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A Cauchy Type Problem for Causal Functional Inclusions in Banach Spaces

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Abstract: In this paper, we study a Cauchy type problem in Banach spaces for various classes of functional inclusions with causal multioperators. Based on the topological degree theory for condensing multimaps, we prove a global theorem on the existence of trajectories for systems governed by functional inclusions. As an application, we obtain generalizations of existence theorems for a Cauchy type problem for semilinear second order differential inclusions of this type and semilinear differential inclusions of the fractional order $1 < q < 2$.

Keywords: causal multioperator, functional inclusion, Cauchy type problem, differential inclusion, fractional derivative, measure of non-compactness, fixed point, topological degree, condensing multioperator

1. INTRODUCTION

Recently, the attention of many researchers (see [2], [19], [20], [23] and references therein) has been attracted to generalizations of differential equations and inclusions, namely to the class of functional equations and inclusions with causal operators. The term of a causal or Volterra operator in the sense of A.N. Tikhonov (see [29]), was used in mathematical physics to solve problems of differential equations, integro-differential equations, functional-differential equations with finite or infinite delay, integral equations of Volterra type, functional equations of neutral type, etc. (see, for example, [6]). The papers [5], [7], [8], [13] among others are devoted to the study of equations and inclusions with causal multioperators of various types, theorems on the existence of solutions, description of qualitative properties of solutions and various applications.

In recent decades, the interest to the theory of fractional-order differential equations has significantly increased, thanks to applications in various branches of applied mathematics, physics, engineering, biology, economics, etc. (see, for example, monographs [17], [27] papers [1], [4], [9], [21], etc.). The boundary value problems of various types for fractional differential equations and inclusions were considered in the works [10], [15], [16], [24], [25].

In this paper, we study a Cauchy type problem in Banach spaces for various classes of functional inclusions with causal multioperators. Based on the topological degree theory for condensing multimaps, we prove a global theorem on the existence of trajectories for systems governed by functional inclusions. As an application, we obtain generalizations of existence theorems for solutions of a Cauchy type problem for second order semilinear differential inclusions and semilinear differential inclusions of the fractional order $1 < q < 2$. In this paper we use standard notation, the symbol "$\multimap$" denotes a multimap.

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2. PRELIMINARIES

2.1. Measures of Noncompactness

We denote by $\mathcal{E}$ a Banach space and introduce the following notation:

- $P(\mathcal{E}) = \{A \subseteq \mathcal{E} : A \neq \varnothing\}$ is the collection of all non-empty subsets of $\mathcal{E}$;
- $Pb(\mathcal{E}) = \{A \in P(\mathcal{E}) : A \text{ is bounded}\}$;
- $Pv(\mathcal{E}) = \{A \in P(\mathcal{E}) : A \text{ is convex}\}$;
- $C(\mathcal{E}) = \{A \in P(\mathcal{E}) : A \text{ is closed}\}$;
- $Cv(\mathcal{E}) = Pv(\mathcal{E}) \cap C(\mathcal{E})$;
- $K(\mathcal{E}) = \{A \in P(\mathcal{E}) : A \text{ is compact}\}$;
- $Kv(\mathcal{E}) = Pv(\mathcal{E}) \cap K(\mathcal{E})$.

Definition 2.1:
(See [3]). Let $(\mathcal{A}, \geq)$ be a partially ordered set. A function $\beta : Pb(\mathcal{E}) \to \mathcal{A}$ is called the measure of noncompactness (MNC) in $\mathcal{E}$ if for each $\Omega \in Pb(\mathcal{E})$ we have:

$$\beta(\overline{\mathrm{co}}\, \Omega) = \beta(\Omega),$$

where $\overline{\mathrm{co}}\, \Omega$ denotes the closure of the convex hull of Ω.

A measure of noncompactness β is called:

1) *monotone*, if for each $\Omega_0, \Omega_1 \in Pb(\mathcal{E})$, from $\Omega_0 \subseteq \Omega_1$ it follows $\beta(\Omega_0) \leq \beta(\Omega_1)$;
2) *nonsingular*, if for each $a \in \mathcal{E}$ and each $\Omega \in Pb(\mathcal{E})$ we have $\beta(\{a\} \cup \Omega) = \beta(\Omega)$.

If $\mathcal{A}$ is a cone in $\mathcal{E}$, the MNC β is called:

3) *regular*, if $\beta(\Omega) = 0$ is equivalent to the relative compactness of $\Omega \in Pb(\mathcal{E})$;
4) *real*, if $\mathcal{A}$ is the set of all real numbers $\mathbb{R}$ with the natural ordering.

As the example of a real MNC satisfying all above properties, we can consider the Hausdorff MNC $\chi(\Omega)$:

$$\chi(\Omega) = \inf\{\varepsilon > 0, \text{for which } \Omega \text{ has a finite } \varepsilon\text{-net in } \mathcal{E}\ \}.$$

As other examples, consider the measures of noncompactness defined in the space of continuous functions $C([a, b]; \mathcal{E})$ with values in the Banach space $\mathcal{E}$:

(1) *the modulus of fiber noncompactness:*

$$\varphi(\Omega) = \sup_{t \in [a,b]} \chi_{\mathcal{E}}(\Omega(t)),$$

where $\chi_{\mathcal{E}}$ is the Hausdorff MNC in $\mathcal{E}$ and $\Omega(t) = \{y(t) : y \in \Omega\}$;

(2) *the fading modulus of fiber noncompactness:*

$$\gamma(\Omega) = \sup_{t \in [a,b]} e^{-Lt} \chi_{\mathcal{E}}(\Omega(t)),$$

where $L > 0$ is a given number;

(3) *the modulus of equicontinuity:*

$$\mathrm{mod}_C\,(\Omega) = \lim_{\delta \to 0} \sup_{y \in \Omega} \max_{|t_1 - t_2| \leq \delta} \|y\,(t_1) - y\,(t_2)\|\,.$$

These measures of noncompactness satisfy all the above properties, except for the regularity.

2.2. Multivalued Maps

Let X be a metric space and Y be a normed space. Let us recall some notations (see, for example, [14], [22]).

Definition 2.2:
A multivalued map (multimap) $\mathcal{F} : X \to P(Y)$ is said to be upper semicontinuous (u.s.c.) at a point $x \in X$, if for every open set $V \subset Y$ such that $\mathcal{F}(x) \subset V$, there exists a neighborhood $U(x)$ of x such that $\mathcal{F}(U(x)) \subset V$.

