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Stability Analysis of a Delayed Epidemic Model with General Incidence and Treatment Functions

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Abstract: In this paper, we study an SIR epidemic model with general nonlinear incidence function, general function of treatment and two discrete time delays, the first described the time delay due to the latent period of the disease and the second is the time delay due to the period that the infected individuals use to move into the recovered class. