Adv Syst Sci Appl 2023; **04**:60–75
Published online at https://ijassa.ipu.ru.

Bayesian Estimation and Prediction from a Mixture of Weibull and Gompertz Distributions

Abdulqader Al-Dugin[1,2*], Mostafa Mohie El-Din[1], Amr Sadek[1]

[1]*Department of Mathematics, Al-Azhar University, Cairo, Egypt*
[2]*Department of Mathematics, Abyan University, Abyan, Yemen*

Abstract: We study different methods for estimation the parameters of a mixture of Weibull and Gompertz distributions as a lifetime model, based on a complete sample. Maximum likelihood estimation and Bayes estimation under informative and non-informative priors have been obtained using the symmetric squared error (SE) loss function, the asymmetric Linear exponential (LINEX) loss function and general entropy (GE) loss function. Also, we discuss two-sample Bayesian prediction intervals of the proposed model. For the illustration of the developing results, some computation results for the proposed model is presented.

Keywords: mixture model; loss function; weibull; Gompertz; maximum likelihood estimation; Bayesian estimation and prediction

1. INTRODUCTION

Mixtures models have received great attention from analysts in the recent years due to their important role in life testing and reliability. In many applications, mixture models are used to analyze random duration in possibly heterogeneous populations, statistical analysis and machine learning such as modeling, classification, and survival analysis. Attention has been paid by some authors to the finite mixtures to discuss lifetime distributions, [see, Everitt and Hand (1981),Titterington et al. (1985), Mclachlan and Basford (1988), Lindsay (1995), Mclachlan and Peel (2000)]. Also, mixture distributions have been considered extensively by several researchers using both classical and Bayesian techniques, for example, Shawky and Bakoban (2009), Abu-Zinadah (2010), Erisoglu et al.(2011), Feroze and Aslam (2014), Daniyal and Rajab (2015), Elshahat and Mahmoud (2016) and Mahmoud et al.(2017). The Weibull distribution has been widely used in modeling of lifetime event data; this is due to the variety of shapes of the probability density function (pdf) based on its parameters, Mohie El-Din et al. (2018) presented a new study on progressive-stress accelerated life testing for power generalized Weibull distribution under progressive Type-II censoring. The Weibull distribution has been shown to be useful for lifetime modeling and data analysis in applied engineering sciences, and also the static stress accelerated life test of the generalized Weibull distribution has been studied under step-wise type-II [see, Mohie El-Din et al.(2019), Murthy et al.(2003)]. The Gompertz distribution is used to model human survival and mortality times and actuarial tables. It has many real-life applications, particularly in medical and actuarial studies, and for statistical inference of the Gompertz distribution on the basis of Type- II Hybrid Progressively censored data [see, Mohie El-Din et al. (2017)]. The Gompertz distribution is also used as a survival model in reliability. It has an increased risk rate for the

*Corresponding author: abdulqader@azhar.edu.eg

life of the systems. Because of its complex shape, it has not received enough attention in the past. However, recently, this distribution has received a lot of attention from demographers and actuaries, and several studies have been presented in this field, including statistical inference and prediction of the Gompertz distribution based on multiply type- I censored data. Mohie El-Din and Abu- Moussa,(2018). The characterization for Gompertz distribution based on general progressively type-II right censored order statistics is done by Mohie El-Din,et al.(2017). Pollard and Valkovics (1992) were the first to deal with the Gompertz distribution thoroughly. However, their results are true only in cases where the initial level of mortality is very close to zero. Willemse and Koppelaar (2000) proposed a new formulation of epistemic elicitation of Gompertz's law of mortality.

If X has the same units and the same rates of variation of two distributions, then a random variable X is said to have a mixture distribution.

Let a random variable X is to have a mixture of two components Weibull and Gompertz distribution, with the probability density function (pdf) is given by:

$$f(x) = \sum_{j=1}^{2} p_j f_j(x); \; ; \quad j = 1, 2, \tag{1.1}$$

where

$$\begin{aligned} f_1(x) &= \alpha_1 \theta x^{\theta-1} e^{-\alpha_1 x^{\theta}}, \quad x > 0, \alpha_1 > 0, \theta > 0, \\ f_2(x) &= \alpha_2 e^{x-\alpha_2(e^x-1)}, \quad x > 0, \alpha_2 > 0. \end{aligned}$$

The mixing proportions p_j, are such that $0 \leq p_j \leq 1, \quad \sum_{j=1}^{2} p_j = 1.$

The corresponding cumulative distribution function(cdf) and reliability function, receptively are given by:

$$F(x) = \sum_{j=1}^{2} p_j F_j(x), \quad j = 1, 2, \tag{1.2}$$

where $F_1(x) = 1 - e^{-\alpha_1 x^{\theta}}$, $F_2(x) = 1 - e^{-\alpha_2(e^x-1)}$, and

$$R(x) = \sum_{j=1}^{2} p_j R_j(x), \quad j = 1, 2, \tag{1.3}$$

where $R_1(x) = e^{-\alpha_1 x^{\theta}}$, $R_2(x) = e^{-\alpha_2(e^x-1)}$.

The objective of this work is to apply the Bayesian procedure to estimate the parameters and obtain two sample prediction bounds for future observations from the proposed model, based on complete sample. The rest of this paper is organized as follows: In Section 2, we obtain maximum likelihood estimators of the proposed parameters. The Bayesian estimation is discussed in Section 3. Bayesian prediction presented in Section 4. Simulation study and real data presented in Section 5 to compare the performance of different estimation methods of the parameters. Finally, conclusions are presented in Section 6.

2. MAXIMUM LIKELIHOOD ESTIMATION

In this section, we present maximum likelihood estimators (MLEs) for the unknown parameters α_1, α_2 and p based on complete sample. Suppose a sample of n units are put on operation in life testing experiment and that the test is terminated if all n items taken from this population have failed. After the n units have failed each item can be attributed to the appropriate sub-population. Thus if the n units have failed during the interval $(0, x_{(n)})$; r_1 from the first sub-population and r_2 from the second sub-population. Let x_{ij} denote the failure of the jth unit that belongs to the ith sub-population and $x_{ij} \leq x_{(n)}$; $j = 1, 2, ..., r_i$; $i = 1, 2; n = r_1 + r_2$, where $x_{(n)}$ denotes the failure time of the nth unit.

For a two-component mixture model, the likelihood function is given by:

$$L(\alpha_1, \alpha_2, \theta, p \,|\underline{x}) = n!\, [\prod_{j=1}^{r_1} p_1\, f_1(x_{1j})][\prod_{j=1}^{r_2} p_2\, f_2(x_{2j})], \tag{2.4}$$

where

$$f_1(x_{1j}) = \alpha_1 \theta x_{1j}^{\theta-1}\, e^{-\alpha_1 x_{1j}^{\theta}}, \qquad x_{1j} > 0,\ \alpha_1 > 0, \quad \theta > 0,$$

$$f_2(x_{2j}) = \alpha_2 e_{2j}^{x} {-\alpha_2 (e^{x_{2j}-1)}}, \qquad x_{2j} > 0,\ \alpha_2 > 0,$$

then,

$$L(\alpha_1, \alpha_2, \theta, p|\underline{x}) \propto \prod_{j=1}^{r_1} p_1\, \alpha_1\, \theta\, x_{1j}^{\theta-1}\, e^{-\alpha_1 x_{1j}{}^{\theta}} \prod_{j=2}^{r_2} p_2\, \alpha_2\, e^{x_{2j} - \alpha_2 (e^{x_{2j}} - 1)}. \tag{2.5}$$

