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Synchronization of Inbound and Outbound Flows at Stations in the Model of Freight Transportation Organization

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Abstract: This article presents a model for organizing railway freight transportation between two node stations based on the interaction of neighboring stations depending on their technical capabilities and the demand for freight transportation. The main objective of such interaction is to synchronize the inbound and outbound flow at the stations, specifically reducing the degree of inconsistency between the receipt and dispatch of goods at the stations. This characteristic represents the imbalance between the volume of incoming and outgoing goods at the stations per unit of time and is described by a non-negative function bounded above by unity. Its dynamics are described by a system of differential equations containing a set of parameters characterizing the infrastructure of the stations, the mode of distribution of goods from the final node station, and the demand for freight transportation. The range of parameter variations for which the specified system has a solution has been determined, as well as such a set of these parameters for which the task of synchronizing the inbound and outbound flow at the stations is best solved. It has been found that regardless of the initial value, starting from a certain point in time, the degree of inconsistency between the receipt and dispatch of goods at all stations except the initial node station becomes zero.

***Keywords**:* freight transportation organization, flow synchronization, system of differential equations, model parameters, stationary solutions, stability of solutions.

1. INTRODUCTION

One of the largest foundational sectors of any country's economy is transportation. It provides geographical connectivity across the nation's territories and coordinates the functioning of all sectors of the economy. Transportation creates the conditions for the effective operation of a country, and its development is a vital component of economic modernization. Additionally, transportation contributes to the development of international economic relations, the exploration of new economic regions, and ensures the country's defense capabilities.

Two major groups of mathematical models for transportation systems can be distinguished:

1. The first group focuses on modeling transportation networks and their utilization. It includes models for calculating correspondences, such as the gravity model [63], entropy model [28, 52], models within the competing centers framework [21], as well as models for flow distribution within the network [57, 47, 3].

2. The second group involves modeling the dynamics of transportation flow [56]. It comprises fundamental classes of dynamic models: macroscopic (hydrodynamic), kinetic (gas dynamic), and microscopic models. Macroscopic models [17, 43, 34, 24] describe averaged characteristics of transportation flow and are sometimes referred to as hydrodynamic models because they liken the flow itself to the movement of compressible fluid. Macroscopic diagrams, which illustrate the relationships between performance parameters such as traffic density, traffic flow, and vehicle speed, are used to represent traffic states and system configurations [16, 26, 12]. Microscopic

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models explicitly describe the movement of each individual vehicle. They provide a more detailed description of movement on specific segments of the transportation network but require significantly greater computational resources for practical implementation. The first microscopic models were introduced in the 1950s [51, 56]. Examples of such models include car-following models [25, 7], optimal velocity models [2, 59], the Treiber model [60], as well as cellular automaton models [15, 14]. Kinetic models occupy an intermediate position between macroscopic and microscopic models. In kinetic models, traffic flow is characterized by the density distribution of vehicles in phase space, and the dynamics of the phase density are described by kinetic equations. These equations result from averaging the effects of interactions among individual vehicles [30, 50].

It is worth noting that the models mentioned above are most suitable for studying automobile traffic. For a vast country like Russia, however, railway transportation plays a pivotal role. It ensures reliable and cost-effective delivery of goods, especially when rapid transportation of large volumes of cargo is required. Publications related to railway logistics can be categorized into three main groups, depending on the types of problems under investigation.

The first group deals with the design of railway infrastructure [32, 42, 20].

The second group focuses on the management of locomotives and wagons. Depending on regulatory and market characteristics, different regions may have their own models to account for specific factors. For instance, the work of R. Fukasawa et al. [22] presents a model used by one of the largest railway operators in Latin America. Another example is the study by A. Chezelli et al. [13], which examines multiple optimization models for freight delivery by Swiss Federal Railways' Cargo Express Service. Several publications focus on models designed to address the specifics of the freight transport market in Italy [49, 11]. Some works present cost-minimization models for transporting goods across multiple European countries via railway [1, 33]. There are also models created for the Russian railway transport market [54, 45, 6].

The third group addresses railway planning tasks, which traditionally centered on scheduling freight train movements [44, 10, 46]. In recent years, publications in this group have been complemented by studies that apply macroscopic traffic theory to describe processes in railway transportation. N. Weik's work [61], for example, provides a theoretical analysis of traffic flow properties on unidirectional railway lines. It constructs macroscopic fundamental diagrams and demonstrates how they can be used to determine flow regimes and various phases of train movement, which is valuable for system design and operational planning.

Another area of research that has been actively developing in recent years is related to predicting delays in railway systems. Trains in this system follow predefined schedules, which allows for the efficient use of routes and tracks. Deviations from such planned operations manifest as delays and can reduce the system's efficiency. Minor delays are often absorbed by built-in buffers and thus do not impact larger scales [63, 18]. However, logistical disruptions, often caused by external factors such as weather, occasionally lead to congestion or even large-scale stoppages with negative consequences for society and the economy [48, 9]. Most models that study delays are based on railway system schedules and typically use trains as agents that can incur delays [27, 23, 29]. In contrast, in the work by Dekker et al. [19], delays are treated as variables associated not with trains but with nodes (stations) and edges of the railway network, which remain in place. The spread of delays between these nodes does not necessarily have to be described in terms of discrete trains and events but can be based solely on common (or even systemic) quantities, such as network topology and schedules. The authors draw an analogy with hydrodynamics: while delays are traditionally considered Lagrangian particles (i.e., following trains like a fluid carrying particles), they suggest considering delays from an Eulerian perspective (i.e., defining incoming and outgoing delays in a fixed spatial system). They refer to this representation of delays as diffusion-like spreading. When examining the microscale, it is expected that this unconventional approach to delay handling may be less accurate than more detailed models, but on a large scale, the performance of such a model increases. The model contains only basic schedule information (e.g., train frequencies and travel times), and all the model's information is embedded in a single matrix, facilitating the analysis of system properties.

Another crucial issue in railway planning is the study of freight transportation modes and their corresponding cargo flows within a dynamic system that describes the transportation process as the interaction of key elements of railway infrastructure, primarily stations. This problem has been addressed in the works of L.A. Beklaryan and N.K. Khachatryan [4, 5, 35–41]. They present dynamic models in which the organization of freight transportation involves forming cargo flows based on station interactions. The rules of station interaction depend on the nature of demand for freight transportation. In cases of consistently high demand for freight transportation, the focus is on utilizing the station's technical capabilities to their fullest. In the absence of consistently high demand for freight transportation, the primary goal of station interaction is to synchronize incoming and outgoing flows, allowing for more efficient freight transportation, minimizing delays, and ensuring a smooth flow of cargo. The work [41] describes a model of organizing freight transportation between major node stations when there is no consistently high demand for freight transportation. It explores the relationship between the degree of mismatch between cargo arrival and departure at stations and the parameters that characterize the demand for freight transportation, the technical capacity of stations, and the degree of its utilization. It assumes that the stations are identical, meaning that the specified parameters are the same for all stations. This article is dedicated to advancing this model and examines the case where stations have different characteristics.

2. PROBLEM STATEMENT

The movement of cargo on a segment of the railway network between two terminal stations, connected by multiple intermediate stations, is being considered. The primary characteristic of station i at time t, where $i \in \{0, 1, .., m, m+1\}$, is the degree of mismatch between cargo reception and dispatch $z_i(t)$, which varies in the range $[0, 1]$. The precise definition is provided in the work [41].

The technical capacity of station i is determined by the maximum permissible increase in the degree of mismatch between cargo reception and dispatch per unit of time and is defined by a non-negative decreasing function $\varphi_i(z)$ defined on the interval $[0, 1]$, satisfying the condition $\varphi_i(1) = 0$.

The initial node station ($i = 0$) receives cargo based on the demand for transportation within its technical capacity and dispatches it to the next station within its technical capacity. Each of the intermediate stations ($i = 1, 2, \dots m$) receives cargo within its technical capacity and dispatches it within the technical capacity of the next station. The final node station ($i = m + 1$) receives cargo within its technical capacity and distributes it according to a specific regime.

The dynamics of the degrees of mismatch between cargo reception and dispatch at the stations are described by the following system of differential equations

$$\dot{z}_0(t) = \min\left(d_0, \varphi_0\big(z_0(t)\big)\right) - \lambda_1 \varphi_1\big(z_1(t)\big), t \in [t_0, +\infty); \tag{2.1}$$

$$\dot{z}_i(t) = \lambda_i \varphi_i\big(z_i(t)\big) - \lambda_{i+1} \varphi_{i+1}\big(z_i(t)\big), i = 1, 2, \dots, m, t \in [t_0, +\infty); \tag{2.2}$$

$$\dot{z}_{m+1}(t) = \lambda_{m+1} \varphi_{m+1}\big(z_{m+1}(t)\big) - d_{m+1}, t \in [t_0, +\infty); \tag{2.3}$$

$$0 \le z_i(t) \le 1, i = 0, 1, \dots, m+1, t \in [t_0, +\infty). \tag{2.4}$$

Here $d_0 > 0, 0 < \lambda_i \le 1, d_{m+1} > 0$ are model parameters:

d_0 is a characteristic of the demand for transportation;

λ_i is a characteristic of the degree of utilization of the technical potential of station number i;

d_{m+1} is a characteristic of the distribution mode of goods from the final node station.

Next, consider the function $\varphi_i(z)$, that defines the technical potential of station number i, of the following form

$$\varphi_i(z) = a_i(1 - z), a_i > 0. \tag{2.5}$$

The parameter $a_i > 0$, which is involved in defining the function $\varphi_i(z)$, represents the capability characteristic of station number i in increasing the flow of goods. Parameter d_0, which is a characteristic of the demand for transportation and is involved in equation (1), is represented as follows:

$$d_0 = \mu a_0, 0 < \mu \leq 1. \tag{2.6}$$

Let's rewrite the system (2.1)–(2.4), where the function $\varphi_i(z)$ is defined according to (2.5), and the parameter d_0 is defined according to (2.6).

$$\dot{z}_0(t) = \min\left(\mu a_0, a_0\left(1 - z_0(t)\right)\right) - \lambda_1 a_1\left(1 - z_1(t)\right), t \in [t_0, +\infty), \tag{2.7}$$

$$\dot{z}_i(t) = \lambda_i a_i \left(1 - z_i(t)\right) - \lambda_{i+1} a_{i+1} \left(1 - z_{i+1}(t)\right), i = 1, 2, \dots, m, t \in [t_0, +\infty), \tag{2.8}$$

$$\dot{z}_{m+1}(t) = \lambda_{m+1} a_{m+1}\left(1 - z_{m+1}(t)\right) - d_{m+1}, t \in [t_0, +\infty), \tag{2.9}$$

$$0 \leq z_i(t) \leq 1, i = 0, 1, \dots, m + 1, t \in [t_0, +\infty). \tag{2.10}$$

Here $\mu, a_i, \lambda_i, d_{m+1}$ are model parameters:

μ $(0 < \mu \leq 1)$ is a characteristic of the demand range for transportation that can be satisfied with the existing technical capacity of the stations;

a_i $(a_i > 0)$ is a characteristic of the ability of station number i to increase freight flow i;

λ_i $(0 < \lambda_i \leq 1)$ is a characteristic of the degree of utilization of the technical capacity of station number i;

d_{m+1} $(d_{m+1} > 0)$ is a characteristic of the distribution mode of goods from the final node station.

Let's outline the main research tasks:

- determine the ranges of variation of the parameters $\mu, a_i, \lambda_i, d_{m+1}$, for which the cargo transportation system can operate smoothly, i.e., the system (2.7)–(2.10) has a solution.
- for a given value of the demand characteristic for cargo transportation (parameter μ) establish the most acceptable achievable levels of inconsistency between receiving and sending goods at all stations, by managing the values of the following characteristics: the capabilities of stations to increase freight flow (parameter a_i), the degree of utilization of the technical capacity of stations (parameter λ_i) and the mode of distribution of goods from the final node station (parameter d_{m+1}).

3. INVESTIGATION OF SOLUTIONS OF THE SYSTEM (2.7)–(2.10)

Studying the solution set of the system (2.7)–(2.10), let's begin by investigating all the solutions of the system of differential equations (2.7)–(2.9).

First and foremost, let's identify the stationary solutions of the system (2.7)–(2.9). By direct examination, we can verify the validity of the following statement.

Statement 3.1:

The system (2.7)–(2.9) for any values of the parameters

$$0 < \mu \leq 1, a_i > 0, 0 < \lambda_i \leq 1, i = 0, 1, \dots, m + 1, d_{m+1} > 0$$

such that $d_{m+1} \leq \mu a_0$ *has stationary solutions:*

$$z_0(\cdot) \equiv 1 - \frac{d_{m+1}}{a_0}, z_i(\cdot) \equiv 1 - \frac{d_{m+1}}{\lambda_i a_i}, i = 1, \dots, m + 1, when\ d_{m+1} < \mu a_0 \tag{3.1}$$

$$z_0(\cdot) \leq 1 - \mu, z_i(\cdot) \equiv 1 - \frac{d_{m+1}}{\lambda_i a_i}, i = 1, \dots, m + 1, when\ d_{m+1} = \mu a_0 \tag{3.2}$$

For $d_{m+1} > \mu a_0$ the system (2.7)–(2.9) has no stationary solutions.

Let's proceed to investigate the remaining solutions of the system (2.7)–(2.9).

Theorem 3.1:

Any solution of the system (2.7)–(2.9) with $d_{m+1} < \mu a_0$ eventually converges to a stationary trajectory (3.1), and when $d_{m+1} = \mu a_0$ it converges to one of the stationary trajectories (3.2). For $d_{m+1} > \mu a_0$ the coordinates $z_i(\cdot), i = 1,\dots,m+1$ of the solution of the system (2.7)–(2.9) eventually converge to the stationary mode described in (3.1), while the function $z_0(\cdot)$ decreases linearly.

