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Nonuniqueness of Equilibrium in Closed Market Model

Alexander Kotyukov[1*], Natal'ya Pavlova[1,2,3,4]

[1] *V.A. Trapeznikov Institute of Control Sciences of Russian Academy of Sciences, Moscow, Russia*
[2] *Moscow Institute of Physics and Technology, Moscow, Russia*
[3] *RUDN University, Moscow, Russia*
[4] *Derzhavin Tambov State University, Tambov, Russia*

Abstract: In this paper, we consider a closed market model. In this model the supply and the demand functions are restored by their price elasticities. We obtain sufficient conditions on nonuniqueness of equilibrium in this model. For several special cases of closed market models we get a criteria of equilibrium uniqueness. The obtained results are illustrated with the example of the market with two goods.

Keywords: supply, demand, equilibrium, coincidence point, covering map

1. INTRODUCTION

Equilibrium is one of the main concepts of modern economics. It is a state of the market in which all the produced goods are sold and every market participant does not want to change their position.

Consider an economic system which consists of producers and consumers. The producers manufacture goods which are sold to the consumers. We assume that the consumers work for the producers and spend all their salaries to purchase the goods and do not make savings. The total amount of the goods produced is called a supply. The total amount of the goods sold is called a demand.

If the supply exceeds the demand, the producers suffer profit loss due to the goods unsold. This makes them cut salaries of the consumers. If the consumers have less money, they purchase less goods which, in turn, decrease the profit of the producers even more. It is easy to see that this process leads to a catastrophic economical situation.

On the other hand, if the demand exceeds the supply, some consumers cannot purchase the goods they need since these goods are not available for purchase. This leads to an unfavorable situation in the region, e.g. a massive hunger, a pandemic, lack of building resources etc.

Therefore, we are interested in maintaining the market state in which the total amount of the goods produced is equal to the total amount of the goods needed, i.e., if the supply equals to the demand. Such market state is called an equilibrium. The prices on the goods on such market are called equilibrium prices.

Equilibrium as an economical concept appeared in the second half of 18th century in the works of Stewart [14] and Smith [13]. The first mathematical notion of the concept was introduced by Walras [15] in 1874. At that time there were no developed mathematical theory

*Corresponding author: amkotyukov@mail.ru

to obtain the substantial results on equilibrium existence. Such results were first obtained by Arrow and Debreu [1] in 1956.

In this work, we obtain sufficient conditions on equilibrium nonuniqueness in a closed market model. Here the demand and the supply functions are restored by their price elasticities. The elasticity shows how one economic variable changes with a change of another. An equilibrium in this model is considered as the coincidence point of the supply and the demand functions. To obtain the mentioned conditions we use the results in the theory of covering maps and coincidence points [2]– [5]. Let us formalize the problem.

2. PROBLEM STATEMENT

Denote $\mathbb{R}^n_+ = \{p = (p_1, ..., p_n) \in \mathbb{R}^n : p_i > 0 \ \forall i = \overline{1, n}\}$. Let the market consist of n goods. The prices on these goods are described by a vector $p \in \mathbb{R}^n_+$ which satisfies the inequalities:

$$c_{1i} \leq p_i \leq c_{2i} \ \ \forall i = \overline{1, n}.$$

Here $c_1 = (c_{11}, ..., c_{1n}), c_2 = (c_{21}, ..., c_{2n}) \in \mathbb{R}^n_+$ are natural constraints on the prices p (denote $P = [c_{11}; c_{21}] \times ... \times [c_{1n}; c_{2n}]$).

Let the demand function $D : \mathbb{R}^n_+ \to \mathbb{R}^n_+, \ D(p) = (D_1(p), ..., D_n(p))$, and the supply function $S : \mathbb{R}^n_+ \to \mathbb{R}^n_+, \ S(p) = (S_1(p), ..., S_n(p))$, be given. We assume that for some known prices $p^* \in P$ we know the demand $D^* \in \mathbb{R}^n_+, \ D^* = (D^*_1, ..., D^*_n)$, and the supply $S^* \in \mathbb{R}^n_+, \ S^* = (S^*_1, ..., S^*_n)$, i.e., $D^* = D(p^*), S^* = S(p^*)$.