Definition 2.3:
A multivalued map $\mathcal{F} : X \to P(Y)$ is called closed if its graph $G_{\mathcal{F}} = \{(x, y) : x \in X, y \in \mathcal{F}(x)\}$ is a closed subset of $X \times Y$.

Definition 2.4:
A multivalued map $\mathcal{F} : X \to P(Y)$ is called quasicompact if its restriction to each compact subset $A \subset X$ is compact.

Definition 2.5:
For a given $p \geq 1$, a multifunction $G : [0, T] \to K(Y)$ is called:

- *L^p–integrable* if it admits an L^p–Bochner integrable selection, i.e., there exists a function $g \in L^p([0, T]; Y)$ such that $g(t) \in G(t)$ for a.e. $t \in [0, T]$;
- *L^p–integrably bounded* if there exists a function $\xi \in L^p([0, T])$ such that
$$\|G(t)\|_Y \leq \xi(t)$$
for a.e. $t \in [0, T]$.

Definition 2.6:
A multimap $\mathcal{F} : X \subseteq \mathcal{E} \to K(\mathcal{E})$ is called condensing with respect to a MNC β (or β–condensing) if for each bounded set $\Omega \subseteq X$ which is not relatively compact, we have:
$$\beta(F(\Omega)) \ngeq \beta(\Omega).$$

Let $\mathcal{D} \subset \mathcal{E}$ be a non-empty closed convex subset, V be a non-empty bounded open subset of $\mathcal{D}$, β is a monotone nonsingular MNC in $\mathcal{E}$ and $\mathcal{F} : \overline{V} \to Kv(\mathcal{D})$ be a u.s.c. β-condensing map such that $x \notin \mathcal{F}(x)$ for all $x \in \partial V$, where $\overline{V}$ and ∂V denote the closure and the boundary of the set V in the relative topology of $\mathcal{D}$.

In such a setting, the (relative) topological degree
$$\deg_{\mathcal{D}}\left(i - \mathcal{F}, \overline{V}\right)$$
of the corresponding vector field $i - \mathcal{F}$, satisfying the standard properties is defined (see, for example, [14], [22]). In particular, the condition
$$\deg_{\mathcal{D}}\left(i - \mathcal{F}, \overline{V}\right) \neq 0$$
implies that the fixed points set $Fix\mathcal{F} = \{x : x \in \mathcal{F}(x)\}$ is a nonempty subset of V.

Application of topological degree theory leads to the following fixed point principles, which will be used in the what follows.

Theorem 2.1:
([14], Corollary 3.3.1). Let $\mathcal{M}$ be a convex closed bounded subset of $\mathcal{E}$ and $\mathcal{F} : \mathcal{M} \to Kv(\mathcal{M})$ be a β–condensing multimap, where β is a monotone nonsingular MNC in $\mathcal{E}$. Then the fixed point set $\mathrm{Fix}\,\mathcal{F}$ is non-empty.

Theorem 2.2:
([14], Theorem 3.3.4). Let $V \subset \mathcal{D}$ be a bounded open neighborhood of a point $a \in V$ and $\mathcal{F} \colon \overline{V} \to Kv(\mathcal{D})$ be a u.s.c. β-condensing multimap, where β is a monotone nonsingular MNC in $\mathcal{E}$, satisfying the boundary condition

$$x - a \notin \lambda(\mathcal{F}(x) - a)$$

for all $x \in \partial V$ and $0 < \lambda \leq 1$. Then $Fix\mathcal{F} \neq \emptyset$ is a non-empty compact set.

2.3. Family of Cosine Operator Functions

Definition 2.7:
(See [12], [18], [28]) A family of bounded operators $\{C(t)\}_{t\in\mathbb{R}}$ in a Banach space $\mathcal{E}$ is called a strongly continuous family of cosine operator functions if:

(1) $C(0) = I$;
(2) $C(s+t) + C(s-t) = 2C(s)C(t)$ for all $t, s \in \mathbb{R}$;
(3) $t \to C(t)x$ is continuous for all $x \in \mathcal{E}$.

The family of strongly continuous sine operator functions associated with the family of cosine operator functions $\{C(t)\}_{t\in\mathbb{R}}$ is the family of operators $\{S(t)\}_{t\in\mathbb{R}}$ such that

$$S(t)x = \int_0^t C(s)x ds,\ x \in \mathcal{E}, t \in \mathbb{R}.$$

The operator A generates a family of cosine operator functions $\{C(t)\}_{t\in\mathbb{R}}$ if:

$$Ax = \frac{d^2}{dt^2}C(t)x\Big|_{t=0},$$

for all $x \in D(A)$ for which the last expression is defined.

2.4. Causal Multioperators

Let E be a separable Banach space. By $L^p([0,T];E)$, $1 \leq p \leq \infty$, we denote the Banach space of all Bochner summable functions $f : [0,T] \to E$ with the usual norm.

For each subset $\mathcal{N} \subset L^p([0,T];E)$ and $\tau \in (0,T)$ we define restriction $\mathcal{N}$ on $[0,\tau]$ as

$$\mathcal{N} \mid_{[0,\tau]} = \{f \mid_{[0,\tau]} : f \in \mathcal{N}\}.$$

Definition 2.8:
A multivalued map $\mathcal{Q} : C([0,T];E) \multimap L^p([0,T];E)$ is said to be a causal multioperator, if for each $\tau \in (0,T)$ and for every $u, v \in C([0,T];E)$ the condition $u \mid_{[0,\tau]} = v \mid_{[0,\tau]}$ implies that $\mathcal{Q}(u) \mid_{[0,\tau]} = \mathcal{Q}(v) \mid_{[0,\tau]}$.

Let us give some examples of causal multioperators.

Example 2.1:
We assume that the multimap $F : [0,T] \times E \to Kv(E)$ satisfies the following conditions:

$(F1)$ *for each $\phi \in E$ the multifunction $F(\cdot,\phi) : [0,T] \to Kv(E)$ admits a measurable selection;*
$(F2)$ *for a.e. $t \in [0,T]$ the multifunction $F(t,\cdot) : E \to Kv(E)$ is u.s.c.;*
$(F3)$ *there exists a function $\alpha \in L^p_+[0,T],\ 1 \leq p \leq \infty$, such that*

$$\|F(t,\phi)\|_E := \sup\{\|z\|_E : z \in F(t,\phi)\} \leq \alpha(t)(1 + \|\phi\|_E)$$

for a.e. $t \in [0,T]$ and $\phi \in E$.