Put $p_1 = p,\ p_2 = 1 - p,$ and assuming that the parameter θ is known, the likelihood function (2.5) reduces to

$$L(\alpha_1, \alpha_2, p|\underline{x}, \theta) \propto \prod_{i=1}^{2} (p_i \alpha_i)^{r_i}\, e^{-\alpha_1 \sum_{j=1}^{r_1} x_{1j}^{\theta}}\; e^{-\alpha_2 \sum_{j=1}^{r_2} (e^{x_{2j}} - 1)}. \tag{2.6}$$

Thus, the log-likelihood function of parameters α_1, α_2 and p are given by:

$$\ln L = \ln L(p, \alpha_1, \alpha_2|\underline{x}) \propto \sum_{i=1}^{2} \{r_i \ln p_i {+} r_i \ln \alpha_i\} - \alpha_1 \sum_{j=1}^{r_1} x_{1j}^{\theta} - \alpha_2 \sum_{j=1}^{r_2} (e^{x_{2j}} {-} 1). \tag{2.7}$$

Taking derivatives with respect to $\alpha_1,\ \alpha_2$ and p in Equation (2.7), and Equating by zero, the maximum likelihood estimators of the three parameters are obtained as follows

$$\hat{\alpha}_1 = \frac{r_1}{\sum_{j=1}^{r_1} x_{ij}^{\theta}}, \qquad \hat{\alpha}_2 = \frac{r_2}{\sum_{j=1}^{r_2} (e^{x_{2j}} - 1)}, \qquad \hat{p} = \frac{r_1}{r_1 + r_2}.$$

3. BAYESIAN ESTIMATION

In this section, we derive Bayesian estimators of the parameters α_1, α_2 and p of the considered model by using various priors based on different symmetric and asymmetric loss functions.

3.1. Loss function

In decision theory, the loss criterion is specified in order to obtain the best estimator. Three loss functions are proposed, symmetric squared error loss function (SE) and asymmetric (LINEX and general entropy) loss functions, as follows:

• Squared error loss function (ES): A simple, and very common loss function is defined by

$$L_1(\hat{\theta}\,,\theta) = c\,(\hat{\theta}\ -\theta)^2,$$

where c is constant which is symmetrical in nature and gives equal weight to overestimation as well as underestimation.

However, in real applications, estimation of reliability and failure rate functions, an overestimate is more serious than the underestimate. The use of symmetric loss function might be inappropriate as has been recognized by Basu and Ebrahimi (1991).

• Linear exponential loss function (LINEX): One of the most commonly used asymmetric loss functions, introduced by Varian (1975) under the assumption that the minimal loss occurs at $\hat{\theta} = \theta$, it can be expressed as

$$L_2(\hat{\theta}\,,\theta) \propto\ e^{-q(\ \hat{\theta}\ -\theta\ )} - q(\ \hat{\theta}\ -\theta) - 1, \quad q \neq 0,$$

where q determines the shape of the loss function. It is used in both overestimation and underestimation, if $q > 0$ means overestimation and if $q < 0$, means underestimation but in a situation where $q = 0$, the LINEX loss function is almost symmetric and approaches squared error loss function.

Under the above loss function, the Bayes estimator $\hat{\theta}_{LINEX}$ of θ can be obtained as

$$\hat{\theta}_{LINEX} = -\frac{1}{q}\ \ln\ [E\ (e^{-q\theta}|\underline{x})],$$

provided that the expected value with respect to the posterior function of $\theta, E(e^{-q\theta}|\underline{x})$ exists and is finite.

• General entropy loss function (GE): Another commonly asymmetric loss function is the modified LINEX loss function called a general entropy loss function proposed by Calabria and Pulcini (1996).

$$L_3(\hat{\theta},\theta) \propto \left(\frac{\hat{\theta}}{\theta}\right)^h - h\ln\left(\frac{\hat{\theta}}{\theta}\right) - 1, \quad h \neq 0,$$

which has a minimum at $\hat{\theta} = \theta$. Also, this loss function used by several authors, in the original form having the shape parameter $h = 1$, for $h > 0$, a positive error has a more effect than a negative error. In this case, the Bayes estimate of θ is given by:

$$\hat{\theta}_{GE} = \left[E\left(\theta^{-h}|\underline{x}\right)\right]^{-\frac{1}{h}},$$

provided that the expected value with respect to the posterior function of $\theta, E\ \left(\theta^{-h}|\underline{x}\right)$ exists and is finite.

3.2. The posterior distribution under the informative prior

Assume the prior distribution of the parameters α_1, α_2 and p are $\alpha_1 \sim \Gamma(a_1, b_1)$, $\alpha_2 \sim \Gamma(a_2, b_2)$, and $p \sim \beta(c, d)$ for the mixing parameter, $p_i;\ i = 1, 2$ where $p_1 = p$ and $p_2 = 1 - p$. Assuming, now, the independence of parameters, the joint distribution prior for α_1, α_2 and p is:

$$\pi(\alpha_1, \alpha_2\ ,\ p) = \pi_1(\alpha_1)\pi_2(\alpha_2)\pi_3(p), \tag{3.8}$$

where

$$\pi_i(\alpha_i) = \frac{b_i^{a_i}}{\Gamma(a_i)}\alpha_i^{a_i-1}e^{-b_i\alpha_i}, \qquad \alpha_i > 0,\ a_i, b_i > 0, \qquad i = 1, 2,$$
$$\pi_3(p) = \frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)}p^{c-1}(1-p)^{d-1}, \qquad 0 < p < 1,\ c, d > 0.$$

Then, the joint prior distribution of α_1, α_2 and p can be written as follows

$$\pi\,(\alpha_1, \alpha_2\,,\,p) \propto \left[\prod_{i=1}^{2} \alpha_i^{a_i-1}\; e^{-b_i\alpha_i}\right] \times\; p^{\,c-1}\,(\,1-p)^{\,d-1}. \tag{3.9}$$

It follows from (2.6) and (3.9), that the joint posterior density function of α_1, α_2 and p is given by

$$g\,(\alpha_1, \alpha_2, p|\underline{x}) = k_1^{-1}\; p^{r_1+c-1}\,(1-p)^{r_2+d-1}\; \alpha_1^{r_1+a_1-1}\; \alpha_2^{r_2+a_2-1}\; e^{-\alpha_1\phi_1}\; e^{\,-\alpha_2\phi_2}, \tag{3.10}$$

where, k_1 is the normalizing constant given by

$$k_1 = \beta\,(r_1{+}c\,,\; r_2{+}d)\;\frac{\Gamma\,(r_1{+}a_1)\;\Gamma\,(r_2{+}a_2)}{(\phi_1)^{r_1+a_1}\,(\phi_2)^{r_2+a_2}},$$

with $\phi_1 = b_1 + \sum_{j=1}^{r_1} x_{1j}^{\,\theta}$, and $\phi_2 = b_2 + \sum_{j=1}^{r_2}(e^{x_{2j}} - 1)$.