Next, suppose that we have a matrix $\mathcal{E} = (E_{ij})_{i,j=\overline{1,n}}$ where $E_{ij} \in \mathbb{R}$ is the jth good price elasticity of the ith good demand. This quantity shows how the demand on the ith good changes with the change of the price on the jth good. Similarly, we have a matrix $\tilde{\mathcal{E}} = (\tilde{E}_{ij})_{i,j=\overline{1,n}}$ where $\tilde{E}_{ij} \in \mathbb{R}$ is the jth good price elasticity of the ith good supply. This quantity shows how the demand on the ith good changes with the change of the price on the jth good.

Definition 2.1. A closed market model is a following set of parameters:

$$\sigma = (c_1, c_2, p^*, D^*, S^*, \mathcal{E}, \tilde{\mathcal{E}}).$$

This set uniquely defines the demand function D by:

$$D_i(p_1, ..., p_n) = D^*_i \prod_{j=1}^{n} (p^*_j)^{-E_{ij}} p_j^{E_{ij}}, \ \ i = \overline{1, n}; \tag{2.1}$$

and the supply function S by:

$$S_i(p_1, ..., p_n) = S^*_i \prod_{j=1}^{n} (p^*_j)^{-\tilde{E}_{ij}} p_j^{\tilde{E}_{ij}}, \ \ i = \overline{1, n}. \tag{2.2}$$

Formulas (2.1), (2.2) are the solutions to the following systems of equations:

$$E_{ij} = \frac{\partial D_i}{\partial p_j} \frac{p_j}{D_i}, \ \ i, j = \overline{1, n}; \tag{2.3}$$

$$\tilde{E}_{ij} = \frac{\partial S_i}{\partial p_j} \frac{p_j}{S_i}, \ \ i, j = \overline{1, n} \tag{2.4}$$

with initial conditions

$$D_i(p^*) = D_i^*, S_i(p^*) = S_i^*, \ \ i = \overline{1, n}. \tag{2.5}$$

Formulas (2.3), (2.4) are the definitions of ith good price elasticity of jth good demand and supply respectively.

Definition 2.2. A vector $p^0 \in P$ is called an equilibrium prices vector (an equilibrium) in the model σ if

$$D(p^0) = S(p^0). \tag{2.6}$$

3. ON THE UNIQUENESS AND NONUNIQUENESS OF EQUILIBRIUM IN A CLOSED MARKET MODEL

Sufficient conditions on equilibrium existence were obtained in [3]. Substitute (2.1), (2.2) into (2.6) and obtain the following system of linear equations:

$$D_i^* \prod_{j=1}^{n} (p_j^*)^{-E_{ij}} p_j^{E_{ij}} = S_i^* \prod_{j=1}^{n} (p_j^*)^{-\tilde{E}_{ij}} p_j^{\tilde{E}_{ij}}, \ \ i = \overline{1, n}.$$

By taking a logarithm of the left-hand and right-hand sides of the last equation we get:

$$\sum_{j=1}^{n} \left(E_{ij} - \tilde{E}_{ij} \right) \ln p_j = \ln S_i^* - \ln D_i^* + \sum_{j=1}^{n} (E_{ij} - \tilde{E}_{ij}) \ln p_j^*, \ \ i = \overline{1, n}. \tag{3.7}$$

Note that this system is linear by $\ln p_j, \ \ j = \overline{1, n}$.

Theorem 3.1. *Let the parameters of the model σ satisfy the condition*

$$E_{ij} = \tilde{E}_{ij} \quad \forall i, j = \overline{1, n}. \tag{3.8}$$

Then $\forall p \in P$ is an equilibrium in the model σ iff $S_i^* = D_i^* \ \ \forall i = \overline{1, n}$.

Proof
Indeed, let the parameters of the model σ satisfy (3.8). Hence, System (3.7) is equivalent to the following system:

$$\ln S_i^* - \ln D_i^* = 0, \quad i = \overline{1, n}. \tag{3.9}$$

It is obvious that any vector $p \in P$ is an equilibrium in the model σ iff $S_i^* = D_i^* \ \ \forall i = \overline{1, n}$. □

Corollary 3.1. Let the parameters of the model σ satisfy (3.8). Then the condition:

$$\exists i = \overline{1, n} : \quad S_i^* \neq D_i^*$$

is a criteria of equilibrium absence in the model σ.

Remark 3.1. The condition (3.8) means that the consumers and the producers react to the price changes in the same way.

Example 3.1. Let us describe several examples of the market models satisfying the conditions of Lemma 3.1.