From the above conditions $(F1)-(F3)$ *it follows that the multimap* $\mathcal{P}_F : C([0;T];E) \to P(L^p([0,T];E))$, *given in the following way*

$$\mathcal{P}_F(x) = \{f \in L^p([0,T];E) : f(t) \in F(t,x(t)) \text{ for a.e. } t \in [0,T]\}$$

is well defined (see, for example, [14], [22]). It is clear that the multioperator $\mathcal{P}_F$ *is causal.*

Example 2.2:
Let $F : [0,T] \times E \to Kv(E)$ *be a multimap satisfying conditions* $(F1)-(F3)$ *from Example 2.1. Suppose that* $\{K(t,s) : 0 \le s \le t \le T\}$ *is a continuous (with respect to the corresponding norm) family of bounded linear operators in* E *and* $m \in L^1([0,T];E)$ *is a given function. Consider the Volterra integral multioperator* $\mathcal{V} : C([0,T];E) \multimap L^1([0,T];E)$ *defined as*

$$\mathcal{V}(u)(t) = m(t) + \int_0^t K(t,s)F(s,u)ds,$$

i.e.,

$$\mathcal{V}(u) = \{y \in L^1([0,T];E) : y(t) = m(t) + \int_0^t K(t,s)f(s)ds : f \in \mathcal{P}_F(u)\}.$$

It is also clear that the multioperator $\mathcal{V}$ *is causal.*

3. CAUCHY TYPE PROBLEM FOR FUNCTIONAL INCLUSIONS WITH THE CAUSAL MULTIOPERATORS

We will assume that the causal operator $\mathcal{Q} : C([0,T];E) \to C(L^p([0,T];E))$ satisfies the following conditions:

$(\mathcal{Q}1)$ $\mathcal{Q}$ is weakly closed in the following sense: conditions $\{u_n\}_{n=1}^{\infty} \subset C([0,T];E)$, $\{f_n\}_{n=1}^{\infty} \subset L^p([0,T];E)$, $1 \le p \le \infty$, $f_n \in \mathcal{Q}(u_n)$, $n \ge 1$, $u_n \to u_0$, $f_n \overset{L^1}{\rightharpoonup} f_0$ imply $f_0 \in \mathcal{Q}(u_0)$;

$(\mathcal{Q}2)$ there exists a function $\alpha \in L^{\infty}_{+}([0,T])$ such that

$$\|\mathcal{Q}(u)(t)\|_E \le \alpha(t)(1 + \|u(t)\|_E), \quad \text{for a.e. } t \in [0,T],$$

for all $u \in C([0,T];E)$;

$(\mathcal{Q}3)$ there exists a function $\omega : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

- $(\omega1)$ for all $x \in \mathbb{R}_+ : \omega(\cdot,x) \in L^p_+([0,T])$, $1 \le p \le \infty$, ;
- $(\omega2)$ for a.e. $t \in [0,T]$ a function $\omega(t,\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, nondecreasing and quasihomogeneous in the sense that $\omega(t,\lambda x) \le \lambda\omega(t,x)$ for all $x \in \mathbb{R}_+$ and $\lambda \ge 0$;
- $(\omega3)$ for each bounded set $\Delta \subset C([0,T];E)$ we have

$$\chi(\mathcal{Q}(\Delta)(t)) \le \omega(t, \varphi(\Delta(s))) \text{ for a.e. } t \in [0,T],$$

where the set $\Delta(s) = \{y(s) : y \in \Delta\} \subset E$ and φ is the modulus of fiber noncompactness in $C([0,T];E)$.

Note that the condition $(\omega2)$ means that $\omega(t,0) = 0$ for a.e. $t \in [0,T]$ and as an example of such a function we can consider $\omega(t,x) = k(t) \cdot x$, where $k(\cdot) \in L^p_+([0,T])$.

Consider a linear operator $\mathcal{S} : L^p([0,T];E) \to C([0,T];E)$, which is causal in the sense that for every $\tau \in (0,T]$ and $f,g \in L^p([0,T];E)$ condition $f(t) = g(t)$ for a.e. $t \in [0,\tau]$ implies $(\mathcal{S}f)(t) = (\mathcal{S}g)(t)$ for all $t \in [0,\tau]$. Following [14], we impose the next conditions on operator $\mathcal{S}$:

$(\mathcal{S}1)$ for $1 \leq p < \infty$ there exists $D \geq 0$ such that

$$\|\mathcal{S}f(t) - \mathcal{S}g(t)\|_E^p \leq D \int_0^t \|f(s) - g(s)\|_E^p ds$$

for all $f, g \in L^p([0,T]; E)$ and $0 \leq t \leq T$;
if $p = \infty$ then there exists $D_1 \geq 0$ such that

$$\|\mathcal{S}f(t) - \mathcal{S}g(t)\|_E \leq D_1 \int_0^t \|f(s) - g(s)\|_E ds$$

for all $f, g \in L^\infty([0,T]; E)$ and $0 \leq t \leq T$.

$(\mathcal{S}2)$ for an arbitrary compact set $K \subset E$ and a sequence $\{f_n\}_{n=1}^\infty \subset L^p\left([0,T]; E\right)$, $1 \leq p \leq \infty$, such that $\{f_n(t)\}_{n=1}^\infty \subset K$ for all $t \in [0,T]$ the weak convergence $f_n \stackrel{L^1}{\rightharpoonup} f_0$ implies $\mathcal{S}f_n \to \mathcal{S}f_0$ in $C([0,T]; E)$.

Also we suppose that $\mathcal{S}$ satisfies the relation:

$(\mathcal{S}3)$ $\left(\mathcal{S}f\right)(0) = 0$ for each function $f \in L^p([0,T]; E)$.

Notice, that condition $(\mathcal{S}1)$ implies that the operator $\mathcal{S}$ satisfies the Lipschitz condition:

$(\mathcal{S}1')$ $\|\mathcal{S}f - \mathcal{S}g\|_C \leq D\|f - g\|_{L^1}$.

Consider the following important examples.

(i) Let a closed linear operator $A : D\left(A\right) \subset E \to E$ generates a family of strongly continuous cosine operator functions $\{\cos(At)\}_{t \geq 0}$. The operator $\mathcal{L} : L^1([0,T]; E) \to C([0,T]; E)$ defined as

$$\mathcal{L}f(t) = \int_0^t \sin(At)\; f(s)ds$$

is a special case of the operator $\mathcal{S}$.
Note that taking $A = 0$ we obtain, as a special case, the usual integral operator $\mathcal{L}_I : L^1([0,T]; E) \to C([0,T]; E)$,

$$\mathcal{L}_I f(t) = \int_0^t f(s)ds.$$

It is easy to verify the following statement.