3.2.1. Bayes estimator under squared error loss function (SE)

The Bayes estimators of α_1, α_2 and p based on the squared error loss function are given by:

$$\hat{\alpha}_{1_{SE}} = k_1^{-1}\;\beta(r_1{+}c\;\;,\;r_2{+}d)\frac{\Gamma(r_1{+}a_1{+}1)\quad\Gamma\,(r_2{+}a_2)}{(\phi_1)^{r_1+a_1+1}(\phi_2)^{r_2+a_2}}\;,$$

$$\hat{\alpha}_{2_{SE}} = k_1^{-1}\;\beta(r_1{+}c\;\;,\;r_2{+}d)\frac{\Gamma\,(r_1{+}a_1)\,\Gamma(r_2{+}a_2{+}1)}{(\phi_1)^{r_1+a_1}(\phi_2)^{r_2+a_2+1}},$$

$$\hat{p}_{SE} = k_1^{-1}\;\beta(r_1{+}c+1\;,r_2{+}d\,)\frac{\Gamma\,(r_1{+}a_1)\,\Gamma\,(r_2{+}a_2)}{(\phi_1)^{r_1+a_1}(\phi_2)^{r_2+a_2}}.$$

3.2.2. Bayes estimator under Linex loss function (LINEX)

The Bayes estimators of α_1, α_2 and p based on the LINEX loss function are given by:

$$\hat{\alpha}_{1_{LINEX}} = -\frac{1}{q}\,\ln\,\left[k_1^{-1}\;\beta(r_1{+}c\;\;,r_2{+}d\,)\frac{\Gamma\,(r_1{+}a_1)\,\Gamma\,(r_2{+}a_2)}{(\phi_1{+}q)^{r_1+a_1}(\phi_2)^{r_2+a_2}}\right],$$

$$\hat{\alpha}_{2_{LINEX}} = -\frac{1}{q}\,\ln\,\left[k_1^{-1}\;\beta\,(r_1{+}c\,,\;r_2{+}d)\;\;\frac{\Gamma\,(r_1{+}a_1)\;\Gamma\,(r_2{+}a_2)}{(\phi_1)^{r_1+a_1}(\phi_2{+}q)^{r_2+a_2}}\right],$$

$$\hat{p}_{LINEX} = -\frac{1}{q}\ln\,\left[k_1^{-1}\;\frac{\Gamma\,(r_1{+}a_1)\,\Gamma\,(r_2+a_2)}{(\phi_1)^{r_1+a_1}(\phi_2)^{r_2+a_2}}\;\sum_{k=0}^{\infty}\frac{(-q)^k}{k!}\beta(r_1{+}c+k\;,\;r_2{+}d)\right].$$

3.2.3. Bayes estimator under general entropy loss function (GE)

The Bayes estimator of α_1, α_2 and p based on general entropy loss function are given by:

$$\hat{\alpha}_{1_{GE}}=\left[k_1^{-1}\,\beta\,(r_1+c\ ,\ r_2+d)\,\frac{\Gamma(r_1+a_1-h\ )\Gamma\,(r_2+a_2)}{(\phi_1)^{r_1+a_1-h}(\phi_2)^{r_2+a_2}}\right]^{-\frac{1}{h}},$$

$$\hat{\alpha}_{2_{GE}}=\left[k_1^{-1}\,\beta\,(r_1+c, r_2+d)\ \frac{\Gamma\,(r_1+a_1)\,\Gamma(r_2+a_2-h)}{(\phi_1)^{r_1+a_1}\,(\phi_2)^{r_2+a_2-h}}\right]^{-\frac{1}{h}},$$

$$\hat{p}_{GE}=\left[k_1^{-1}\,\beta(r_1+c-h\ \ ,\ r_2+d)\frac{\Gamma\,(r_1+a_1)\,\Gamma\,(r_2+a_2)}{(\phi_1)^{r_1+a_1}(\phi_2)^{r_2+a_2}}\right]^{-\frac{1}{h}}.$$

3.3. The posterior distribution under the non-informative prior

Jeffery's prior is a formal rule for obtaining a non-informative prior. The non-informative priors are recommended when there is no formal prior information about the parameters. This is defined as the distribution of the parameters proportional to the square root of the determinants of the Fisher information matrix. The prior distribution for the mixing parameter $p, i.e., p \sim$ uniform(0,1). Assuming independence, the joint prior distribution is given by:

$$\begin{aligned}\pi_i(\alpha_i) &\propto \frac{1}{a_i}, && \alpha_i > 0, \qquad i = 1, 2,\\ \pi_3(p) &= 1, && 0 < p < 1.\end{aligned} \tag{3.11}$$

Then, the joint prior distribution of α_1, α_2 and p can be written as follows

$$\pi\ (\alpha_1, \alpha_2, p)\propto\frac{1}{\alpha_1\alpha_2}\ \ , \alpha_1, \alpha_2> 0\ , 0 < p < 1. \tag{3.12}$$

It follows from (2.6) and (3.12) that, the joint posterior density function of α_1, α_2 and p is given by

$$g\ (p, \alpha_1, \alpha_2|\underline{x}) \propto k_2^{-1}\ p^{r_1}\ (1-p)^{r_2}\ \alpha_1^{r_1-1}\ \alpha_2^{r_2-1}\ \ e^{-\alpha_1\phi_1^*}\ e^{-\alpha_2\phi_2^*}, \tag{3.13}$$

where, k_2 is the normalizing constant given by $k_2 = \beta(r_1+1\ ,\ r_2+1\ )\ \frac{\Gamma\,(r_1)\,\Gamma\,(r_2)}{(\phi_1^*)^{r_1}\,(\phi_2^*)^{r_2}}$, with $\phi_1^*= \sum_{j=1}^{r_1} x_{1j}^{\theta}$ and $\phi_2^*= \sum_{j=1}^{r_2}(e^{x_{2j}}-1)$.

3.3.1. Bayes estimator under squared error loss function (SE)

The Bayes estimators of α_1, α_2 and p based on the squared error loss function are given by:

$$\hat{\alpha}_{1_{SEN}} = k_2^{-1}\ \beta(r_1+1, r_2+1)\ \ \frac{\Gamma(r_1+1)\Gamma\,(r_2)}{(\phi_1^*)^{r_1+1}\,(\phi_2^*)^{r_2}},$$

$$\hat{\alpha}_{2_{SEN}} = k_2^{-1}\ \beta(r_1+1\ ,\ \ r_2+1)\ \frac{\Gamma\,(r_1)\,\Gamma\,(r_2+1)}{(\phi_1^*)^{r_1}\,(\phi_2^*)^{r_2+1}},$$

$$\hat{p}_{SEN} = k_2^{-1}\ \beta(r_1+2\ ,\ r_2+1)\ \frac{\Gamma\,(r_1)\,\Gamma\,(r_2)}{(\phi_1^*)^{r_1}\,(\phi_2^*)^{r_2}}.$$

3.3.2. Bayes estimator under Linex loss function (LINEX)

The Bayes estimators of α_1, α_2 and p based on the LINEX loss function are given by:

$$\hat{\alpha}_{1_{LINEXN}} = -\tfrac{1}{q} \ln \left[k_2^{-1} \beta\,(r_1{+}1,\ r_2{+}1)\ \frac{\Gamma(r_1)\Gamma\,(r_2)}{(\phi_1^{*}{+}q)^{r_1}\ (\phi_2^{*})^{r_2}} \right],$$

$$\hat{\alpha}_{2_{LINEXN}} = -\tfrac{1}{q} \ln \left[k_2^{-1}\ \beta\,(r_1{+}1\ , r_2{+}1)\ \frac{\Gamma\,(r_1)\,\Gamma(r_2)}{(\phi_1^{*})^{r_1}\ (\phi_2^{*}{+}q)^{r_2}} \right],$$

$$\hat{p}_{LINEXN} = -\tfrac{1}{q} \ln \left[k_2^{-1} \frac{\Gamma(r_1)\Gamma(r_2)}{(\phi_1^{*})^{r_1}(\phi_2^{*})^{r_2}} \sum_{k=0}^{\infty} \frac{(-q)^k}{k!} \beta(r_1{+}k+1\ ,\ r_2{+}1) \right].$$