1. Consider a market model of electric energy inside the given country. There is only one good – electric energy, and only one producer – a number of state energetic companies which possess the natural monopoly on this good. In the short term the price elasticity of demand equals zero [9]. This is conditioned by the fact that the companies cannot rearrange their technological processes and change the volume of production rapidly. In this model the price elasticity of demand also equals zero in the short term [7].

2. The models of raw material market have zero price elastitcities of both supply and demand [6]. For consumers the cost of raw material has small influence to the goods' prices. For producers raw material are inelastic for several reasons, such as the load capacity, production factors interchangeability or raw material availability [11].

Now we introduce the following notation:

$$A = (a_{ij})_{i,j=\overline{1,n}}, \;\; a_{ij} = E_{ij} - \tilde{E}_{ij};$$
$$\omega = (\omega_1, ..., \omega_n), \quad \omega_i = \ln S_i^* - \ln D_i^* + \sum_{j=1}^{n} (E_{ij} - \tilde{E}_{ij}) \ln p_j^*.$$

Theorem 3.2. *Let the parameters of the model* σ *satisfy the condition* $\det A \neq 0$. *Then there exists a unique equilibrium prices vector*

$$p_i^0 = \exp^{(A^{-1}\omega)_i}, \;\; i = \overline{1,n} \tag{3.10}$$

iff the parameters of the model satisfy the following condition:

$$\max_{i=\overline{1,n}} \frac{2}{\ln c_{2i} - \ln c_{1i}} \left| (A^{-1}\omega)_i - \frac{\ln c_{1i} + \ln c_{2i}}{2} \right| \leq 1 \tag{3.11}$$

where $(A^{-1}\omega)_i$ *is the* i*th component of the vector* $A^{-1}\omega$.

Proof
In conditions of Lemma the system (3.7) is consistent by Rouché-Capelli theorem [12, Theorem 2.14] since $\det A \neq 0$. Moreover, by Theorem [12, Theorem 2.15] its solution is unique.

Firstly we prove the necessity. Let p^0 defined by (3.10) be an equilibrium in the model σ. Then p^0 satisfies (3.7). Moreover, $p^0 \in P$, i.e.:

$$\ln c_{1i} \leq (A^{-1}\omega)_i, \; (A^{-1}\omega)_i \leq \ln c_{2i}, \quad i = \overline{1,n}, \tag{3.12}$$

where $(A^{-1}\omega)_i$ is the ith component of the vector $A^{-1}\omega$. Subtract $(\ln c_{1i} + \ln c_{2i})/2$ from the inequalities (3.12):

$$\ln c_{1i} - \frac{\ln c_{1i} + \ln c_{2i}}{2} \leq (A^{-1}\omega)_i - \frac{\ln c_{1i} + \ln c_{2i}}{2},$$
$$(A^{-1}\omega)_i - \frac{\ln c_{1i} + \ln c_{2i}}{2} \leq \ln c_{2i} - \frac{\ln c_{1i} + \ln c_{2i}}{2}, \quad i = \overline{1,n}.$$

Thus,

$$-\frac{\ln c_{2i} - \ln c_{1i}}{2} \leq (A^{-1}\omega)_i - \frac{\ln c_{1i} + \ln c_{2i}}{2} \leq \frac{\ln c_{2i} - \ln c_{1i}}{2}, \quad i = \overline{1,n}.$$

Hence, we have:

$$\left| (A^{-1}\omega)_i - \frac{\ln c_{1i} + \ln c_{2i}}{2} \right| \leq \frac{\ln c_{2i} - \ln c_{1i}}{2}, \quad i = \overline{1,n}.$$

By dividing this inequality by $(\ln c_{2i} - \ln c_{1i})/2 > 0$ we obtain:

$$\frac{2}{\ln c_{2i} - \ln c_{1i}} \left| (A^{-1}\omega)_i - \frac{\ln c_{1i} + \ln c_{2i}}{2} \right| \leq 1, \quad i = \overline{1,n}.$$

Therefore, if p^0 defined by (3.10) is an equilibrium in the model σ, the following condition holds:

$$\max_{i=\overline{1,n}} \frac{2}{\ln c_{2i} - \ln c_{1i}} \left| (A^{-1}\omega)_i - \frac{\ln c_{1i} + \ln c_{2i}}{2} \right| \leq 1.$$

Now we prove the sufficiency. Let the condition (3.11) be satisfied. Since $\det A \neq 0$, the matrix A is invertible and, hence, the solution to the system (3.7) is unique and has the form:

$$p = \exp{(A^{-1}\omega)}.$$

From the condition (3.11) we obtain that $p \in P$. Therefore, there exists a unique equilibrium p^0 in the model σ. □

Corollary 3.2. Let the parameters of the model σ satisfy Lemma 3.2 and the condition $S_i^* = D_i^*\ \forall i = \overline{1,n}$. Then $p^0 = p^*$.