Lemma 3.1:
The operator $\mathcal{L}$ satisfies conditions $(\mathcal{S}1) - (\mathcal{S}3)$.

(ii) Consider the operator functions

$$\mathcal{G}(t) = \int_0^\infty \xi_q(\theta) C(t^q\theta) d\theta, \;\; \mathcal{K}(t) = \int_0^t \mathcal{G}(s)ds, \;\; \mathcal{T}(t) = q\int_0^\infty \theta\xi_q(\theta) S(t^q\theta) d\theta,$$

where

$$\xi_q(\theta) = \frac{\theta^{-1-\frac{1}{q}}}{q}\Psi_q(\theta^{-1/q}), \;\; \Psi_q(\theta) = \sum_{n=1}^\infty (-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{\pi n!}\sin(n\pi q), \;\; \theta \in \mathbb{R}^+.$$

Lemma 3.2:
(see [30]). The operator functions $\mathcal{G}, \mathcal{K}$ and $\mathcal{T}$ possess the following properties:

1) *For each* $t \in [0,T]$, $\mathcal{G}(t), \mathcal{K}(t)$ *and* $\mathcal{T}(t)$ *are linear bounded operators, more precisely, for each* $x \in E$ *we have*

$$\|\mathcal{G}(t)x\|_E \le M\|x\|_E, \quad \|\mathcal{K}(t)x\|_E \le M\|x\|_E t, \quad \|\mathcal{T}(t)x\|_E \le \frac{M}{\Gamma(2q)}\|x\|_E t^q,$$

where $M = \sup\{\|C(t)\|; t \in [0,T]\}$.

2) *the operator functions* $\mathcal{G}(\cdot), \mathcal{K}(\cdot)$ *and* $t^{q-1}\mathcal{T}(\cdot)$ *are strongly continuous, i.e., functions* $t \in [0,T] \to \mathcal{G}(t)x$, $t \in [0,T] \to \mathcal{K}(t)x$ *and* $t \in [0,T] \to t^{q-1}\mathcal{T}(t)x$ *are continuous for each* $x \in E$.

By using Lemma 3.2 it is easy to see that the operator functions $\mathcal{G}, \mathcal{K}$ and $\mathcal{T}$ are also satisfy the conditions $(\mathcal{S}1) - (\mathcal{S}3)$.

(iii) Let $A : D(A) \subset E \to E$ be a closed linear operator E generating a family of strongly continuous cosine operator functions $\{C(t)\}_{t\ge 0}$. The operator $G : L^p([0,T];E) \to C([0,T];E)$, $p \ge 1$, defined as

$$Gf(t) = \int_0^t (t-s)^{q-1}\mathcal{T}(t-s)f(s)ds, \quad 1 < q \le 2,$$

is a special case of the operator $\mathcal{S}$.

Lemma 3.3:
The operator G *satisfies conditions* $(\mathcal{S}1) - (\mathcal{S}3)$.

Proof
$(\mathcal{S}1)$ Let $1 \le p < \infty$. By using the Holder inequality, we get:

$$\|G(f)(t) - G(g)(t)\|_E \le \int_0^t (t-s)^{q-1}\|\mathcal{T}(t-s)(f(s)-g(s))\|_E ds \le$$

$$\le \frac{M}{\Gamma(2q)}\left[\int_0^t (t-s)^{\frac{(2q-1)p}{p-1}} ds\right]^{\frac{p-1}{p}} \left[\int_0^t \|f(s)-g(s)\|_E^p ds\right]^{\frac{1}{p}}.$$

Then

$$\|G(f)(t) - G(g)(t)\|_E^p \le D\int_0^t \|f(s)-g(s)\|_E^p ds,$$

where

$$D = \left[\frac{p-1}{2qp-1}\right]^{p-1} \frac{M^p T^{2qp-1}}{\Gamma^p(2q)}.$$

For $p = \infty$, we have:

$$\|G(f)(t) - G(g)(t)\|_E \le \int_0^t (t-s)^{q-1}\|\mathcal{T}(t-s)(f(s)-g(s))\|_E ds \le$$

$$\le \frac{MT^{q-1}}{\Gamma(2q)}\int_0^t \|f(s)-g(s)\|_E ds.$$

Then

$$\|G(f)(t) - G(g)(t)\|_E \le D_1\int_0^t \|f(s)-g(s)\|_E ds,$$

where

$$D_1 = \frac{MT^{q-1}}{\Gamma(2q)}.$$

$(\mathcal{S}2)$ Let $K \subset E$ be a compact set and a sequence $\{f_n\}_{n=1}^{\infty} \subset L^p([0,T];E)$, $1 \leq p \leq \infty$, such that $\{f_n(t)\}_{n=1}^{\infty} \subset K$ for all $t \in [0,T]$ and $f_n \overset{L^1}{\rightharpoonup} f_0$. Applying Lemma 3.2, we obtain:

$$\chi\bigg(\Big\{G(f_n)(t)\Big\}\bigg) \leq \int_0^t (t-s)^{q-1} \chi_E(\{\mathcal{T}(t-s) f_n\}) ds = 0.$$

This means that the sequence $\{G(f_n)(t)\}_{n=1}^{\infty} \subset E$ is relatively compact for each $t \in [0,T]$.

From the other side, if we take $\epsilon > 0$ and $t_1, t_2 \in [0,T]$ such that $0 < t_1 < t_2 \leq T$, then for each f_n, we have:

$$\left\| G(f_n)(t_2) - G(f_n)(t_1) \right\|_E =$$

$$\left\| \int_0^{t_2} (t_2-s)^{q-1} \mathcal{T}(t_2-s) f_n(s) ds - \int_0^{t_1} (t_1-s)^{q-1} \mathcal{T}(t_1-s) f_n(s) ds \right\|_E \leq$$

$$\left\| \int_{t_1}^{t_2} (t_2-s)^{q-1} \mathcal{T}(t_2-s) f_n(s) ds \right\|_E +$$

$$+ \left\| \int_0^{t_1} \left((t_2-s)^{q-1} \mathcal{T}(t_2-s) - (t_1-s)^{q-1} \mathcal{T}(t_1-s) \right) f_n(s) ds \right\|_E = Z_1 + Z_2,$$

where

$$Z_1 = \left\| \int_{t_1}^{t_2} (t_2-s)^{q-1} \mathcal{T}(t_2-s) f_n(s) ds \right\|_E,$$

$$Z_2 = \left\| \int_0^{t_1} \left((t_2-s)^{q-1} \mathcal{T}(t_2-s) - (t_1-s)^{q-1} \mathcal{T}(t_1-s) \right) f_n(s) ds \right\|_E.$$

Since $\{f_n(t)\}_{n=1}^{\infty} \subset K$ then there exists a constant $N > 0$, such that $\|f_n(t)\|_E \leq N$.

