 2.13 are met. Then optimal criterion value for the Problem 2.6 are continuous function of the parameter .

Now, study the conditions of continuous dependence of optimal criterion value for the variational calculus problems on choice of parametric control laws on uncontrollable parameters.

Theorem 2.22. Let given any $a \in A$ conditions of the Theorem 2.14 are met. Then for chosen number value of the law j K_j optimal criterion values for the Problem 2.7 are continuous function of the parameter .

Consequence 2.23. When conditions of the Theorem 2.22 are met for all $j = 1, ..., r$, optimal criterion value $K = max_{j=1,...,r} K_j$ for the Problem 2.7 are continuous functions of the parameter $a \in A$.

Obtained results will be used in the next section in proving the existence of corresponding bifurcation points of extremals of the variational problems.

2.3 Sufficient Conditions for the Existence of Extremals' Bifurcation Points of the Problems on Choice of Optimal Parametric Control Laws

Let us introduce a notion of extremals' bifurcation point of the variational calculus problems on choice (among given finite algorithms set) of optimal parametric control laws.Existence of such a bifurcation point for some uncontrollable function $a(\cdot)$ means that in its neighborhood for the problem in question occurs transfer from one optimal parametric control law to another.

Consider abstract variational calculus problem on choice (among given finite algorithms set) of optimal parametric control laws, generalizing the Problems 2.3, 2.5, and 2.7.

Given: the set of uncontrollable functions (or parameters), set of the acceptable controls sets (adjustable coefficients values) V_a^j, $a \in A, j = 1, ..., r$ and set of functionals (optimality criteria) $K_j = K_j(a, v)$ where $v \in V_a^j$, $a \in A$, $j = 1, ..., r$. Give definition for extremals' bifurcation point for set of the maximization problems for given functionals on corresponding sets of acceptable controls, i.e. abstract variational calculus problem on choice (among given finite algorithms set) of optimal parametric control laws.

Definition. The value $a \in A$ call asextremals' bifurcation point for the maximization problem for mappings $u \to K_k(a, v)$ on the sets V_a^k, $k = 1, ..., r$ if there are exist two different numbers $i, j \in 1, ..., r$ such that the following relation is true

$$\max_{v \in V_a^l} K_i(a, v) = \max_{v \in V_a^j} K_i(a, v) = \max_{k=1,...,r} \max_{v \in V_a^k} K_i(a, v)$$

and in any neighborhood of point there is such a point $b \in A$ for which the value

$$\max_{k=1,...,r} \max_{v \in V_b^j} K_j(b, v)$$

reachs for the only value of k.

Theorem 2.24. Assume that when conditions of the Theorem 2.17(or 2.20, or 2.23) are met, is a connected set, and there are two different values $a_0, a_1 \in A$, such that the maximal values by $j = 1, ..., r$ of function maxima $v \to K_j(a, v)$ on the sets are reached for the different values j_0, j_1 that is the following inequalities are true:

$$\max_{j=1,...,r, j \neq j_0} \max_{v \in V_{a_0}^j} K_j(a_0, v) < \max_{v \in V_{a_0}^{j_0}} K_{j_0}(a_0, v)$$
$$\max_{j=1,...,r, j \neq j_1} \max_{v \in V_{a_1}^j} K_j(a_1, v) < \max_{v \in V_{a_1}^{j_1}} K_{j_1}(a_1, v)$$

Then there exists a bifurcation point of extremals of the Problem 2.3 (or 2.5, or 2.7) on choice (among given finite algorithms set) of optimal parametric control law.

Proof. Under connectivity of the set points a_0, a_1 can be connected by continuous line $a = a(s), s \in [0, 1]$, lying in the set , and the following equalities are true

$$a(0) = a_0, a(1) = a_1$$

Denote

$$K_j(s) = \max_{v \in V_{a(s)}^j} K_{j_1}(a(s), v), s \in [0, 1]$$

From the Theorems 2.17 (or 2.20, or2.23) follows continuity of functions $s \to K_j(s)$ in the segment $[0, 1]$, and therefore, continuity in this segment of function as well, where

Determine the set

$$\Delta(j) = \{ s \in [0, 1] |\, K_j(s) = K^*(s) \}, j = 1, ..., r$$

It is closed, being complete preimageof closed set, consisting of the only point (zero) for continuous function $y = y(s)$ where $y(s) = K_j(s) - K * (s)$ Thereby, we present the segment $[0, 1]$ in terms of the following sum

$$[0, 1] = \bigcup_{j=1,...,r} \Delta(j)$$

consisting, according to the theorem conditions, as minimum, of two non-empty closed sets.

From theorem conditions it follows the following relations as well:

$$0 \in \Delta(j_0), 1 \notin \Delta(j_0)$$

Then the set of boundary points of the set $\Delta(j_0)$ which are situated in the interval $(0, 1)$, is not empty. Consequently, there exists lower boundary s_0 of such boundary points. This value is a boundary point of some another set $\Delta(j_2)$ and is a part of it as well. Thereby, at $a = a(s_0)$ the maximum by $j = 1, ..., r$ the values is reached, as minimum, for two different numbers j_0 and j_2. At the same time, at $0 \leq s \leq s_0$ this maximum is reached for the only value j_0. Thus, $a(s_0)$ actually corresponds to desired bifurcation point. The theorem is proved.

The following statement is direct consequence of the Theorem 2.24.

Consequence 2.25. Assume that when conditions of the Theorem 2.17 (or 2.20, or 2.23) are met, is connected set, and at value control using the law provides solution of the Problem 2.3 (or 2.5, or 2.7), and at , (control using this law does not provide solution of the problem in question, that is the following inequalities are true

$$\max_{j=1,...,r, j \neq j_0} \max_{v \in V^j_{a_0}} K_j(a_0, v) < \max_{v \in V^{j_0}_{a_0}} K_{j_0}(a_0, v)$$
$$\max_{j=1,...,r, j \neq j_1} \max_{v \in V^j_{a_1}} K_j(a_1, v) > \max_{v \in V^{j_0}_{a_1}} K_{j_0}(a_1, v)$$

Then there is at least one bifurcation point of extremals of mentioned problem 2.3 (or 2.5, or 2.7).

At the end we present description of the numerical algorithm for finding bifurcational value of function (or parameter) a of one of the problems 2.3 or 2.5 or

2.7 on choice (among given finite algorithms set) of parametric control laws and when conditions of the theorem 2.24 are met.

Connect the points and by smooth curve. Divide this curve to equal parts with sufficiently small step. For obtained values (points) are determined the numbers of parametric control laws- bringing solution of the problem 2.3 or 2.5 or 2.7 given values. Then we find the first value i, at which corresponding law number differs from. In this case bifurcational value lies on arc of the curve.

For found part of the curve, the algorithm for determining bifurcation point with given accuracy consists in using the method of bisections. Consequently we find the point, on the one hand from which on this arc within the range of deviation from the value the optimal law is, and on the other hand-within the range of deviation from the value this law is not optimal. From the Consequence 2.25 follows that extremal's bifurcation point for the problem under solution lies on mentioned arc, and as its estimate can be taken any point of the arE.

References

[1] Ekeland I., Temam R. (1976), "Convex analysis and variational problems", *Amsterdam: Nord-Holland publishing company.*

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