Now we consider the case $\operatorname{rang} A = \operatorname{rang}(A|\omega) = k < n$.

Theorem 3.3. *In the model σ there exist an infinite number of equilibrium prices vectors iff the parameters of the model σ satisfy the following conditions:*

1) $\operatorname{rang} A = \operatorname{rang}(A|\omega) = k < n$*;*

2)

$$\max_{i=\overline{1,n}} \frac{2}{\ln c_{2i} - \ln c_{1i}} \left| \sum_{j=1}^{n-k} C_j x_{ji} - \frac{\ln c_{1i} + \ln c_{2i}}{2} \right| < 1, \tag{3.13}$$

where $x_{ji} \in \mathbb{R}$ is the ith component of the vector $X_j \in X,\ i = \overline{1,n}$ and $X = \{X_1, ..., X_{n-k}\}$ is the fundamental system of solutions to the system (3.7).

Proof

First let an infinite number of equilibrium prices vectors exist in the model σ. Since any equilibrium $p \in P$ is a solution to the system (3.7), we have $\operatorname{rang} A = \operatorname{rang}(A|\omega) = k$ (since the system (3.7) is consistent) and $k < n$ (since system (3.7) has more than one solution). Let

$$X = \{X_1, ..., X_{n-k}\},\ \ X_i \in \mathbb{R}^n \tag{3.14}$$

be the fundamental system of solutions for the system (3.7). Then any equilibrium p (which is the solution to the system (3.7)) can be written in the following form:

$$\ln p_i = \sum_{j=1}^{n-k} C_j X_j \tag{3.15}$$

where $C_j \in \mathbb{R},\ j = \overline{1, n-k}$, are some constants.

From the inclusion $p \in P$ we get

$$\ln c_{1i} \leq \sum_{j=1}^{n-k} C_j X_j,\ \sum_{j=1}^{n-k} C_j X_j \leq c_{2i},\ i = \overline{1,n}.$$

By repeating the steps conducted in the proof of Lemma 3.2 we obtain that if the vector p is an equilibrium in the model σ, then

$$\max_{i=\overline{1,n}} \frac{2}{\ln c_{2i} - \ln c_{1i}} \left| \sum_{j=1}^{n-k} C_j x_{ji} - \frac{\ln c_{1i} + \ln c_{2i}}{2} \right| \leq 1$$

where x_{ji} is the ith component of the vector X_j, $i = \overline{1, n}$. From this we easily obtain what we want.

Now let conditions 1) and 2) of the Theorem be satisfied. Then by Rouché-Capelli theorem this system is consistent, but its solution is not unique since $\operatorname{rang} A \neq n$. Using (3.14) we get that any solution $p \in \mathbb{R}^n_+$ to the system (3.7) can be written in the form (3.15). Moreover, from condition 2) inverting the conclusions made in the first part of the proof we obtain that $p \in P$. Hence, any solution to the system (3.7) satisfying condition 2) is an equilibrium in the model σ. □

To demonstrate these results consider the following example.

Example 3.2. Let $n = 2, p^* = (1; 1), c_1 = (1; e), c_2 = (1/(4e); 1)$. In Table 3.1 we can see the existence and uniqueness and non-uniqueness for different parameters $D^*,\ S^*, \omega$.

Table 3.1. Existence, uniqueness and non-uniqueness of the solution in the case $n = 2, p^* = (1; 1), c_1 = (1; e), c_2 = (1/(4e); 1)$.