3.3.3. Bayes estimator under general entropy loss function (GE)

The Bayes estimators of α_1, α_2 and p based on general entropy loss function are given by:

$$\hat{\alpha}_{1_{GEN}} = \left[k_2^{-1}\ \beta(r_1{+}1, r_2{+}1)\ \frac{\Gamma(r_1{-}h)\Gamma(r_2)}{(\phi_1^{*})^{r_1-h}(\phi_2^{*})^{r_2}} \right]^{-\frac{1}{h}},$$

$$\hat{\alpha}_{2_{GEN}} = \left[k_2^{-1}\ \beta(r_1{+}1\ ,\ r_2{+}1) \frac{\Gamma(r_1)\Gamma(r_2{-}h)}{(\phi_1^{*})^{r_1}(\phi_2^{*})^{r_2-h}} \right]^{-\frac{1}{h}},$$

$$\hat{p}_{GEN} = \left[k_2^{-1} \beta(r_1 - h + 1, r_2 + 1) \frac{\Gamma(r_1)\Gamma(r_2)}{(\phi_1^{*})^{r_1}(\phi_2^{*})^{r_2}} \right]^{-\frac{1}{h}}.$$

4. BAYESIAN PREDICTION

In this section, the Bayesian two-sample prediction of a future order statistics sample considered on the basis of observed data used in the informative and non-informative prior. A random sample of size m of future observation, independent sample of size n, is drawn from the same population with Eq(1.1). Therefore Y_s represents the s^{th} ordered statistic in the future sample of size m, $1 \leqslant s \leqslant m$. The s^{th} order statistic in a sample of size m represents the life length of a $(m - s + 1)$ out of m system. The distribution function of Y_s , the ordered future sample is given by (See, Arnold et al. (1992) and Jaheen (2003)),

$$\begin{aligned} F_{Y_s}(y_s|\alpha_1, \alpha_2, p) &= \sum_{l=s}^{m} \binom{m}{l} [F_X(y_s|\alpha_1, \alpha_2, p)]^l\ [1 - F_X(y_s|\alpha_1, \alpha_2, p)]^{m-l} \\ &= \sum_{l=s}^{m} \sum_{j_1=0}^{l} \binom{m}{l} \binom{l}{j_1} (-1)^{j_1} [R(y_s)]^{m-l+j_1}, \end{aligned} \tag{4.14}$$

where $F_X(y_s|\alpha_1, \alpha_2, p) = 1 - R(y_s)$ is the distribution function of the mixture model and $R(y_s)$ is the reliability function of the mixture model after replacing x by y_s.

Using the binomial expansion for $[R(y_s)]^{m-l+j_1}$ as follows:

$$\begin{aligned} [R(y_s)]^{m-l+j_1} &= \left[p_1\ e^{-\alpha_1 y_s^{\theta}} + p_2\ e^{-\alpha_2 (e^{y_s}-1)} \right]^{m-l+j_1}, \\ &= \sum_{j_2=0}^{m-l+j_1} \binom{m-l+j_1}{j_2} p_1^{\delta_1}\ p_2^{j_2}\ e^{-\alpha_1 \delta_1 y_s^{\theta}}\ e^{-\alpha_2 j_2 (e^{y_s}-1)}. \end{aligned} \tag{4.15}$$

Therefore, we get

$$F_{Y_s}(y_s|\alpha_1,\alpha_2,p) = \sum_{l=s}^{m}\sum_{j_1=0}^{l}\sum_{j_2=0}^{m-L+j_1}\binom{m}{l}\binom{l}{j_1}\binom{m-l+j_1}{j_2}(-1)^{j_1} p_1^{\delta_1}\,p_2^{j_2}\,(e^{-\alpha_1 y_s^{\theta}})^{\delta_1}\,(e^{-\alpha_2(e^{y_s}-1)})^{j_2}, \tag{4.16}$$

where $\delta_1 = m - l + j_1 - j_2$.

$$f^*(y_s|\underline{x}) = \int_0^1\int_0^\infty\int_0^\infty f(y_s|\alpha_1,\alpha_2,p)\; g(\alpha_1,\alpha_2,p|\underline{x})\,d\alpha_1 d\alpha_2 dp, \tag{4.17}$$

where $g(\alpha_1,\alpha_2,p|\underline{x})$ is the joint posterior density for parameters α_1,α_2 and p and $f(y_s|\alpha_1,\alpha_2,p)$ is the pdf of s^{th} component in a future sample. Therefore, Bayesian prediction density of Y_s for a given value v, can be obtained as:

$$P[Y_s \geq v|\underline{x}] = \int_v^\infty f^*(y_s|\underline{x})dy_s$$
$$= 1 - \int_0^1\int_0^\infty\int_0^\infty F_{Y_s}(v|\alpha_1,\alpha_2,p)\,g(\alpha_1,\alpha_2,p|\underline{x})\,d\alpha_1 d\alpha_2 dp, \tag{4.18}$$

Substitution of (3.9) and (4.16) in (4.18), we get Bayes predictive distribution bounds with value v for Y_s in case of informative as

$$P[Y_s \geq v\,|\underline{x}] = 1 - k_1^{-1}\sum B\;\beta(\delta_1+\delta_2,\delta_3)\,\frac{\Gamma(r_1+a_1)\,\Gamma(r_2+a_2)}{(\phi_1^{**})^{r_1+a_1}\,(\phi_2^{**})^{r_2+a_2}}.$$

where $\delta_2 = r_1 + c, \delta_3 = r_2 + d + j_2$.

Substitution of (3.11) and (4.16) in (4.18), we get Bayes predictive distribution bounds with value v for Y_s in case of non- informative as

$$p[Y_s \geq v\,|\underline{x}] = 1 - k_2^{-1}\sum B\;\beta(\delta_1^*+1\,,\;\delta_2^*+1)\,\frac{\Gamma(r_1)\,\Gamma(r_2)}{(\phi_1^{***})^{r_1}\,(\phi_2^{***})^{r_2}}.$$

where

$$\sum = \sum_{l=s}^{m}\sum_{j_1=0}^{l}\sum_{j_2}^{m-l+j_1}\quad,\; B = \binom{m}{l}\binom{l}{j_1}\binom{m-l+j_1}{j_2}(-1)^{j_1}$$
$$\delta_1^* = \delta_1 + r_1 \quad,\quad \delta_2^* = j_2 + r_2$$
$$\phi_1^{**} = \delta_1 v^\theta + \phi_1 \quad,\quad \phi_2^{**} = j_2(e^v - 1) + \phi_2$$
$$\phi_1^{***} = \delta_1 v^\theta + \phi_1^* \quad,\quad \phi_1^{***} = j_2(e^v - 1) + \phi_2^*$$

A $100\,\gamma\%$ prediction interval for Y_s is given by

$$P[L(\underline{x}) < Y_s < U(\underline{x})] = \gamma,$$

where $L(\underline{x})$ and $U(\underline{x})$ are obtained respectively by solving the following two equations:

$$P[Y_s > L(\underline{x})] = \frac{1+\gamma}{2} \text{ and } P[Y_s > U(\underline{x})] = \frac{1-\gamma}{2}. \tag{4.19}$$

5. SIMULATION STUDY AND REAL DATA

In this section, we present some simulation results to compare the performance of various estimates by using the Monte Carlo simulation method from the mixture of Weibull and Gompertz distributions. The following steps were considered:

1. For the parameter we have considered $(\alpha_1, \alpha_2, p) = (0.2, 0.1, 0.7)$ along with θ is known and $\theta = 0.61$. The values of q and h are (0.5, -0.5) for LINEX and general entropy loss functions. The method of choosing the hyper parameters values introduced in Ahmadi et al.(2020). Let $(a_1 = 0.03,\ a_2 = 0.04,\ b_1 = 0.2,\ b_2 = 0.35,\ c = 96.23,$ and $d = 41.167)$ for informative prior. In case of non-informative prior, we take $(a_1 = a_2 = b_1 = b_2 = 0; c = d = 1)$. In all these cases samples of size $n = 20, 40, 60, 80, 100$ and 150, are generated.
2. Generate a uniform random number u from the interval(0,1).
3. If $u \leqslant p$ the observation has been randomly taken from first sub-population and if $u > p$ then the observation have been taken from the second sub-population.
4. The obtained results for maximum likelihood estimates and Bayes estimates are calculated.
5. Bayesian prediction for the future observations of Y_s are obtained by solving numerically, Equations (4.6) with $\gamma = 0.95$.
6. The above steps are repeated 1000 times, and the average of the estimates are calculated and presented in the Tables (5.1)-(5.3). Also the average lower and average upper interval of Y_s when $s = \frac{m}{2}$ is even or $\frac{m+1}{2}$ is odd for different sample size n and different future sample size m, simulated coverage probability and average interval lengths are obtained in Tables (5.4)-(5.5). All results were obtained by using Mathematica 10.

Table 5.1. Average estimates and corresponding MSE of the parameter $\alpha_1 = 0.2$ based on informative and non- informative Prior

n	MLE	Bayes									
		SE		LINEX				GE			
		informative	non-informative	informative		non-informative		informative		non-informative	
				q=-0.5	q=0.5	q=-0.5	q=0.5	h=-0.5	h=0.5	h=-0.5	h=0.5
20	0.217318	0.217035	0.21770	0.217996	0.216086	0.218285	0.216363	0.213051	0.205008	0.21332	0.205247
	(0.00451)	(0.00446)	(0.00475)	(0.00458)	(0.00435)	(0.00463)	(0.0044)	(0.00418)	(0.00372)	(0.00423)	(0.00376)
40	0.209797	0.209695	0.21531	0.210106	0.209287	0.210209	0.209388	0.207823	0.204061	0.207922	0.204154
	(0.00176)	(0.00175)	(0.00273)	(0.00177)	(0.00173)	(0.00178)	(0.0017)	(0.00169)	(0.00158)	(0.00169)	(0.00159)
60	0.205851	0.205791	0.20781	0.206052	0.205531	0.206113	0.205591	0.204558	0.202084	0.204617	0.202141
	(0.00107)	(0.00107)	(0.00116)	(0.00108)	(0.00106)	(0.00108)	(0.0011)	(0.00105)	(0.00101)	(0.00105)	(0.00101)
80	0.202925	0.202884	0.20450	0.203072	0.202696	0.203113	0.202737	0.2019757	0.200154	0.202016	0.200194
	(0.00076)	(0.00076)	(0.00098)	(0.00077)	(0.00076)	(0.00077)	(0.0008)	(0.00075)	(0.00073)	(0.00075)	(0.00074)
100	0.202708	0.202675	0.20293	0.202825	0.202526	0.202858	0.202558	0.201949	0.200494	0.201981	0.200525
	(0.00063)	(0.00062)	(0.00067)	(0.00063)	(0.00062)	(0.00063)	(0.0006)	(0.00062)	(0.0006)	(0.00062)	(0.00061)
150	0.201981	0.200088	0.20288	0.202024	0.20209	0.202079	0.201883	0.200249	0.200195	0.198195	0.200575
	(0.00041)	(0.00078)	(0.00052)	(0.00055)	(0.00042)	(0.00041)	(0.0004)	(0.00064)	(0.00044)	(0.00106)	(0.0004)

Table 5.2. Average estimates and corresponding MSE of the parameter $\alpha_2 = 0.1$ based on informative and non- informative Prior

n	MLE	Bayes									
		SE		LINEX				GE			
		informative	non-informative	informative		non-informative		informative		non-informative	
				q=-0.5	q=0.5	q=-0.5	q=0.5	h=-0.5	h=0.5	h=-0.5	h=0.5
20	0.122528	0.121892	0.12360	0.123059	0.120817	0.123818	0.121374	0.116016	0.103891	0.116521	0.104117
	(0.00532)	(0.004878)	(0.00641)	(0.00532)	(0.00454)	(0.00594)	(0.00488)	(0.00401)	(0.00273)	(0.00432)	(0.00287)
40	0.111112	0.11105	0.11245	0.111384	0.110721	0.111449	0.11078	0.10856	0.103519	0.10861	0.103544
	(0.00183)	(0.001781)	(0.00289)	(0.00183)	(0.00173)	(0.00188)	(0.00178)	(0.00163)	(0.00139)	(0.00168)	(0.00142)
60	0.108024	0.108021	0.10953	0.108201	0.107841	0.108205	0.107844	0.106479	0.103373	0.106479	0.103365
	(0.00079)	(0.00079)	(0.00095)	(0.0008)	(0.00078)	(0.0008)	(0.00079)	(0.00074)	(0.00067)	(0.00075)	(0.00067)
80	0.103682	0.103692	0.10633	0.103813	0.103572	0.103803	0.103562	0.102581	0.100348	0.10257	0.100332
	(0.00043)	(0.000433)	(0.00079)	(0.00044)	(0.00043)	(0.00044)	(0.00043)	(0.00042)	(0.00039)	(0.00042)	(0.00039)
100	0.103475	0.103483	0.10521	0.103578	0.103389	0.103569	0.10338	0.102607	0.100849	0.102598	0.100837
	(0.00041)	(0.000411)	(0.00052)	(0.00041)	(0.00041)	(0.00041)	(0.00041)	(0.0004)	(0.00038)	(0.0004)	(0.00038)
150	0.101673	0.101554	0.10172	0.101613	0.101495	0.101733	0.101614	0.100986	0.0998459	0.101101	0.099953
	(0.00027)	(0.000266)	(0.00029)	(0.00027)	(0.00027)	(0.00027)	(0.00027)	(0.00026)	(0.00026)	(0.00027)	(0.00026)

Table 5.3. Average estimates and corresponding MSE of the parameter $p = 0.7$ based on informative and non- informative Prior