By using Lemma 3.2 and condition $(F2)$, for each $\epsilon > 0$ we can choose $\delta_1 > 0$ such that $|t_2 - t_1| < \delta_1$, implies the following estimate:

$$Z_1 \leq \frac{MN(t_2-t_1)^{2q}}{\Gamma(2q+1)} < \frac{\epsilon}{2}.$$

Taking into account that the family of operators $t^{q-1}\mathcal{T}(t)$ is strongly continuous, for each $\epsilon > 0$ we can choose $\gamma = \gamma(\epsilon) > 0$ and $\delta_2 > 0$ such that the inequality $|t_2 - t_1| < \delta_2$ implies

$$\left\| (t_2-s)^{q-1} \mathcal{T}(t_2-s) x - (t_1-s)^{q-1} \mathcal{T}(t_1-s) x \right\|_E < \gamma, \quad x \in E,$$

we get the following estimate:

$$Z_2 \leq \gamma T < \frac{\epsilon}{2}.$$

Therefore, for each $\epsilon > 0$ we may choose $\delta = \min\{\delta_1, \delta_2\}$ such that

$$\left\| G(f_n)(t_2) - G(f_n)(t_1) \right\|_E \le Z_1 + Z_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So, the sequence $\{G(f_n)\}$ is equicontinuous. By the Arzela-Ascoli theorem, we conclude that the sequence $\{G(f_n)\} \subset C([0,T];E)$ is relatively compact.

From (S_1) it follows that the operator G is linear and bounded, therefore it is continuous with respect to the topology of weak sequential convergence, hence from the weak convergence $f_n \rightharpoonup f_0$ in $L^1([0,T];E)$ implies $G(f_n) \rightharpoonup G(f_0)$. Since the sequence $\{G(f_n)\}$ is relatively compact, we conclude that $G(f_n) \to G(f_0)$ in $C([0,T];E)$.

The satisfaction of the condition (S_3) is obvious.

□

Consider a system governed by a functional inclusion with causal operators $\mathcal{Q}$ and $\mathcal{S}$, of the following form:

$$x(t) \in \mathcal{G}(t)x_0 + \mathcal{K}(t)x_1 + \mathcal{S} \circ \mathcal{Q}(x),\ t \in [0,T] \tag{3.1}$$

$$x(0) = x_0, \quad x'(0) = x_1, \tag{3.2}$$

Definition 3.1:
A function $x \in C([0,T];E)$ is called a mild solution to problem (3.1)-(3.2)*, if it satisfies the conditions:*

(1) the function $x \in C([0,T];E)$ and it satisfies inclusion (3.1)*;*
(2) $x(0) = x_0$, $x'(0) = x_1$.

Consider the multioperator $\Gamma : C([0,T];E) \multimap C([0,T];E)$ defined as

$$\Gamma(x) = \{x \in C([0,T];E) : x(t) = \mathcal{G}(t)x_0 + \mathcal{K}(t)x_1 + \mathcal{S} \circ \mathcal{Q}(x)\}.$$

It is clear that if the function x is a fixed point of the multioperator Γ, then x is a solution to the problem (3.1) - (3.2). Therefore, our goal is to prove the existence of a fixed point of the multioperator Γ.

Definition 3.2:
A sequence of functions $\{\xi_n\} \subset L^p([0,T];E)$ is called L^p–semicompact if it is L^p–integrably bounded, i.e.,

$$\|\xi_n(t)\|_E \le v(t) \text{ for a.e. } t \in [0,T] \text{ and for all } n = 1, 2, ...,$$

where $v \in L^p([0,T])$, and the set $\{\xi_n(t)\}$ is relatively compact in E for a.e. $t \in [0,T]$.

Lemma 3.4:
(see. [14], Proposition 4.2.1.). Every L^p–semicompact sequence is weakly compact in $L^1([0,T];E)$.

We need the following properties of the multioperator $\mathcal{S} \circ \mathcal{Q}$. Since for every $1 < p \le \infty : L^p([0,T];E) \subset L^1([0,T];E)$, we can formulate a modification of Theorem 5.1.1 from [14] in the following form.

Lemma 3.5:
Let $\mathcal{S} : L^p([0,T];E) \to C([0,T];E)$ be an operator satisfying the conditions $(\mathcal{S}1)$ and $(\mathcal{S}2)$. Then for every L^p-semicompact sequence $\{f_n\}_{n=1}^{\infty} \subset L^p([0,T];E)$, the sequence $\{\mathcal{S}f_n\}_{n=1}^{\infty}$ is relatively compact in $C([0,T];E)$ and, moreover, the weak convergence $f_n \overset{L^1}{\rightharpoonup} f_0$ implies that $\mathcal{S}f_n \to \mathcal{S}f_0$ in $C([0,T];E)$.

Theorem 3.1:
(See [2]). Let a multioperator $\mathcal{Q}$ satisfy the conditions $(\mathcal{Q}1)$–$(\mathcal{Q}3)$ and a operator $\mathcal{S}$ satisfy $(\mathcal{S}1)$, $(\mathcal{S}2)$. Then the composition $\mathcal{S} \circ \mathcal{Q} : C([0,T];E) \multimap C([0,T];E)$ is a u.s.c. multimap with compact values.

Let us find conditions for the multioperator $\mathcal{S} \circ \mathcal{Q}$ to be condensing with respect to a corresponding MNC. For this we need the following statements.

Lemma 3.6:
(See [2]) Let a sequence of functions $\{f_n\}_{n=1}^{\infty} \subset L^p([0,T];E)$ be L^p-integrally bounded and there exists a function $\upsilon \in L^p_+([0,T])$ such that

$$\chi\left(\{f_n(t)\}_{n=1}^{\infty}\right) \leq \upsilon(t) \textit{ for a.e. } t \in [0,T].$$

If an operator $\mathcal{S}$ satisfies the conditions $(\mathcal{S}1)$ and $(\mathcal{S}2)$, then for $1 \leq p < \infty$ we have

$$\chi\left(\{\mathcal{S}f_n(t)\}_{n=1}^{\infty}\right) \leq \left(4^p D \int_0^t \upsilon^p(s) ds\right)^{1/p},$$

and for $p = \infty$

$$\chi\left(\{\mathcal{S}f_n(t)\}_{n=1}^{\infty}\right) \leq 2D_1 \int_0^t \upsilon(s) ds,$$

where D, D_1 are the constants from condition $(\mathcal{S}1)$.