		S^*, D^*, ω^*			
		$(1;1)$ $(1;1)$ $(0;0)$	$(2;2)$ $(1;1)$ $(\ln 2;\ln 2)$	$(1;1)$ $(2;2)$ $(-\ln 2;-\ln 2)$	$(2;2)$ $(1;\frac{1}{2})$ $(\ln 2; 2\ln 2)$
A	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	Theorem 3.1 $\forall p\in P$	Corollary 3.1 $\emptyset$	Corollary 3.1 $\emptyset$	Corollary 3.1 $\emptyset$
	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	Corollary 3.2 (1;1)	Theorem 3.2 (2;2)	Theorem 3.2 (1/2;1/2)	Theorem 3.2 (2;4)
	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	Corollary 3.2 (1;1)	Theorem 3.2 (2;2)	Theorem 3.2 (1/2;1/2)	Theorem 3.2 (4;2)
	$\begin{pmatrix}1/2&1/2\\1/2&1/2\end{pmatrix}$	Theorem 3.3 $(p_1,\frac{1}{p_1}),\ p_1\in[1;e]$	Theorem 3.3 $(p_1;\frac{4}{p_1}),p_1\in[1;e]$	Theorem 3.3 $(p_1;\frac{1}{4p_1}),p_1\in[1;e]$	Corollary 3.1 $\emptyset$
	$\begin{pmatrix}1&0\\1/2&1/2\end{pmatrix}$	Corollary 3.2 (1;1)	Theorem 3.2 (2;2)	Theorem 3.2 (1/2;1/2)	Theorem 3.2 (2;8)

Now we consider the case $n = 2$. The system (3.7) is equivalent to the following system:

$$\begin{aligned} a_{11}\ln p_1 + a_{12}\ln p_2 &= \omega_1, \\ a_{21}\ln p_1 + a_{22}\ln p_2 &= \omega_2, \end{aligned} \tag{3.16}$$

which is considered under the conditions

$$\begin{aligned} \ln c_{11} \leq \ln p_1 \leq \ln c_{21}, \\ \ln c_{12} \leq \ln p_2 \leq \ln c_{22}. \end{aligned} \tag{3.17}$$

Theorem 3.4. *Let $n = 2$. Then:*

1. if $\det A \neq 0$ and

$$\begin{aligned} \ln c_{11} \leq \frac{a_{22}\omega_1 - a_{12}\omega_2}{a_{11}a_{22} - a_{12}a_{21}} \leq \ln c_{21}, \\ \ln c_{12} \leq \frac{a_{11}\omega_2 - a_{21}\omega_1}{a_{11}a_{22} - a_{12}a_{21}} \leq \ln c_{22}, \end{aligned}$$

then there exists a unique equilibrium

$$\left(\frac{a_{22}\omega_1 - a_{12}\omega_2}{a_{11}a_{22} - a_{12}a_{21}}, \frac{a_{11}\omega_2 - a_{21}\omega_1}{a_{11}a_{22} - a_{12}a_{21}}\right)$$

in the model σ;

2. *if* $\operatorname{rang} A = \operatorname{rang}(A|\omega) = 1$ *and* $a_{i2} = 0 \quad \forall i = \{1, 2\}$, *then there exists an infinite number of equilibrium prices vectors in the model* σ *which belong to the set:*

$$\ln p_1 = \frac{\omega_1}{a_{11}} = \frac{\omega_2}{a_{21}}, \quad c_{12} \le p_2 \le c_{22}.$$

3. *if* $\operatorname{rang} A = \operatorname{rang}(A|\omega) = 1$ *and* $a_{i1} = 0 \quad \forall i = \{1, 2\}$, *then there exists an infinite number of equilibrium prices vectors in the model* σ *which belong to the set:*

$$\ln p_2 = \frac{\omega_1}{a_{12}} = \frac{\omega_2}{a_{22}}, \quad c_{11} \le p_1 \le c_{21}.$$

4. *if* $\operatorname{rang} A = \operatorname{rang}(A|\omega) = 1$, $a_{11}a_{12} > 0$ *and*

$$\omega_1 = a_{11} \ln c_{11} + a_{12} \ln c_{12},$$

then there exists a unique equilibrium (c_{11}, c_{12}) *in the model* σ;

5. *if* $\operatorname{rang} A = \operatorname{rang}(A|\omega) = 1$, $a_{11}, a_{12} > 0$ *and*

$$\omega_1 = a_{11} \ln c_{21} + a_{12} \ln c_{22},$$

then there exists a unique equilibrium (c_{21}, c_{22}) *in the model* σ;