n	MLE	Bayes									
		SE		LINEX				GE			
		informative	non-informative	informative		non-informative		informative		non-informative	
				q=-0.5	q=0.5	q=-0.5	q=0.5	h=-0.5	h=0.5	h=-0.5	h=0.5
20	0.69115	0.699258	0.69973	0.699589	0.6989326	0.676418	0.671834	0.69878	0.367819	0.670447	0.662696
	(0.01073)	(0.000173)	(0.00947)	(0.00017)	(0.00017)	(0.0093)	(0.00967)	(0.00017)	(0.00018)	(0.009988)	(0.01083)
40	0.7034	0.701061	0.70132	0.701354	0.700767	0.694919	0.692503	0.70064	0.699792	0.691894	0.68816
	(0.0052)	(0.00026)	(0.00474)	(0.00027)	(0.00026)	(0.00471)	(0.0048)	(0.00027)	(0.00027)	(0.00483)	(0.00502)
60	0.697767	0.699586	0.69986	0.69985	0.699321	0.692219	0.690552	0.699205	0.698441	0.690144	0.687615
	(0.00346)	(0.00032)	(0.00331)	(0.00032)	(0.00032)	(0.00329)	(0.0033)	(0.00032)	(0.00032)	(0.00336)	(0.00347)
80	0.700113	0.700282	0.70053	0.700521	0.700042	0.695861	0.6946	0.699937	0.699244	0.694303	0.692422
	(0.00272)	(0.00037)	(0.00261)	(0.00037)	(0.00037)	(0.0026)	(0.0026)	(0.00037)	(0.00037)	(0.00264)	(0.0027)
100	0.69908	0.699832	0.69991	0.700052	0.699612	0.695685	0.694666	0.699516	0.698881	0.69443	0.692921
	(0.00203)	(0.00036)	(0.00197)	(0.00036)	(0.00036)	(0.00197)	(0.002)	(0.00036)	(0.00036)	(0.00199)	(0.00203)
150	0.700313	0.699868	0.69998	0.699949	0.699786	0.69802	0.697335	0.699506	0.698982	0.697181	0.69618
	(0.00136)	(0.00035)	(0.00132)	(0.00035)	(0.00035)	(0.00132)	(0.0013)	(0.00035)	(0.00035)	(0.00133)	(0.00135)

Table 5.4. The 95% Bayesian prediction bounds, length of Bayesian prediction and their simulated coverage probability for Ys based informative prior

(n,m)	Y1			Y4			Ym		
	(Lower, Upper)	Length	Coverage	(Lower, Upper)	Length	Coverage	(Lower, Upper)	Length	Coverage
(15,8)	(0.000303,0.4654)	0.465106	0.934	(0.103358,1.96123)	1.85788	0.934	(0.620589,34.0201)	33.3995	0.977
(20,8)	(0.000275,0.45017)	0.449896	0.936	(0.106456,1.90021)	1.79375	0.935	(0.760848,32.4311)	31.6703	0.967
(30,8)	(0.000265,0.44291)	0.442654	0.945	(0.109328,1.86521)	1.75588	0.945	(0.463508,30.5226)	30.0591	0.981
(50,8)									
(n,m)	Y1			Y5			Ym		
	(Lower, Upper)	Length	Coverage	(Lower, Upper)	Length	Coverage	(Lower, Upper)	Length	Coverage
(15,10)	(0.000216,0.36387)	0.363661	0.946	(0.130549,1.82091)	1.69036	0.933	(0.879757,39.4524)	38.5727	0.958
(20,10)	(0.0002006,0.3515)	0.351314	0.941	(0.137363,1.78166)	1.6443	0.944	1.05329,34.2686)	33.2153	0.957
(30,10)	(0.000190,0.34323)	0.343049	0.951	(0.141384,1.72623)	1.58484	0.936	(0.936194,32.7176)	31.7814	0.963
(50,10)	(0.000174,0.33177)	0.331604	0.946	(0.145217,1.69538)	1.55017	0.937	(0.779346,34.6254)	33.8461	0.983
(n,m)	Y1			Y6			Ym		
	(Lower, Upper)	Length	Coverage	(Lower, Upper)	Length	Coverage	(Lower, Upper)	Length	Coverage
(15,12)	(0.000165,0.29767)	0.297512	0.935	(0.1511,1.69131)	1.54021	0.929	(1.06404,39.1433)	38.0793	0.968
(20,12)	(0.000154,0.28254)	0.282393	0.945	(0.150129,1.61944)	1.46931	0.936	(0.900551,38.0184)	37.1179	0.965
(30,12)	(0.000139,0.27243)	0.272296	0.954	(0.165389,1.63105)	1.46566	0.946	(0.861137,34.6709)	33.8098	0.978
(50,12)	(0.000129,0.26469)	0.264562	0.947	(0.169259,1.56466)	1.3954	0.947	(0.949135,33.8375)	32.8884	0.978
(n,m)	Y1			Y7			Ym		
	(Lower, Upper)	Length	Coverage	(Lower, Upper)	Length	Coverage	(Lower, Upper)	Length	Coverage
(15,13)	(0.000152,0.27705)	0.277058	0.938	(0.193216,1.9467)	1.75349	0.936	(0.964702,42.2606)	41.2959	0.973
(20,13)	(0.000141,0.26195)	0.26181	0.941	(0.208467,2.03536)	1.82689	0.935	(1.17018,39.2434)	38.0732	0.976
(30,13)	(0.0001234,0.24464)	0.244524	0.945	(0.210109,1.90317)	1.69306	0.931	(1.17348,35.6211)	34.4476	0.966
(50,13)	(0.0001143,0.23821)	0.238096	0.957	(0.211075,1.81535)	1.60427	0.958	(1.18585,33.1661)	31.9803	0.982

Table 5.5. The 95% Bayesian prediction bounds,length of Bayesian prediction and their simulated coverage probability for Ys based non - informative prior

(n,m)	Y1			Y4			Ym		
	(Lower, Upper)	Length	Coverage	(Lower, Upper)	Length	Coverage	(Lower, Upper)	Length	Coverage
(15,8)	(0.000248,0.49451)	0.49427	0.953	(0.111943,2.86765)	2.75571	0.948	(0.779858,33.9345)	33.1547	0.955
(20,8)	(0.0002421,0.46972)	0.469481	0.951	(0.102928,2.46754)	2.36461	0.949	(0.558534,36.6563)	36.0978	0.988
(30,8)	(0.000240,0.46191)	0.461669	0.963	(0.106012,2.21552)	2.10951	0.961	(0.537214,32.6924)	32.1552	0.979
(50,8)	(0.000235,0.45122)	0.450991	0.948	(0.108414,2.01425)	1.90583	0.952	(0.183484,30.7059)	30.5224	0.988
(n,m)	Y1			Y5			Ym		
	(Lower, Upper)	Length	Coverage	(Lower, Upper)	Length	Coverage	(Lower, Upper)	Length	Coverage
(15,10)	(0.000187,0.38934)	0.389161	0.954	(0.147564,2.78801)	2.64045	0.948	(1.14207,45.6324)	44.4903	0.949
(20,10)	(0.000177,0.36678)	0.366609	0.951	(0.141181,2.34187)	2.20069	0.947	(1.11316,37.6697)	36.5566	0.961
(30,10)	(0.000165,0.34543)	0.345268	0.957	(0.1373962.01525)	1.87786	0.953	(1.05137,32.9594)	31.908	0.968
(50,10)	(0.0001621,0.33803)	0.337875	0.947	(0.143411,1.86337)	1.71996	0.953	(1.06816,39.8492)	38.7811	0.973
(n,m)	Y1			Y6			Ym		
	(Lower, Upper)	Length	Coverage	(Lower, Upper)	Length	Coverage	(Lower, Upper)	Length	Coverage
(15,12)	(0.0001403,0.3082)	0.30812	0.961	(0.16975,2.64656)	2.47681	0.923	(1.31578,45.0772)	43.7614	0.953
(20,12)	(0.0001340,0.291961)	0.291827	0.952	(0.173968,2.29415)	2.12018	0.939	(1.17472,40.9827)	39.808	0.967
(30,12)	(0.0001238,0.275178)	0.275055	0.947	(0.179458,2.0297)	1.85024	0.961	(1.35481,38.1587)	36.8039	0.968
(50,12)	(0.0001214,0.265905)	0.265784	0.962	(0.171414,1.74186)	1.57044	0.959	(1.33663,33.6541)	32.3175	0.974
(n,m)	Y1			Y7			Ym		
	(Lower, Upper)	Length	Coverage	(Lower, Upper)	Length	Coverage	(Lower, Upper)	Length	Coverage
(15,13)	(0.0001372,0.28479)	0.284626	0.941	(0.205341,2.78688)	2.58154	0.939	(1.01779,49.069)	48.0512	0.983
(20,13)	(0.000191,0.266334)	0.266213	0.949	(0.206602,2.48416)	2.27756	0.956	(1.37849,43.4506)	42.0721	0.967
(30,13)	(0.0001098,0.24637)	0.246269	0.956	(0.217119,2.22986)	2.01274	0.961	(1.1813,37.8901)	36.7088	0.976
(50,13)	(0.000125,0.23432)	0.234541	0.954	(0.224957,1.99452)	1.76956	0.958	(1.4746,33.8247)	32.3501	0.958