Consider the measure of noncompactness ν in the space $C([0,T];E)$ with values in the cone $\mathbb{R}^2_+$. On a bounded subset of $\Omega \subset C([0,T];E)$ we define the values of ν as follows:

$$\nu(\Omega) = \left(\gamma(\Omega), mod_C(\Omega)\right),$$

where mod_C is the modulus of equicontinuity, γ is the fading modulus of the fiber noncompactness

$$\gamma(\Omega) = \sup_{t \in [0,T]} e^{-Lt} \chi(\Omega(t)).$$

The constant $L > 0$ is chosen so that

$$\max\{q_1, q_2\} < 1,$$

where

$$q_1 = \sup_{t \in [0,T]} \left(4D^{1/p} \left(\int_0^t e^{-pL(t-s)} \omega^p(s,1) ds\right)^{1/p}\right),$$

$$q_2 = \sup_{t \in [0,T]} \left(2D_1 \int_0^t e^{-L(t-s)} \omega(s,1) ds\right),$$

where the constants D, D_1 are from the condition $(\mathcal{S}1)$, ω is a function from the condition $(\mathcal{Q}3)$.

It is easy to see that the MNC ν is monotone, nonsingular, and algebraically semi-additive. It follows from the Arzela–Ascoli theorem that it is also regular.

Theorem 3.2:
Let a causal multioperator $\mathcal{Q} : C([0,T];E) \multimap L^p([0,T];E)$ satisfy the conditions $(\mathcal{Q}2)$ and $(\mathcal{Q}3)$ and for a causal operator $\mathcal{S} : L^p([0,T];E) \to C([0,T];E)$ the conditions $(\mathcal{S}1)$–$(\mathcal{S}3)$ be satisfied. Then the multioperator Γ is ν-condensing.

Proof
By Lemma 3.2, it suffices to prove the assertion of the theorem for the multioperator $\mathcal{S} \circ \mathcal{Q}$. Let $\Omega \subset C([0,T];E)$ be a bounded set such that

$$\nu\left(\mathcal{S} \circ \mathcal{Q}\left(\Omega\right)\right) \geq \nu\left(\Omega\right). \tag{3.3}$$

Let us show that the set Ω is relatively compact.

Inequality (3.3) means that

$$\gamma(\{\mathcal{S} \circ \mathcal{Q}\left(\Omega\right)\}) \geq \gamma(\Omega). \tag{3.4}$$

Applying the condition $(\mathcal{Q}3)$ and by using the properties of the function ω, we obtain for a.e. $t \in [0,T]$

$$\chi\left(\{f(t) : f \in \mathcal{Q}\left(\Omega\right)\}\right) \leq \omega\left(t, \sup_{s\in[0,t]} \chi\left(\{y(s) : y \in \Omega\}\right)\right) = \omega\left(t, \varphi\left(\{y : y \in \Omega\}\right)\right) =$$

$$\omega\left(t, e^{Lt}e^{-Lt}\varphi\left(\{y : y \in \Omega\}\right)\right) \leq \omega\left(t, e^{Lt}\gamma\left(\Omega\right)\right) \leq \omega\left(t, e^{Lt}\right) \cdot \gamma\left(\Omega\right).$$

First, we consider the case $1 \leq p < \infty$. By Lemma 3.6 we have for each $t \in [0,T]$:

$$\chi\left(\{\mathcal{S}f(t) : f \in \mathcal{Q}\left(\Omega\right)\}\right) \leq \left(4^p D \int_0^t \omega^p\left(s, e^{Ls}\right) ds \cdot \gamma^p\left(\Omega\right)\right)^{1/p} \leq$$

$$4D^{1/p}\left(\int_0^t e^{pLs}\omega^p\left(s,1\right) ds\right)^{1/p} \cdot \gamma\left(\Omega\right).$$

Inequality (3.4) and the last inequality imply the following

$$\gamma(\Omega) \leq \sup_{t\in[0,T]}\left(4D^{1/p}\left(\int_0^t e^{-pL(t-s)}\omega^p\left(s,1\right) ds\right)^{1/p}\right)\gamma\left(\Omega\right) = q_1 \cdot \gamma\left(\Omega\right).$$

Therefore

$$\gamma\left(\Omega\right) = 0,$$

and

$$\varphi\left(\Omega\right) = 0$$

for all $t \in [0,T]$.

Let us turn to the case $p = \infty$. By Lemma 3.6 we have for each $t \in [0,T]$:

$$\chi\left(\{\mathcal{S}f(t) : f \in \mathcal{Q}\left(\Omega\right)\}\right) \leq 2D_1 \int_0^t \omega\left(s, e^{Ls}\right) ds \cdot \gamma\left(\Omega\right) \leq 2D_1 \int_0^t e^{Ls}\omega\left(s,1\right) ds \cdot \gamma\left(\Omega\right);$$

Inequality (3.4) and the last inequality imply the following

$$\gamma(\Omega) \leq \sup_{t\in[0,T]}\left(2D_1 \int_0^t e^{-L(t-s)}\omega\left(s,1\right) ds\right)\gamma\left(\Omega\right) = q_2 \cdot \gamma\left(\Omega\right).$$

Therefore

$$\gamma\left(\Omega\right) = 0,$$

and

$$\varphi\left(\Omega\right) = 0$$

for each $t \in [0, T]$.

Now we will show that the set Ω is equicontinuous. We take sequences $\{y_n\}_{n=1}^{\infty} \subset \Omega,\ n \geq 1$, and $\{f_n\}_{n=1}^{\infty}$, $f_n \in \mathcal{Q}(y_n)$. From the conditions $(\mathcal{Q}2)$ and $(\mathcal{Q}3)$ it follows that the sequence $\{f_n\}_{n=1}^{\infty}$ is L^p-semicompact, and therefore by Lemma 3.5 the sequence $\{\mathcal{S}f_n\}_{n=1}^{\infty}$ is relatively compact. Hence

$$mod_C(\{\mathcal{S}f_n\}_{n=1}^{\infty}) = 0.$$

Thus

$$\nu\left(\mathcal{S} \circ \mathcal{Q}\left(\Omega\right)\right) = (0, 0),$$

but then it follows from the inequality (3.3) that

$$\nu(\Omega) = (0, 0),$$

and the last expression yields that the set Ω is relatively compact. □

To prove the main theorem of the paper, we need the following statements, known as the Gronwall - Bellman Lemma and the generalized Gronwall - Bellman Lemma.