6. *if* $\operatorname{rang} A = \operatorname{rang}(A|\omega) = 1$, $a_{11}a_{12} < 0$ *and*

$$\omega_1 = a_{11} \ln c_{21} + a_{12} \ln c_{12},$$

then there exists a unique equilibrium (c_{21}, c_{12}) *in the model* σ;

7. *if* $\operatorname{rang} A = \operatorname{rang}(A|\omega) = 1$, $a_{11}a_{12} < 0$ *and*

$$\omega_1 = a_{11} \ln c_{11} + a_{12} \ln c_{12},$$

then there exists a unique equilibrium (c_{11}, c_{12}) *in the model* σ;

8. *if* $\operatorname{rang} A = \operatorname{rang}(A|\omega) = 1$ *and either:*
   - $a_{11}a_{12} > 0$ *and*

   $$a_{11} \ln c_{11} + a_{12} \ln c_{12} < \omega_1 < a_{11} \ln c_{21} + a_{12} \ln c_{22};$$

   or
   - $a_{11}a_{12} < 0$ *and*

   $$a_{11} \ln c_{21} + a_{12} \ln c_{12} < \omega_1 < a_{11} \ln c_{11} + a_{12} \ln c_{22};$$

   then there exists an infinite number of equilibrium prices vectors in the model σ *defined by the formula:*

   $$\ln p_2 = \frac{\omega_1 - a_{11} \ln p_1}{a_{12}}, \quad \ln c_{11} \le p_1 \le \ln c_{21}.$$

Proof
Consider all the cases consequently.

1. Let $\det A \neq 0$. Then by Rouché-Capelli theorem the system (3.16) is consistent and by Theorem [12, Theorem 2.15] its solution is unique. Let us find this solution using, e.g., Cramer's rule:

$$A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21},$$

$$A_1 = \begin{vmatrix} \omega_1 & a_{12} \\ \omega_2 & a_{22} \end{vmatrix} = a_{22}\omega_1 - a_{12}\omega_2,$$

$$A_2 = \begin{vmatrix} a_{11} & \omega_1 \\ a_{21} & \omega_2 \end{vmatrix} = a_{11}\omega_2 - a_{21}\omega_1.$$

Then the solution to this system has the form:

$$\ln p_1 = A_1/A = \frac{a_{22}\omega_1 - a_{12}\omega_2}{a_{11}a_{22} - a_{12}a_{21}},$$

$$\ln p_2 = A_2/A = \frac{a_{11}\omega_2 - a_{21}\omega_1}{a_{11}a_{22} - a_{12}a_{21}}.$$

This solution must satisfy (3.17). Therefore, if

$$\ln c_{11} \leq \frac{a_{22}\omega_1 - a_{12}\omega_2}{a_{11}a_{22} - a_{12}a_{21}} \leq \ln c_{21},$$
$$\ln c_{12} \leq \frac{a_{11}\omega_2 - a_{21}\omega_1}{a_{11}a_{22} - a_{12}a_{21}} \leq \ln c_{22},$$

then there exists a unique equilibrium in the model σ. Otherwise an equilibrium in this model does not exist.

Now let $\operatorname{rang} A = \operatorname{rang}(A|\omega) = 1$.

2. In the case $a_{i2} = 0,\ \forall i \in \{1, 2\}$ the system (3.16) is equivalent to the following system:

$$a_{11} \ln p_1 = \omega_1,$$

from which it follow that all equilibrium prices vectors has the form:

$$\ln p_1 = \frac{\omega_1}{a_{11}} = \frac{\omega_2}{a_{21}}, \quad c_{12} \leq p_2 \leq c_{22}.$$

3. In the case $a_{i1} = 0,\ \forall i \in \{1, 2\}$ all equilibrium prices vectors belong to the set:

$$\{(p_1, p_2) \in P | \ln p_2 = \frac{\omega_1}{a_{12}} = \frac{\omega_2}{a_{22}}, \quad c_{11} \leq p_1 \leq c_{21}\}.$$

Hereinafter we suppose that $a_{ij} \neq 0,\ i, j = \overline{1, 2}$. Then all the solutions to the system (3.16) belong to the line:

$$\ln p_2 = \frac{\omega_1 - a_{11} \ln p_1}{a_{12}}. \tag{3.18}$$

Now we find the conditions under which an equilibrium is unique even in the case when the system (3.16) has an infinite number of solutions. Note that the angle of the line (3.18) equals $-a_{11}/a_{12}$. We consider several cases.