Proof.

Let's find the general solution of the system (2.7)–(2.9). We will start with the last equation, rewritten as follows

$$\dot{z}_{m+1}(t) + \lambda_{m+1}a_{m+1}z_{m+1}(t) = \lambda_{m+1}a_{m+1} - d_{m+1}. \tag{3.3}$$

The linear equation (3.3) has the following general solution

$$z_{m+1}(t) = 1 - \frac{d_{m+1}}{\lambda_{m+1}a_{m+1}} + c_{m+1}e^{-\lambda_{m+1}a_{m+1}t}. \tag{3.4}$$

Using this, let's find the solution to the penultimate equation of the system (2.7)–(2.9). It is easy to verify that if $\lambda_m a_m \neq \lambda_{m+1}a_{m+1}$, then it has the following form

$$z_m(t) = 1 - \frac{d_{m+1}}{\lambda_m a_m} + \frac{\lambda_{m+1}a_{m+1}c_{m+1}}{\lambda_m a_m - \lambda_{m+1}a_{m+1}}e^{-\lambda_{m+1}a_{m+1}t} + c_m e^{-\lambda_m a_m t}. \tag{3.5}$$

Otherwise ($\lambda_m a_m = \lambda_{m+1}a_{m+1}$) the solution will be as follows

$$z_m(t) = 1 - \frac{d_{m+1}}{\lambda_m a_m} + \lambda_{m+1}a_{m+1}c_{m+1}te^{-\lambda_m a_m t} + c_m e^{-\lambda_m a_m t}. \tag{3.6}$$

Similarly, we can find solutions to all the other equations of the system (2.7)–(2.9), except for the initial one. If all $\lambda_i a_i, i = 1, 2, \dots, m+1$ are pairwise distinct, then we obtain the following solutions:

$$z_{m-1}(t) = 1 - \frac{d_{m+1}}{\lambda_{m-1}a_{m-1}} + \frac{\lambda_m a_m \lambda_{m+1}a_{m+1}c_{m+1}}{(\lambda_m a_m - \lambda_{m+1}a_{m+1})(\lambda_{m-1}a_{m-1} - \lambda_{m+1}a_{m+1})}e^{-\lambda_{m+1}a_{m+1}t} +$$

$$\frac{\lambda_m a_m c_m}{\lambda_{m-1}a_{m-1} - \lambda_m a_m}e^{-\lambda_m a_m t} + c_{m-1}e^{-\lambda_{m-1}a_{m-1}t};$$

… …

$$z_{m-k}(t) = 1 - \frac{d_{m+1}}{\lambda_{m-k}a_{m-k}} +$$
$$\frac{\lambda_{m-k+1}a_{m-k+1}\lambda_{m-k+2}a_{m-k+2} \dots \lambda_{m+1}a_{m+1}c_{m+1}e^{-\lambda_{m+1}a_{m+1}t}}{(\lambda_m a_m - \lambda_{m+1}a_{m+1})(\lambda_{m-1}a_{m-1} - \lambda_{m+1}a_{m+1}) \dots (\lambda_{m-k}a_{m-k} - \lambda_{m+1}a_{m+1})} +$$
$$\frac{\lambda_{m-k+1}a_{m-k+1}\lambda_{m-k+2}a_{m-k+2} \dots \lambda_m a_m c_m e^{-\lambda_m a_m t}}{(\lambda_{m-1}a_{m-1} - \lambda_m a_m)(\lambda_{m-2}a_{m-2} - \lambda_m a_m) \dots (\lambda_{m-k}a_{m-k} - \lambda_m a_m)} + \quad (3.7)$$
$$\frac{\lambda_{m-k+1}a_{m-k+1}\lambda_{m-k+2}a_{m-k+2} \dots \lambda_{m-1}a_{m-1}c_{m-1}e^{-\lambda_{m-1}a_{m-1}t}}{(\lambda_{m-2}a_{m-2} - \lambda_{m-1}a_{m-1})(\lambda_{m-3}a_{m-3} - \lambda_{m-1}a_{m-1}) \dots (\lambda_{m-k}a_{m-k} - \lambda_{m-1}a_{m-1})} +$$
$$\dots + \frac{\lambda_{m-k+1}a_{m-k+1}c_{m-k+1}e^{-\lambda_{m-k+1}a_{m-k+1}t}}{(\lambda_{m-k}a_{m-k} - \lambda_{m-k+1}a_{m-k+1})} + c_{m-k}e^{-\lambda_{m-k}a_{m-k}t};$$