Lemma 3.7:
Let $v(t)$ and $f(t)$ be nonnegative continuous functions on the segment $[a, b]$, moreover

$$v(t) \leq c + \int_a^t f(s)v(s)ds,\ t \in [a, b],$$

where c is a positive constant. Then for each $t \in [a, b]$ the inequality

$$v(t) \leq ce^{\int_a^t f(s)ds},$$

holds.

Lemma 3.8:
Let $h(t),\ u(t)$ and $v(t)$ be nonnegative functions integrable on $[a, b]$ satisfying the inequality:

$$v(t) \leq u(t) + \int_a^t h(s)v(s)ds, \quad t \in [a, b].$$

Then the following inequality holds:

$$v(t) \leq u(t) + \int_a^t e^{\int_a^t h(\theta)d\theta} h(s)u(s)ds, \quad t \in [a, b].$$

Theorem 3.3:
Let a causal multioperator $\mathcal{Q} : C([0, T]; E) \to Cv(L^p([0, T]; E)),\ 1 \leq p \leq \infty$, satisfy the conditions $(\mathcal{Q}1)$–$(\mathcal{Q}3)$ and a linear causal operator $\mathcal{S} : L^p([0, T]; E) \to C([0, T]; E)$ satisfy the conditions $(\mathcal{S}1)$–$(\mathcal{S}3)$. Then the set Σ of all solutions to problem (3.1)-(3.2) is a non-empty compact set.

Proof

Let us show that the set of all solutions $x \in C([0, T]; E)$ of a one-parameter inclusion

$$x \in \lambda\Gamma(x), \quad \lambda \in [0, 1], \tag{3.5}$$

is a priori bounded. We divide the proof into three cases: $p = 1$, $1 < p < \infty$, $p = \infty$.

Let $p = 1$, if a function $x \in C([0, T]; E)$ satisfies condition (3.5). Then by Lemma 3.2 for each $t \in [0, T]$, we have the following estimates:

$$\|x(t)\|_E \leq \|\mathcal{G}(t)x_0\|_E + \|\mathcal{K}(t)x_1\|_E + D\int_0^t \|f(s)\|_E ds \leq$$

$$M\|x_0\|_E + Mt\|x_1\|_E + D\int_0^t \|f(s)\|_E ds \leq M\|x_0\|_E + MT\|x_1\|_E + D\int_0^t \|f(s)\|_E ds,$$

where $f \in \mathcal{Q}(x)$ and, therefore, by condition $(\mathcal{Q}2)$:

$$\|f(s)\|_E \leq \alpha(s)(1 + \|x(s)\|_E).$$

Then

$$\|x(t)\|_E \leq M\|x_0\|_E + MT\|x_1\|_E + D\int_0^t \alpha(s)(1 + \|x(s)\|_E)ds \leq$$

$$M\|x_0\|_E + MT\|x_1\|_E + D\|\alpha\|_{L^\infty}T + D\|\alpha\|_{L^\infty}\int_0^t \|x(s)\|_E ds.$$

The last expression is a non-decreasing function of t, so we get

$$\|x(t)\|_E \leq M\|x_0\|_E + MT\|x_1\|_E + D\|\alpha\|_{L^\infty}T + \int_0^t D\|\alpha\|_{L^\infty}\|x(t)\|_E ds.$$

This means that the function $v(t) = \|x(t)\|_E$ satisfies the inequality

$$v(t) \leq M\|x_0\|_E + MT\|x_1\|_E + D\|\alpha\|_{L^\infty}T + \int_0^t D\|\alpha\|_{L^\infty}v(s)ds.$$

Applying Lemma 3.7, we obtain the required a priori boundedness:

$$v(t) = \|x(t)\|_E \leq Ue^{D\|\alpha\|_{L^\infty}} = \gamma_1,$$

where

$$U = M\|x_0\|_E + MT\|x_1\|_E + D\|\alpha\|_{L^\infty}T.$$

Then $\|x\|_C = \sup_{t\in[0,T]}\|x(t)\|_E \leq \gamma_1$.

For the case $1 < p < \infty$, we have

$$\|x(t)\|_E \leq M\|x_0\|_E + MT\|x_1\|_E + \Big(D\int_0^t \|f(s)\|_E^p ds\Big)^{1/p}$$

$$\leq M\|x_0\|_E + MT\|x_1\|_E + \Big(D\int_0^t \alpha^p(s)(1 + \|x(s)\|_E)^p ds\Big)^{1/p}$$

$$\leq M\|x_0\|_E + MT\|x_1\|_E + \Big(D\int_0^t \alpha^p(s)ds + D\int_0^t \alpha^p(s)\|x(s)\|_E^p ds\Big)^{1/p} \leq$$

$$\leq M\|x_0\|_E + MT\|x_1\|_E + \Big(D\int_0^t \alpha^p(s)ds\Big)^{1/p} + \Big(D\int_0^t \alpha^p(s)\|x(s)\|_E^p ds\Big)^{1/p}.$$

Let us introduce the following notation:

$$c_0 = M\|x_0\|_E + MT\|x_1\|_E + D^{1/p}\|\alpha\|_{L^p},$$

$$h(s) = D^{1/p}\alpha(s).$$

Then we get:

$$\|x(t)\|_E \leq c_0 + \left(\int_0^t h^p(s)\|x(s)\|_E^p ds\right)^{1/p}.$$

Let $v(t) = \|x(t)\|_E^p$, then from the last inequality we obtain the estimate:

$$v(t) \leq 2^p c_0^p + 2^p \int_0^t h^p(s)v(s)ds.$$

Now applying Lemma 3.8 to the last inequality, we get

$$v(t) = \|x(t)\|_E^p \leq 2^p c_0^p \left(1 + \int_0^t e^{2^p \int_0^t h^p(\theta)d\theta} h^p(s)ds\right).$$

Then we have the final estimate for $1 < p < \infty$:

$$\|x(t)\|_E \leq 2c_0 \sqrt[p]{1 + \int_0^t e^{2^p \int_0^t h^p(\theta)d\theta} h^p(s)ds} = \gamma_2.$$

Then $\|x\|_C = \sup_{t\in[0,T]} \|x(t)\|_E \leq \gamma_2$.