5.1. Numerical Example

Now, we presents a numerical example to illustrate the methodology for the proposed estimates based on real data. Consider the following data set is from Kotz and Johhnson (1983) and represents the survival times (in years) after diagnosis of 43 patient with certain kind of leukemia. The data has been classified into two sets using probabilistic mixing weights for p=0.7, which produced $r_1 = 30$ and $r_2 = 13$,, as follows:

population-I

$0.019, 0.159, 0.636, 0.748, 1.175, 1.206,$
$1.282, 1.356, 1.362, 1.458, 1.564, 1.586,$
$1.592, 1.781, 1.923, 1.959, 2.134, 2.548,$
$2.652, 2.951, 3.038, 3.655, 3.754, 4.690,$
$4.888, 5.143, 5.167, 5.603, 6.192, 6.874$

population-II

$6.655, 0.129, 0.485, 2.466, 0.203, 4.203,$
$2.413, 1.219, 1.219, 0.781, 0.869, 3.6, 5.633$

The estimated and prediction results are presented in Tables [(5.6)-(5.11)].

Table 5.6. Average estimates corresponding to real data set in case informative Prior

Parameter	MLE	Bayes								
		Loss Function								
		SE	LINEX				GE			
			q=0.5	q=-0.5	q=1	q=-1	h=0.5	h=-0.5	h=1	h=-1
α_1	0.241776	0.241112	0.240629	0.241597	0.240148	0.242086	0.235093	0.239111	0.233075	0.241112
α_2	0.010971	0.0109709	0.0109686	0.0109733	0.0109663	0.0109756	0.0103402	0.010762	0.0101271	0.0109709
p	0.697674	0.65625	0.655097	0.657399	0.653939	0.658542	0.650806	0.654462	0.648936	0.65625

Table 5.7. Average estimates corresponding to real data set in case non- informative Prior

Parameter	MLE	Bayes								
		Loss Function								
		SE	LINEX				GE			
			q=0.5	q=-0.5	q=1	q=-1	h=0.5	h=-0.5	h=1	h=-1
α_1	0.241776	0.241776	0.24129	0.242264	0.240807	0.242755	0.23574	0.239769	0.233716	0.241776
α_2	0.010971	0.0109713	0.010969	0.0109736	0.0109667	0.010976	0.0103405	0.0107624	0.0101274	0.0109713
p	0.697674	0.688889	0.687721	0.690051	0.686547	0.691206	0.683628	0.687163	0.681818	0.688889

Table 5.8. Bayesian prediction bounds Ys, length of the Bayesian prediction corresponding 90% in case informative prior for the real data set

(n,m)	Y1		Y4		Ym	
	(Lower, Upper)	Length	(Lower, Upper)	Length	(Lower, Upper)	Length
(63,8)	(0.0897561,2.00536)	1.9156039	(1.23083,4.29157)	3.06074	(4.01878,9.16364)	5.14486
(n,m)	Y1		Y5		Ym	
	(Lower, Upper)	Length	(Lower, Upper)	Length	(Lower, Upper)	Length
(63,10)	(0.0759191,1.69281)	1.6168909	(1.38585,4.2249)	2.83905	(4.29473,9.51146)	5.21673
(n,m)	Y1		Y6		Ym	
	(Lower, Upper)	Length	(Lower, Upper)	Length	(Lower, Upper)	Length
(63,12)	(0.0662061,1.4729)	1.4066939	(1.50481,4.17095)	2.66614	(4.49234,9.79406)	5.30172
(n,m)	Y1		Y7		Ym	
	(Lower, Upper)	Length	(Lower, Upper)	Length	(Lower, Upper)	Length
(63,13)	(0.0623424,1.38559)	1.3232476	(1.72943,4.32339)	2.59396	(4.57268,9.9177)	5.34502

Table 5.9. Bayesian prediction bounds Ys, length of the Bayesian prediction corresponding 90% in case non-informative prior for the real data set

(n,m)	Y1		Y4		Ym	
	(Lower, Upper)	Length	(Lower, Upper)	Length	(Lower, Upper)	Length
(63,8)	(0.0870625,1.93105)	1.8439875	(1.18344,4.27051)	3.08707	(3.97367,9.4282)	5.45453
(n,m)	Y1		Y5		Ym	
	(Lower, Upper)	Length	(Lower, Upper)	Length	(Lower, Upper)	Length
(63,10)	(0.0736783,1.62945)	1.5557717	(1.33145,4.19961)	2.86816	(4.27084,9.78129)	5.51045
(n,m)	Y1		Y6		Ym	
	(Lower, Upper)	Length	(Lower, Upper)	Length	(Lower, Upper)	Length
(63,12)	(0.0642801,1.41804)	1.3537599	(1.44492,4.14233)	2.69741	(4.4837,10.0682)	5.5845
(n,m)	Y1		Y7		Ym	
	(Lower, Upper)	Length	(Lower, Upper)	Length	(Lower, Upper)	Length
(63,13)	(0.0605408,1.33419)	1.2736492	(1.65938,4.30482)	2.64544	(4.57015,,10.1938)	5.62365

Table 5.10. Bayesian prediction bounds Ys, length of the Bayesian prediction corresponding 95% in case informative prior for the real data set

(n,m)	Y1		Y4		Ym	
	(Lower, Upper)	Length	(Lower, Upper)	Length	(Lower, Upper)	Length
(63,8)	(0.0528,2.36052)	2.30772	(1.03033,4.54519)	3.51486	(3.69989,10.2809)	6.58101
(n,m)	Y1		Y5		Ym	
	(Lower, Upper)	Length	(Lower, Upper)	Length	(Lower, Upper)	Length
(63,10)	(0.0446395,1.99962)	1.9549805	(1.18738,4.46186)	3.27448	(4.01896,10.6249)	6.60594
(n,m)	Y1		Y6		Ym	
	(Lower, Upper)	Length	(Lower, Upper)	Length	(Lower, Upper)	Length
(63,12)	(0.0389124,1.74192)	1.7030076	(1.31001,4.39521)	3.0852	(4.24416,10.9047)	6.66054
(n,m)	Y1		Y7		Ym	
	(Lower, Upper)	Length	(Lower, Upper)	Length	(Lower, Upper)	Length
(63,13)	(0.0366346,1.63903)	1.6023954	(1.52286,4.53485)	3.01199	(4.33457,11.0273)	6.69273

Table 5.11. Bayesian prediction bounds Ys, length of the Bayesian prediction corresponding 95% in case non-informative prior for the real data set

(n,m)	Y1		Y4		Ym	
	(Lower, Upper)	Length	(Lower, Upper)	Length	(Lower, Upper)	Length
(63,8)	(0.0513029,2.27814)	2.2268371	(0.991506,4.54152)	3.550014	(3.63099,10.5643)	6.93331
(n,m)	Y1		Y5		Ym	
	(Lower, Upper)	Length	(Lower, Upper)	Length	(Lower, Upper)	Length
(63,10)	(0.0434,1.92644)	1.88304	(1.14161,4.45264)	3.31103	(3.97304,10.9139)	6.94086
(n,m)	Y1		Y6		Ym	
	(Lower, Upper)	Length	(Lower, Upper)	Length	(Lower, Upper)	Length
(63,12)	(0.0378516,1.67744)	1.6395884	(1.25864,4.38166)	3.12302	(4.21589,11.1984)	6.98251
(n,m)	Y1		Y7		Ym	
	(Lower, Upper)	Length	(Lower, Upper)	Length	(Lower, Upper)	Length
(63,13)	(0.0356443,1.57834)	1.5426957	(1.46169,4.53063)	3.06894	(4.31345,11.323)	7.00955

6. CONCLUSION

In this paper, we have addressed the estimation and prediction problems of the mixture of Weibull and Gompertz distributions. Different estimators of the parameters are obtained using maximum likelihood and Bayesian methods, under the informative and non-informative prior distributions, we conclude.

