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Lyapunov Functions for Periodic Selector-Linear Difference Inclusions

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Abstract: The paper considers periodic selector-linear difference inclusions. A class of time-periodic quasi-quadratic Lyapunov functions is distinguished, as well as parametric classes of piecewise-quadratic and piecewise-linear Lyapunov functions. These functions establish necessary and sufficient conditions for asymptotic stability. An example leading to periodic selector-linear differential and difference inclusions is given. The results can find applications in the stability analysis of control systems with periodic parameters, in particular, servomechanisms whose elements operate on alternating current, control systems with amplitude-frequency modulation.

Keywords: periodic selector-linear difference inclusion, asymptotic stability, Lyapunov functions.

1. INTRODUCTION

The problem of the stability of periodic difference inclusions arises in the study of discrete control systems with periodic parameters and uncertainty, in particular, tracking systems that operate on alternating current and systems with amplitude-frequency modulation. In some cases, such as the absolute stability problem, the study of linear nonstationary systems whose right part matrix satisfies interval constraints can be used selector-linear difference inclusions. In [5] periodic difference inclusions are considered. The definitions of asymptotic, uniform asymptotic and uniform exponential stability are given and the equivalence of these properties for selector-linear difference inclusions is proved. On the basis of the variational approach, a necessary and sufficient condition for uniform asymptotic stability in the form of some limit relation is obtained. In this paper we obtain asymptotic stability criteria for periodic selector-linear difference inclusions based on the Lyapunov function method.

The remainder of this paper is structured as follows. In Section 2, we consider periodic selector-linear difference inclusions and set the problem of obtaining asymptotic stability criteria for such inclusions on the basis of the Lyapunov function method. In section 3 we distinguish a class of periodic in time Lyapunov functions of quasi-quadratic form and parametric classes of piecewise quadratic and piecewise linear Lyapunov functions. Theorems 1-3 are formulated and proven. They establish necessary and sufficient conditions for asymptotic stability of the considered inclusions. In Section 4, an example of a system leading to periodic differential and difference inclusions is given. In Section 5, we offer concluding remarks.

2. STATEMENT OF THE PROBLEM

Consider the dynamic systems described by periodic selector-linear difference inclusion

$$x(s+1) \in F(s,x), s = 0, 1, \dots, \ \ x \in R^n, \tag{2.1}$$

where the set-valued map $F: R^n \to R^n$ has the form

$$F(s,x) = \{y: y = B(s)x, \ B(s) \in \Omega(s)\},$$

Here $\Omega(s), \Omega(s+N) = \Omega(s)$ ($s = 0,1, \dots$, $N$is a natural number) is a convex, compact set of real $(n \times n)$ – matrices B. Such set-valued maps are called selector-linear, since the right-hand side is a union of linear maps. The sequence of vectors $\{x(s)\}$, satisfying for all $s = 0,1, \dots$ inclusion (2.1), is the *solution* of the inclusion (2.1). Let $x(s, s_0, x_0)$ be *the solution of inclusion (2.1) with initial conditions* (s_0, x_0). Due to the periodicity of the multivalued function $F(s,x)$ in s without generality restriction we can assume that $0 \le s_0 \le N$. The equivalence of the properties of asymptotic stability, uniform asymptotic stability, and uniform exponential stability for inclusion (2.1) was proved in [5]. Hereinafter we will refer to the asymptotic stability of inclusion (2.1).

The problem is to identify the parametric classes of Lyapunov functions establishing necessary and sufficient conditions for asymptotic stability of inclusion (2.1) and to construct the stability criteria for inclusion (2.1) using a discrete analogue of the direct Lyapunov method.

3. RESULTS

Theorem 3.1:

The following conditions are equivalent:

1. Inclusion (2.1) is asymptotically stable.

2. There exists the Lyapunov function $v(s,x)$ *of the quasiquadratic form*

$$v(s,x) = x' L(s,x)x,\ L(s,x) = (l_{ij}(s,x))_{i,j=1}^{n}, L(s+N,x) = L(s,x),$$

$$L'(s,x) = L(s,x) = L(s,\mu x), x \neq 0, \mu \neq 0, v(s,0) \equiv 0 \tag{3.1}$$

which is N*-periodic in* s*, homogeneous (of second order), strictly convex in* x*, and it satisfies the following inequality:*

$$\max_{y \in F(s,x)} v(s,y) \le \theta v(s,x), x \in R^n, s = 0,1, \dots \tag{3.2}$$

for some θ $(0 < \theta < 1)$.

In (3.1) the prime means the transposition operation.

The sufficiency of the conditions of Theorem 3.1 is established (using the lemma given in [4]) by the reasoning used in the proof of exponential stability of discrete systems in [2]. The proof of the necessity follows the same scheme as the proof of the corresponding conditions of the theorem in [4], where exponential estimates for solutions of inclusion (2.1) are used. Theorem 3.1 is an extension of the classical theorem for the discrete analogue of direct Lyapunov method in [2] to selector-linear periodic difference inclusions (2.1).

The reasoning used in the proof of the lemma in [4] proves that, under Theorem 1, there exist constants such $\lambda_2 \ge \lambda_1 > 0$ that for all $s \ge 0$ and $x \in R^n$, the inequalities

$$\lambda_1 \|x\|^2 \le v(s,x) \le \lambda_2 \|x\|^2. \tag{3.3}$$

(3.3) implies positive definiteness of the function $v(s,x)$.

The problem of constructing the Lyapunov function is simplified if this function is selected from a certain parametric class of functions that depend on a finite number of parameters. The class of quasi-quadratic Lyapunov functions (3.1) is not parametric. The existence of a parametric class of Lyapunov functions, also defining necessary and sufficient conditions for the asymptotic stability of inclusion (3.1), can be formulated as a theorem.

Theorem 3.2:

Inclusion (2.1) is asymptotically stable iff, for some integer $M \ge n$*, there exists a periodic in* s *(of period* N*), piecewise-quadratic Lyapunov function*

$$v_M(s,x) = \max_{1\le j\le M}\langle l^j(s),x\rangle^2, (l^j(s+N) = l^j(s), \tag{3.4}$$

for which the inequality (3.2) is satisfied for all $s \ge 0$ *and* $x \in R^n$*, and the* n*-dimensional periodic vectors* $(l^j(s)\ \ (l^j(s+N) = l^j(s)), j = \overline{1,M}$ *satisfy the condition*

$$rank L(s) = n \le M, L(s) = \big(l^1(s),\dots,l^M(s)\big), s = 0,1,\dots \tag{3.5}$$

(i.e., periodic $(n \times M)$ *matrix* $L(s)(L(s+N) = L(s))$ *has a maximum rank for all* $s \ge 0$*).*

We denote by $\langle\cdot,\cdot\rangle$ a scalar product of vectors.

The set of Lyapunov functions (3.4) forms a parametric class. The parameters defining this class are the components of periodic vectors $l^j(s)$ $(j = \overline{1,M}, s \ge 0\ )$ and the integer $M \ge n$. If the rank condition (3.5) is satisfied, the function $v_M(s,x)$ is positively defined in R^n and its level surfaces at any fixed one $s \ge 0$ are centrally symmetric convex polyhedrons. The vectors $l^j(s),\ j = \overline{1,M}$ defining the norms to their faces.

Proof. Sufficiency. The sufficiency of the conditions of Theorem 3.2 is established according to the standard scheme of the proof of exponential stability in [2] using inequality (3.2) and estimates $\lambda_1\|x\|^2 \le v_M(s,x) \le \lambda_2\|x\|^2,\ \lambda_2 \ge \lambda_1 > 0$ for function (3.4) under condition (3.5).