… …

$$z_1(t) = 1 - \frac{d_{m+1}}{\lambda_1 a_1} +$$
$$\frac{\lambda_2 a_2 \lambda_3 a_3 \dots \lambda_{m+1}a_{m+1}c_{m+1}e^{-\lambda_{m+1}a_{m+1}t}}{(\lambda_m a_m - \lambda_{m+1}a_{m+1})(\lambda_{m-1}a_{m-1} - \lambda_{m+1}a_{m+1}) \dots (\lambda_1 a_1 - \lambda_{m+1}a_{m+1})} +$$
$$\frac{\lambda_2 a_2 \lambda_3 a_3 \dots \lambda_m a_m c_m e^{-\lambda_m a_m t}}{(\lambda_{m-1}a_{m-1} - \lambda_m a_m)(\lambda_{m-2}a_{m-2} - \lambda_m a_m) \dots (\lambda_1 a_1 - \lambda_m a_m)} +$$
$$\frac{\lambda_2 a_2 \lambda_3 a_3 \dots \lambda_{m-1}a_{m-1}c_{m-1}e^{-\lambda_{m-1}a_{m-1}t}}{(\lambda_{m-2}a_{m-2} - \lambda_{m-1}a_{m-1})(\lambda_{m-3}a_{m-3} - \lambda_{m-1}a_{m-1}) \dots (\lambda_1 a_1 - \lambda_{m-1}a_{m-1})} +$$
$$\dots + \frac{\lambda_2 a_2 c_2 e^{-\lambda_2 a_2 t}}{(\lambda_1 a_1 - \lambda_2 a_2)} + c_1 e^{-\lambda_1 a_1 t}.$$

If $\lambda_1 a_1 = \lambda_2 a_2 = \cdots \lambda_{m+1}a_{m+1}$, then the solutions take the following form:

$$z_{m-1}(t) = 1 - \frac{d_{m+1}}{\lambda_{m-1}a_{m-1}} + e^{-\lambda_{m-1}a_{m-1}t}\left(\frac{\lambda_{m+1}a_{m+1}c_{m+1}}{2}t^2 + c_m t + c_{m-1}\right),$$

… …

$$z_{m-k}(t) = 1 - \frac{d_{m+1}}{\lambda_{m-k}a_{m-k}} + e^{-\lambda_{m-k}a_{m-k}t}\left(\frac{\lambda_{m+1}a_{m+1}c_{m+1}}{(k+1)!}t^{k+1} + c_m t^k + \cdots + c_2 t + c_1\right), \quad (3.8)$$

… …

$$z_1(t) = 1 - \frac{d_{m+1}}{\lambda_1 a_1} + e^{-\lambda_1 a_1 t}\left(\frac{\lambda_{m+1}a_{m+1}c_{m+1}}{m!}t^m + c_m t^{m-1} + \cdots + c_2 t + c_1\right).$$

In all other cases, the solutions have the following form

$$z_i(t) = 1 - \frac{d_{m+1}}{\lambda_i a_i} + G_i(t), \quad (3.9)$$

where $G_i(t)$ is the sum of functions of the form

$$b_p e^{-\lambda_p a_p t} \text{ and } c_l e^{-\lambda_j a_j t} t^k, p = i, \dots, m+1; j = i, \dots, m; k = 1, \dots, m-i+1 \quad (3.10)$$

From (3.4)–(3.10), it follows that

$$\lim_{t \to +\infty} z_i(t) = 1 - \frac{d_{m+1}}{\lambda_i a_i}, i = 1, 2, \dots, m.$$

We now need to solve the first equation of the system (2.7)–(2.9). To do this, let's rewrite it in the following form:

$$\dot{z}_0(t) = \begin{cases} \mu a_0 - \lambda_1 a_1(1 - z_1(t)), \text{if } z_0(t) < 1-\mu, t \in [t_0, +\infty), \\ a_0(1 - z_0(t)) - \lambda_1 a_1(1 - z_1(t)), \text{if } z_0(t) \geq 1-\mu\,, t\, \in [t_0, +\infty). \end{cases} \quad (3.11)$$

Let's consider the following two equations

$$\dot{z}_0(t) = \mu a_0 - \lambda_1 a_1(1 - z_1(t)), t \in [\bar{t}, +\infty), \quad (3.12)$$

$$\dot{z}_0(t) = a_0(1 - z_0(t)) - \lambda_1 a_1(1 - z_1(t)), t \in [\bar{t}, +\infty), \quad (3.13)$$

where $\bar{t} \geq t_0$.