For the case $p = \infty$, in the same way as in the case $p = 1$, the following estimate holds:

$$\|x\|_C \leq U_1 e^{D_1\|\alpha\|_{L^\infty}} = \gamma_3,$$

where

$$U_1 = M\|x_0\|_E + MT\|x_1\|_E + D_1\|\alpha\|_{L^\infty}T.$$

Now, if we take $R \geq \max\{\gamma_1, \gamma_2, \gamma_3\}$, then we can guarantee that the set $V \subset C([0,T]; E)$, given as

$$V = \{x \in C([0,T]; E) : \|x\|_C < R\},$$

contains all solutions of inclusion (3.5). Thus, the multioperator Γ satisfies on ∂V the condition of Theorem 2.2 with $a = 0$, hence the set of its fixed points is non-empty and compact. □

4. CAUCHY TYPE PROBLEMS FOR SEMILINEAR DIFFERENTIAL INCLUSIONS

4.1. A Cauchy Type Problem for a Second Order Semilinear Differential Inclusions

Consider the following system governed by a differential inclusion in a separable Banach space E :

$$y''(t) \in Ay(t) + F(t, y(t)), \quad t \in [0,T], \tag{4.6}$$

$$y(0) = y_0, \qquad y'(0) = y_1, \tag{4.7}$$

where $F : [0,T] \times E \to Kv(E)$ is a multivalued map, $y_0, y_1 \in E$ are given functions. Suppose that

(A) $A : D(A) \subset E \to E$ generates a family of strongly continuous cosine operator functions $\{\cos(At)\}_{t\geq 0}$.

A multimap $F : [0,T] \times E \to Kv(E)$ is such that:

$(F1)$ for each $\psi \in E$ the multifunction $F(\cdot, \psi) : [0,T] \to Kv(E)$ admits a measurable selection;

$(F2)$ for a.e. $t \in [0,T]$ the multimap $F(t,\cdot): E \to Kv(E)$ is upper semicontinuous (u.s.c.);
$(F3)$ there exists a function $\alpha \in L^\infty_+[0,T]$ such that

$$\|F(t,\psi)\|_E := \sup\{\|z\|_E : z \in F(t,\psi)\} \le \alpha(t)(1+\|\psi\|_E)$$

for a.e. $t \in [0,T]$ and for all $\psi \in E$;
$(F4)$ there exists a function $\omega_F : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the conditions $(\omega 1)$-$(\omega 3)$ such that for each bounded set $\Omega \subset C([0,T];E)$ we have

$$\chi(F(t,\Omega)) \le \omega_F(t,\varphi(\Omega)) \text{ for a.e. } t \in [0,T].$$

Notice that when $q=2$:

$$\mathcal{G}(t) = \cos(At), \quad \mathcal{K}(t) = \sin(At), \quad \mathcal{T}(t) = \frac{\sin(At)}{t}.$$

In accordance with [11], a function $y \in C([0,T];E)$ is a mild solution to problem (4.6)-(4.7), if it can be represented in the form:

$$y(t) = \cos(At)y_0 + \sin(At)y_1 + \int_0^t \sin(A(t-s))f(s)ds,$$

where $f \in \mathcal{P}_F(y)$, $\mathcal{P}_F$ is a superposition multioperator (see Example 2.1).

The fact that the superposition multioperator $\mathcal{P}_F : C([0,T];E) \multimap L^1([0,T];E)$ satisfies the condition $(\mathcal{Q}1)$ can be verified by Lemma 5.1.1 from [14]. Conditions $(\mathcal{Q}2)$ and $(\mathcal{Q}3)$ for $\mathcal{P}_F$ follow from $(F3)$ and $(F4)$, respectively. Taking into account Lemma 3.1, we can consider relation (4.6) as a special case of functional inclusion (3.1) with $\mathcal{Q} = \mathcal{P}_F$, and $\mathcal{S} = \mathcal{L}$ is the Cauchy operator.

As a direct consequence of Theorem 3.3, we obtain the following result.

Theorem 4.1:
Suppose that the conditions (A), $(F1)$–$(F4)$, hold true. Then the set of solutions to problem (4.6)-(4.7) *is a non-empty compact subset of the space $C([0,T];E)$.*

4.2. A Cauchy Type Problem for Fractional Semilinear Differential Inclusions

Let us recall the notion of the Caputo fractional derivative.

Definition 4.1:
The Caputo fractional derivative of the order $q \in (1,2)$ of a function $g \in C^2([0,T];E)$ is the function ${}^C D_0^q g$ of the following form:

$$^C D_0^q g(t) = \frac{1}{\Gamma(2-q)} \int_0^t (t-s)^{1-q} g''(s)\, ds,$$

where Γ is Euler's gamma-function

$$\Gamma(q) = \int_0^\infty x^{q-1} e^{-x} dx.$$

Consider the following system governed by a functional differential inclusion in a separable Banach space E :

$$^C D_0^q y(t) \in Ay(t) + F(t, y(t)), \quad t \in [0,T], \tag{4.8}$$

$$y(0) = y_0, \qquad y'(0) = y_1. \tag{4.9}$$

Suppose that

(A) $A : D(A) \subset E \to E$ is a closed linear operator in E generating a family of strongly continuous cosine operator functions $\{C(t)\}_{t \geq 0}$.

Assume also that a multimap $F : [0, T] \times E \to Kv(E)$ satisfies the conditions $(F1)$–$(F4)$ from section 4.1.

A function $y \in C([0, T]; E)$ is a mild solution to problem (4.8)–(4.9), if it can be presented in the form:

$$y(t) = \mathcal{G}(t)y_0 + \mathcal{K}(t)y_1 + \int_0^t (t-s)^{q-1}\mathcal{T}(t-s)f(s)ds,$$

where $f \in \mathcal{P}_F(y)$.

The fact that the superposition multioperator $\mathcal{P}_F : C([0, T]; E) \multimap L^p([0, T]; E)$, $p > 1/q$ satisfies the condition $(\mathcal{Q}1)$ can be verified as in the paper [26]. Conditions $(\mathcal{Q}2)$ and $(\mathcal{Q}3)$ for $\mathcal{P}_F$ follow from $(F3)$ and $(F4)$, respectively. Taking into account Lemma 3.3, we can consider the relation (4.8) as a special case of functional inclusion (3.1) with $\mathcal{Q} = \mathcal{P}_F$, and $\mathcal{S} = G$ is the Cauchy type operator.

As a direct consequence of Theorem 3.3, we obtain the following result.

Theorem 4.2:
Suppose that the conditions (A), $(F1)$–$(F4)$ hold true. Then the set of solutions to problem (4.8)-(4.9) *is a non-empty compact subset of the space $C([0, T]; E)$.*

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