Necessity. It follows from Theorem 3.1 that for inclusion (2.1) there exists a periodic in s Lyapunov function $v(s,x)$ of the quasiquadratic form satisfying the conditions of Theorem 3.1. Consider centrally symmetric convex bodies

$$P_1(s) = \{x: v(s,x) \le 1\}, P_r(s) = \{x: v(s,x) \le r\}, s = 0,1,\dots,0 < r < 1.$$

Let $int\,A$ be the set of interior points of the set A. Since $0 < r < 1$, then $P_r(s) \subset int\,P_1\,(s)$. It follows from Theorem 20.4 in [6] that there exists a centrally symmetric convex polyhedron $D_1(s)$, that the relations

$$P_r(s) \subset int\,D_1\,(s) \subset D_1(s) \subset int\,P_1\,(s) \subset P_1(s). \tag{3.6}$$

Let $2M$ be the number of faces of the polyhedron $D_1(s)$, then there are M pairs of centrally symmetric faces. Let $r_j > 0$ be the distance from a point $x = 0$ to the faces of the j -pair, and let $\pm n_j$ $(\|n_j\| = 1), j = \overline{1,M}$ be the unit external normals to the faces of this pair. In order the surface of the polyhedron $D_1(s)$ to be the surface of the level $\Omega_{D_1}(s) = \{x: v_M(l(s),x) = 1\}$ for function $v_M(l(s),x)$ (3.4), choose $l_j(s) = r_j^{-1}n_j, j = \overline{1,M}, s = 0,1,\dots$ Then the vectors $l_j(s), j = \overline{1,M},\ s = 0,1,\dots$ satisfy rank condition (3.5), because otherwise the surfaces of the level function $v_M(l(s),x)$ will not be bounded.

Let $\delta(K,z) = \max_{x\in K}(z,x),\ z \in R^n$ be the reference function of compact $K \subset R^n$ (see [6]). Since $\delta(P_r(s),z) = \sqrt{r}\delta(P_1(s),z)$, it follows from (3.6) and property 9 in [1]

$$\sqrt{r}\delta(P_1(s),z) < \delta(D_1(s),z) < \delta(P_1(s),z),\ z \ne 0. \tag{3.7}$$

Let us show that there exists a number r_1 $(r < r_1 < 1)$, such that the inequality

$$\sqrt{r}\delta(P_1(s),z) < \sqrt{r_1}\delta(D_1(s),z),\ z \ne 0. \tag{3.8}$$

Suppose that inequality (3.8) is not satisfied, i.e., for any r_1 $(r < r_1 < 1)$

$$\sqrt{r}\delta(P_1(s),z) \ge \sqrt{r_1}\delta(D_1(s),z) \text{ or } \sqrt{\frac{r}{r_1}}\delta(P_1(s),z) \ge \delta(D_1(s),z),\ z \ne 0,$$

that contradicts inequality (3.7).

Function $\sqrt{r_1}\delta(D_1(s),z) = \delta(D_{r_1},z)$ is the reference function of the polyhedron $D_{r_1}(s) = \{x: v_M(l(s),x) \le r_1\}$, which is similar to polyhedron $D_1(s) = \{x: v_M(l(s),x) \le 1\}$. Therefore (3.8) is equivalent to the relation

$$P_r(s) \subset int\,D_{r_1}\,(s) \subset D_{r_1}(s). \tag{3.9}$$

It follows from inequality (3.2) for the function $v(s,x)$ that $F(P_1(s)) \subset P_r(s)$, where $F(P_1(s)) = \bigcup_{x\in P_1(s)} F(s,x)$. (3.6) implies that $D_1(s) \subset P_1(s)$. Therefore

$$F\big(D_1(s)\big) \subset F\big(P_1(s)\big) \subset P_r(s). \tag{3.10}$$

It follows from (3.9) and (3.10) $F(D_1(s)) \subset int\, D_{r_1}(s)$ and therefore the condition

$$\max_{y\in F(s,x)} v_M(l(s),y) < r_1, x \in D_1(s), s = 0,1,... \tag{3.11}$$

Since

$$v_M(l(s),\mu x) = \mu^2 v_M(l(s),x) \text{ and } F(s,\mu x) = \mu F(s,x)$$

for all $s \geq 0$, $x \in R^n$ and $\mu \in R^1$, then, for the above-constructed function $v_M(l(s),x)$ from (3.11), the inequality

$$\max_{y\in F(s,x)} v_M(l(s),y) \leq r_1 v_M(l(s),x), 0 < r_1 < 1, x \in R^n, s = 0,1,... \tag{3.12}$$

of the form (3.2). This completes the proof of Theorem 3.2. □

Consider piecewise linear Lyapunov functions

$$V_M(s,x) = \max_{1\leq j\leq M} \left|\langle l^j(s),x\rangle\right|. \tag{3.13}$$