Using the expressions for $z_1(t)$, we obtain the solutions to equations (3.12) and (3.13). They can be represented as follows

$$z_0(t) = (\mu a_0 - d_{m+1})t + F_1(t) + c_o \text{ where } F_1(t) \in \mathrm{C}^\infty[\bar{t}, +\infty), \lim_{t\to+\infty} F_1(t) = 0, \quad (3.14)$$

$$z_0(t) = 1 - \frac{d_{m+1}}{a_0} + F_2(t) \text{ where } F_2(t) \in \mathrm{C}^\infty[\bar{t}, +\infty), \lim_{t\to+\infty} F_2(t) = 0. \quad (3.15)$$

Using these solutions, let's investigate the asymptotic behavior of the solution to equation (3.11). When $d_{m+1} < \mu a_0$ the asymptotics of the solution to equation (3.11) is determined by relation (3.15), i.e., $\lim_{t\to+\infty} z_0(t) = 1 - \frac{d_{m+1}}{a_0}$. When $d_{m+1} > \mu a_0$ the asymptotics of the solution to equation (3.11) is determined by relation (3.14), i.e., $z_0(\cdot)$ decreases linearly with $\lim_{t\to+\infty} z_0(t) = -\infty$. However, if $d_{m+1} = \mu a_0$ then the asymptotics of the solution to equation (3.11) may be determined by either relation (3.14) or relation (3.15), depending on the initial conditions. In other words, either $\lim_{t\to+\infty} z_0(t) = \mathrm{c}_0$, where $\mathrm{c}_0 < 1-\mu$ or $\lim_{t\to+\infty} z_0(t) = 1 - \frac{d_{m+1}}{a_0}$. ■

4. INVESTIGATION OF SOLUTIONS OF THE SYSTEM (2.7)–(2.10)

Let's proceed to study the solutions of the system (2.7)–(2.9) that satisfy the constraints (2.10).

Lemma 4.1:

For all parameter values

$$a_i > 0, 0 < \lambda_i \leq 1, i = 1, 2, \dots, m+1, d_{m+1} > 0$$

satisfying the condition

$$\lambda_1 a_1 \geq \lambda_2 a_2 \geq \cdots \lambda_m a_m \geq \lambda_{m+1} a_{m+1} \geq d_{m+1}$$

the components $z_1(\cdot), z_2(\cdot), \dots\, z_{m+1}(\cdot)$ *of any solution of the system (2.7)–(2.10) that satisfy the constraints (2.10) at the initial moment in time will continue to satisfy them at subsequent moments in time.*

Proof.

Let's begin by examining the last component of the solution to the system (2.7)–(2.9), namely $z_{m+1}(\cdot)$. It takes the form (3.4), where c_{m+1} is determined by the condition

$$1 - \frac{d_{m+1}}{\lambda_{m+1} a_{m+1}} + c_{m+1} e^{-\lambda_{m+1} a_{m+1} t_0} = \bar{z}_{m+1}, \text{where } 0 \leq \bar{z}_{m+1} \leq 1,$$

i.e.

$$c_{m+1} = \left(\frac{d_{m+1}}{\lambda_{m+1} a_{m+1}} - 1 + \bar{z}_{m+1}\right) e^{\lambda_{m+1} a_{m+1} t_0}, \text{where } 0 \leq \bar{z}_{m+1} \leq 1 \quad (4.1)$$

From (4.1), it follows that

$$\left(\frac{d_{m+1}}{\lambda_{m+1}a_{m+1}} - 1\right) e^{\lambda_{m+1}a_{m+1}t_0} \le c_{m+1} \le \frac{d_{m+1}}{\lambda_{m+1}a_{m+1}} e^{\lambda_{m+1}a_{m+1}t_0}.$$

Using this estimate for c_{m+1} and the expression (3.4), we obtain an estimate for $z_{m+1}(\cdot)$. It will take the following form

$$1 - \frac{d_{m+1}}{\lambda_{m+1}a_{m+1}} - \left(1 - \frac{d_{m+1}}{\lambda_{m+1}a_{m+1}}\right) e^{\lambda_{m+1}a_{m+1}(t_0 - t)} \le z_{m+1}(t) \le 1 - \frac{d_{m+1}}{\lambda_{m+1}a_{m+1}}\left(1 - e^{\lambda_{m+1}a_{m+1}(t_0 - t)}\right) \quad (4.2)$$

From (4.2), it follows that when the condition $d_{m+1} \le \lambda_{m+1}a_{m+1}$ is met, the inequality holds

$$0 \le z_{m+1}(t) \le 1, t \in [t_0, +\infty). \quad (4.3)$$

Let's demonstrate that similar inequalities to (4.3) hold for the other components of the solution to the system (2.7)–(2.9). We'll start with the component $z_m(\cdot)$. To do this, consider the equation (2.8) for $i = m$:

$$\dot{z}_m(t) = \lambda_m a_m \left(1 - z_m(t)\right) - \lambda_{m+1}a_{m+1}\left(1 - z_{m+1}(t)\right), t \in [t_0, +\infty).$$