1. The Bayes estimates perform better under informative prior that non-informative prior for all different loss functions.
2. The mean squared error of maximum likelihood estimates and Bayesian estimates of the proposed parameters decrease as the sample size increases.
3. The estimates of α_1 and α_2 in the case of asymmetric loss function is better, at the positive values of q and h than with negative values.
4. Tables (5.4)and (5.5) show that the lengths of the Bayesian prediction intervals lengths decrease as the sample size increases, and that the Bayesian simulated coverage probability of Y_s one when to the confidence level. The lengths of the Bayesian prediction intervals increases as s increases.
5. Based on real data, It can be noticed from Tables (5.8)-(5.11) that the length of the Bayesian prediction intervals increase as s increase, also, when we fix n and m increase, the length of intervals increase.

REFERENCES

1. Abu- Zinadah, H. H. (2010). A study on mixture of Exponentiated Pareto and Exponential distributions, *Journal of Applied Sciences Research*, **6**(4), 358–376.
2. Ahmadi, P. A., Matthes, C. & Wang, M. C. (2020). Choosing Prior Hyperparameters: with Applications to Time-Varying Parameter Models, *Journal of Business and Economic Statistics*, **38**, 124–136.
3. Arnold, B. C., Balakrishnan, N. & Nagaraja, H. N. (1992). *A First Course in Order Statistics*. New York, NY: John Wiley & Sons.
4. Basu, A. P. & Ebrahimi, N. (1991). Bayesian approach to life testing and reliability estimation using asymmetric loss function, *Journal of Statistics Planning and Inference*, **29**, 21–31.

5. Calabria, R. & Pulcini, G. (1996). Point estimation under asymmetric loss function for left-truncated exponential samples, *Communications in Statistics-Theory and Methods*, **25**(3), 585–600.
6. Daniyal, M. & Rajab, M. (2015). On some classical properties of the mixture of Burr XII and Lomax distributions, *Journal of Statistics Applications and Probability*, **4**(1),173–181.
7. Elshahat, M. A. T. & Mahmoud, A. A. M. (2016). A study on the mixture of exponentiated- weibull distribution part ii (The Method of Bayesian Estimation), *Pakistan Journal of Statistics and Operation Research*, **XII**(4), 709–737.
8. Erisoglu, U., Erisoglu, M. & Erol, H. (2011). A mixture model of two different distributions approach to the analysis of heterogeneous survival data, *World Academy of Science, Engineering Technology International Journal of Computer, Electrical, Automation, Control and Information Engineering*, **5**(6), 544–548.
9. Everitt, B. S. & Hand, D. J. (1981). *Finite Mixture Distributions*. Cambridge, UK: The University Press.
10. Feroze, N. & Aslam, M.(2014). Bayesian analysis of doubly censored lifetime data using two component mixture of Weibull distribution, *Journal of the National Science Foundation of Sri Lanka*, **42**(4),325–334.
11. Jaheen, Z. F.(2003). Bayesian prediction under a mixture of two-component Gompertz lifetime model, *Sociedad de Estadistica e Investigacion Opertiva Test*, **12**(2), 413–426.
12. Kotz, S. & Johnson, N. L.(1983). *Encyclopedia of statistical sciences*. New York, NY: John Wiley & Sons.
13. Lindsay, B. G.(1995). *Mixture Models: Theory, Geometry and Applications*. CA: Institute of Mathematical Statistics.
14. Mahmoud, M., Nassar, M. M. & Aefa, M. A. (2017). Bayesian Estimation and Prediction for a Mixture Weibull and Lomax Distribution, *International Journal of Innovative Research and Development*, **6**(5), 33–46.
15. Mclachlan, G. J. & Basford, K. E. (1988). *Mixture Models: Inferences and Applications to Clustering*. New York, NY: Marcel Dekker.
16. Mclachlan, G. J. & Peel, D. (2000). *Finite Mixture Models*. New York, NY: John Wiley & Sons.
17. Mohie El-Din, M. M., Abd El-Raheem, A. M. & Abd El-Azeem, S. O. (2018). On progressive-stress accelerated life testing for the power Generalized Weibull distribution under progressive type-II censoring, *Journal of Statistics Applications & Probability Letters*, **5**(3), 131–143.
18. Mohie El-Din, M. M., Abd El-Raheem, A. M. & Abd El-Azeem, S. O. (2019). Inference for a Constant-Stress Accelerated Life Testing for Power Generalized Weibull Distribution under Progressive Type-II Censoring, *Journal of Statistics Applications & Probability*, **8**(3), 201–216.
19. Mohie El-Din, M. M., Abdel-Aty, Y. & Abu-Moussa, M. H. (2017). Statistical inference for the Gompertz distribution based on type-II progressively hybrid censored data, *Communications in Statistics- Simulation and Computation*, **46**(8), 6242–6260.
20. Mohie El-Din, M. M. & Abu-Moussa, M.H. (2018). Statistical inference and prediction for the Gompertz distribution based on multiply Type-I censored data, *Journal of the Egyptian Mathematical Society*, **26**(2), 218–234.
21. Mohie El-Din, M.M., Sadek, A., Mohie El-Din, Marwa M. and Sharawy, A.M., (2017). Characterization for Gompertz distribution based on general progressively type-II right censored order statistics, *International Journal of Advanced Statistics and Probability (IJASP)*, **5**(1), 52–56.
22. Murthy, D. N. P., Xie, M. & Jiang, R. (2003). Weibull models. New York, NY : John Wiley & Sons.
23. Pollard, J. & Valkovics, E. (1992). The Gompertz distribution and its applications, *Genus*, **48**(3–4), 15–29.

24. Shawky, A. I. & Bakoban, R. A. (2009). On finite mixture of two component exponentiated gamma distribution, *Journal Applied Sciences Research*, **5**(10), 1351–1369.
25. Titterington, D. M., Smith, A. F. M. & Makov, U. E. (1985). Statistical Analysis of Finite Mixture Distributions. New York, NY: John Wiley & Sons.
26. Varian, H. R. (1975). A Bayesian approach to real estate assessment. In: Stephen, E.F & Zellner, A.(Eds). *Studies in Bayesian Econometrics and statistics in Honor of Leonard J. Savage* (pp. 195–208). Amsterdam, NL: North Holland.
27. Willemse, W. & Koppelaar, H. (2000). Knowledge elicitation of Gompertz' law of morality, *Scandinavian Actuarial Journal*, **94**, 168–180.