4. Let $a_{11}a_{12} > 0$. Then an equilibrium in the model σ is unique in two cases:
   (a) if:
$$\ln c_{12} = -\frac{a_{11}\ln c_{11}}{a_{12}} + \frac{\omega_1}{a_{12}}; \tag{3.19}$$
   (b) if:
$$\ln c_{22} = -\frac{a_{11}\ln c_{21}}{a_{12}} + \frac{\omega_1}{a_{12}}; \tag{3.20}$$
   In the case 4a from (3.19) we obtain that if
$$\omega_1 = a_{11}\ln c_{11} + a_{12}\ln c_{12},$$
   then in the model σ there exists a unique equilibrium $p^0 = (c_{11}, c_{12})$.
5. In the case 4b from (3.20) we obtain that if:
$$\omega_1 = a_{11}\ln c_{21} + a_{12}\ln c_{22},$$
   then in the model σ there exists a unique equilibrium $p^0 = (c_{21}, c_{22})$.
   In the case when
$$a_{11}\ln c_{11} + a_{12}\ln c_{12} < \omega_1 < a_{11}\ln c_{21} + a_{12}\ln c_{22}, \tag{3.21}$$
   in the model σ there exist an infinite number of equilibrium prices vectors belonging to the set:
$$\ln p_2 = \frac{\omega_1 - a_{11}\ln p_1}{a_{12}}, \quad \ln c_{11} \le p_1 \le \ln c_{21}.$$
6. Let $a_{11}a_{12} < 0$. This case is considered similarly to the previous case. Here an equilibrium in the model σ is unique in two cases:
   (a) if:
$$\ln c_{22} = -\frac{a_{11}\ln c_{11}}{a_{12}} + \frac{\omega_1}{a_{12}}; \tag{3.22}$$
   (b) if:
$$\ln c_{21} = -\frac{a_{11}\ln c_{12}}{a_{12}} + \frac{\omega_1}{a_{12}}; \tag{3.23}$$
   In the case 6a from (3.22) we obtain that if:
$$\omega_1 = a_{11}\ln c_{11} + a_{12}\ln c_{22},$$
   then in the model σ there exists a unique equilibrium $p^0 = (c_{11}, c_{22})$.
7. In the case 6b from (3.23) we obtain that if:
$$\omega_1 = a_{11}\ln c_{12} + a_{12}\ln c_{21},$$
   then in the model σ there exists a unique equilibrium $p^0 = (c_{12}, c_{21})$.
   In the case when:
$$a_{11}\ln c_{12} + a_{12}\ln c_{21} < \omega_1 < a_{11}\ln c_{11} + a_{12}\ln c_{22}, \tag{3.24}$$
   in the model σ there exist an infinite number of equilibrium prices vectors belonging to the set:
$$\ln p_2 = \frac{\omega_1 - a_{11}\ln p_1}{a_{12}}, \quad \ln c_{11} \le p_1 \le \ln c_{21}.$$

8. The last statement is obtained from inequalities (3.21) and (3.24) with the corresponding conditions on the parameters a_{11}, a_{12}.

□

Example 3.3. Consider a market of electric energy in the Russian Federation in 2022. In the table below we can see the data collected from Federal State Statistics Service [8]:

Table 3.2. The production and consumption of electric energy in the Russian Federation in 2021–2022

	2021	2022
Production, trillion kW*hr	1.100	1.167
Consumption, trillion kW*hr	0.965	1.110
Prices, rubles per kW*hr	4.23	4.78

In terms of the model σ we have:

$$p^* = 4.78,\ S^* = 1.167,\ D^* = 1.11.$$

We put $c_1 = 2.69,\ c_2 = 4.81$ as the lowest and the highest mean prices on electric energy in the Russian Federation in 10 years, respectively. Using the economic definition of elasticity we obtain that:

$$E = 1.15,\ \tilde{E} = 0.46.$$

Now we check the conditions of Theorem 3.2. Here $A = 0.69,\ \omega = 1.1295$. From (3.11) we obtain that

$$\frac{2}{\ln 8.5 - \ln 2.63}\left|\frac{1.1295}{0.69} - \frac{\ln 8.5 + \ln 2.63}{2}\right| = 0.1422 < 1.$$

Therefore, the conditions of Theorem 3.2 are satisfied. Hence, there exists a unique equilibrium in the model σ.

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