Let's demonstrate that the function $z_m(\cdot)$ cannot take a value greater than 1. Indeed, otherwise, due to the continuity of the function $z_m(\cdot)$ there must exist a point $t^* > t_0$, such that $z_m(t^*) = 1$. In such a case

$$\dot{z}_m(t^*) = -\lambda_{m+1}a_{m+1}\left(1 - z_{m+1}(t^*)\right)$$

and due to inequality (4.3), it follows that $\dot{z}_m(t^*) \le 0$, i.e. $z_m(\cdot)$ is upper-bounded by one. Let's now show that the function $z_m(\cdot)$ cannot take a value less than 0. Indeed, otherwise, due to the continuity of the function $z_m(\cdot)$ there must exist a poin $t^{**} > t_0$, such that $z_m(t^{**}) = 0$. Then

$$\dot{z}_m(t^{**}) = \lambda_m a_m - \lambda_{m+1}a_{m+1}\left(1 - z_{m+1}(t^{**})\right)$$

and when the condition $d_{m+1} \le \lambda_{m+1}a_{m+1} \le \lambda_m a_m$ is satisfied, the inequality $\dot{z}_m(t^{**}) \ge 0$, holds, i.e., $z_m(\cdot)$ is lower-bounded by zero.

Similarly, the validity of all the other inequalities (2.10) can be proven, except for the first one (related to the function $z_0(\cdot)$).■

Let's formulate a similar lemma for the zeroth component of the solution to the system (2.7)–(2.9).

Lemma 4.2:

For all parameter values $0 < \mu \le 1$, $a_i > 0, i = 0, 1, \dots m + 1$, *satisfying the condition* $a_0 \le a_1$, *there exists* $\tilde{\lambda}_1(\mu, a_0, a_1, z_0(t_0), z_1(t_0), \dots, z_{m+1}(t_0))$, $\frac{a_0}{a_1}\mu \le \tilde{\lambda}_1 \le 1$ *such that for all* λ_1 *in the interval* $\left[\frac{a_0}{a_1}\mu, \tilde{\lambda}_1\right]$, $0 < \lambda_i \le 1$, $i = 2, 3, \dots, m + 1$, $d_{m+1} > 0$, *satisfying the condition*

$$\lambda_1 a_1 \ge \lambda_2 a_2 \ge \cdots \lambda_m a_m \ge \lambda_{m+1}a_{m+1} \ge d_{m+1}$$

the zeroth component $z_0(\cdot)$ *of any solution to the system (2.7)–(2.9) that satisfies the constraint (2.10) at the initial moment in time will continue to satisfy it at subsequent moments in time.*

Proof.

Similarly to the other components, let's show that the function $z_0(\cdot)$ cannot take a value greater than 1. Indeed, otherwise, due to the continuity of the function $z_0(\cdot)$ there must exist a point $\tilde{t} > t_0$, such that $z_0(\tilde{t}) = 1$. Then, from (2.7), it follows that

$$\dot{z}_0(\tilde{t}) = \lambda_1 a_1(z_1(\tilde{t}) - 1),$$

i.e., according to Lemma 1, $\dot{z}_0(\tilde{t}) \leq 0$. This means that the function $z_0(\cdot)$ is upper-bounded by one.

Now, let's estimate the function $z_0(\cdot)$ from below. To do this, we'll investigate the behavior of its derivative as $z_0(\cdot) \to 0+$. According to (2.7), it is described by the equation

$$\dot{z}_0(t) = \mu a_0 - \lambda_1 a_1(1 - z_1(t)).$$

Let's analyze the inequality

$$\mu a_0 - \lambda_1 a_1(1 - z_1(t)) \geq 0.$$

We will show that for any $0 < \mu \leq 1$, $t \in [t_0, +\infty)$ there exists a range of λ_1 values on the half-open interval $(0, 1]$, for which this inequality holds. To do this, rewrite it as

$$z_1(t) \geq 1 - \frac{\mu a_0}{\lambda_1 a_1}. \tag{4.4}$$

According to Lemma 4.1,

$$0 \leq z_1(t) \leq 1, t \in [t_0, +\infty). \tag{4.5}$$

From (4.5), it follows that for any μ, satisfying $0 < \mu \leq 1$, and for any a_0, a_1, satisfying $a_0 \leq a_1$ there exists $\tilde{\lambda}_1$, $\frac{a_0}{a_1}\mu \leq \tilde{\lambda}_1 \leq 1$ such that for any value of λ_1 from the interval $\left[\frac{a_0}{a_1}\mu, \tilde{\lambda}_1\right]$ the inequality (4.4) will hold for all $t \in [t_0, +\infty)$, i.e. $\dot{z}_0(t) \geq 0$ as $z_0(t) \to 0+$. This demonstrates the lower bound of the function $z_0(\cdot)$ as 0. It is evident that $\tilde{\lambda}_1$ depends on the parameters μ, a_0, a_1, as well as the initial conditions, so we denote it as $\tilde{\lambda}_1(\mu, a_0, a_1, z_0(t_0), z_1(t_0), \ldots, z_{m+1}(t_0))$. ■

Let's state the main result of this study.