A corollary of Theorem 3.2 is

Theorem 3.3: *Inclusion (2.1) is asymptotically stable iff, for some integer $M \geq n$ there exists a periodic on s (period N), piecewise-linear Lyapunov function $V_M(s,x)$ (3.13), satisfying condition (3.5) and inequality (3.2) for all $x \in R^n$, $s \geq 0$.*

Proof. Since

$$\sqrt{v_M(s,x)} = \sqrt{\max_{1\leq v\leq M} \langle l^j(s),x\rangle^2} = \max_{1\leq j\leq M} \left|\langle l^j(s),x\rangle\right| = V_M(s,x),$$

then each function $v_M(s,x)$ can be matched with function $V_M(s,x) = \sqrt{v_M(s,x)}$. In the same way each function $V_M(s,x)$ (3.13) can be associated with function $v_M(s,x) = {V_M}^2(s,x)$. From this and Theorem 3.2 follows the validity of Theorem 3.3. □

4. EXAMPLE

Consider the torsional vibrations of the crankshafts of a single-cylinder engine with a flywheel with allowance for the inertia of connecting rods and pistons. Let us assume that the mass of the flywheel is sufficiently large, so that the rotation of the shaft can be considered uniform. Let ω - angular velocity of the flywheel, c - torsional stiffness coefficient of the shaft. In [3] it is shown that vibrations are described by second order differential equation

$$\frac{d^2q}{dt^2} + \frac{c}{\omega^2} p(t) q = 0, \tag{4.14}$$

where q – generalized coordinate associated with the angle of rotation of the crank, and periodic function $p(t)$ of the period 2πcan be determined by the kinetic energy of the crank together with the associated moving masses (connecting rod and piston).

Equation (4.1) is a special case of an equation of more general form

$$\frac{d^2z}{dt^2} + bf(t)z = 0, \tag{4.2}$$

where $f(t)$ is a periodic function of time (with period $T > 0$), and $b \in I, I = [b_1, b_2]$ is a certain parameter. Let us introduce notation

$$x_1 = z,\ x_2 = \frac{dx_1}{dt} = \frac{dz}{dt},\ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ A(t,a,b) = \begin{pmatrix} 0 & 1 \\ -bf(t) & 0 \end{pmatrix}.$$

Equation (4.2) will take the form of second order system

$$\frac{dx}{dt} = A(t,b)x,\ A(t+T) \equiv A(t), \quad b \in I,\ t \geq 0,\ T > 0,\ x \in R^2. \tag{4.3}$$

Consider discrete analogue of system (4.3)

$$x(s+1) = A(s,b)x(s),\ A(s+M) \equiv A(s), \quad b \in I, M \in N,\ x \in R^2, \tag{4.4}$$

where $s = 0,1,\ldots$ is discrete time.

System (4.4) is equivalent to periodic selector-linear difference inclusion (2.1), where the multivalued function $F(s,x)$ $(F(s+M,x) \equiv F(s,x))$ is defined at each point (s,x), $x \in R^2$by the relation

$$F(s,x) = \{y: y = A(s,b)x,\ b \in I\}.$$

5. CONCLUSION

For periodic selector-linear difference inclusion (2.1) asymptotic stability criteria were obtained. They use quasi-quadratic, piecewise quadratic and piecewise linear Lyapunov functions. The example of technical problem leading to consideration of periodic differential and difference inclusions is given.

The obtained results can be used in the study of the stability of control systems with periodic parameters. In particular, such systems are the tracking systems, which elements operate on alternating current and the control systems with pulse-amplitude modulation. The extracted in Theorems 3.2, 3.3 piecewise-quadratic and piecewise-linear Lyapunov functions of the form (3.4), (3.13) establish necessary and sufficient conditions of asymptotic stability for inclusion (2.1). These functions can be used when developing numerical methods of stability analysis for systems equivalent to difference inclusion (2.1).

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