Theorem 4.1:

For any initial values $0 \leq z_i(t_0) \leq 1$, *parameters* $0 < \mu \leq 1$, $a_i > 0$, $i = 0, 1, \ldots, m+1$, *satisfying the condition* $a_0 \leq a_1$, *there exists* $\tilde{\lambda}_1(\mu, a_0, a_1, z_0(t_0), z_1(t_0), \ldots, z_{m+1}(t_0))$, $\frac{a_0}{a_1}\mu \leq \tilde{\lambda}_1 \leq 1$ *such that for all* λ_1 *in the interval* $\left[\frac{a_0}{a_1}\mu, \tilde{\lambda}_1\right]$, $0 < \lambda_i \leq 1$, $i = 2, 3, \ldots, m+1$, $d_{m+1} > 0$, *satisfying the condition*

$$\lambda_1 a_1 \geq \lambda_2 a_2 \geq \cdots \lambda_m a_m \geq \lambda_{m+1} a_{m+1} \geq d_{m+1}$$

the solution to the system (2.7)–(2.10) exists and converges to either the stationary solution (3.1) (when $d_{m+1} < \mu a_0$*) or to one of the stationary solutions (3.2), which is the same for all* d_{m+1} *and* a_i *(when* $d_{m+1} = \mu a_0$*).*

Proof.

The proof directly follows from Theorem 3.1, Lemma 4.1, and Lemma 4.2. ■

Corollary 1:

The system of differential equations (2.7)–(2.10) has a globally stable stationary solution (3.1) and a family of stable solutions of the form (3.2).

Proof.

The proof directly follows from Theorem 4.1. ■

Corollary 2:

For any initial values $0 \leq z_i(t_0) \leq 1$, *parameters* $a_i > 0$, $i = 0, 1, \ldots, m+1$, *satisfying the condition* $a_i \geq a_0$, $0 < \mu \leq 1$, $\bar{\lambda}_i = \frac{a_0}{a_i}\mu$, $i = 1, \ldots, m+1$ *and* $d_{m+1} = \mu a_0$ *the solution to the system (2.7)–(2.10) exists and converges to one of the stationary solutions of the form*

$$z_0(\cdot) \leq 1 - \mu, z_i(\cdot) \equiv 0, i = 1, \ldots, m+1.$$

Proof.

The proof directly follows from Theorems 3.1 and 4.1. ■

Let's move on to the interpretation of the main results obtained in this study. Before doing that, let's remind ourselves that the research on this model, described by the system (2.7)–(2.10), boils down to solving two fundamental tasks.

The first task is to determine the ranges of model parameter variations within which the mentioned system has a solution (ensuring uninterrupted freight flow). Theorem 4.1 defines these specified ranges.

The second task is to determine the most acceptable achievable levels of inconsistency between cargo reception and dispatch on all stations for a given value of the freight demand characteristic using control parameters. The solution to this task is presented in Corollary 2. According to it, for any values of inconsistency levels between cargo reception and dispatch at the initial time, if the capabilities of all stations to increase freight flow, starting from the first one, are not less than the capabilities of the zero station (initial node station), it is always possible to activate station potentials in such a way (by selecting values for the parameter λ_i) and align the cargo distribution mode with the final node station's demand for freight transport (by selecting the value for the parameter d_{m+1}), such that the degree of inconsistency at all stations except the initial node station will gradually become zero over time. The corresponding characteristic at the initial node station will depend on both the freight demand characteristic and the initial conditions

5. CONCLUSION

A dynamic model for organizing cargo transportation has been investigated on a segment of the railway network, which represents a railway line between two node stations. This model is represented by a system of differential equations that describe the dynamics of the discrepancy between cargo reception and dispatch at the stations. It includes a set of parameters that define the characteristics of cargo transportation demand, the technical capabilities of the stations, the extent of their utilization, as well as the mode of cargo distribution from the node station. Ranges of parameter variations have been determined within which the cargo transportation system can operate smoothly without interruptions. For each station, the most suitable level of utilizing its technical potential has been identified, allowing for a smooth and efficient flow of cargo and the ability to respond to changes in cargo transportation demand.

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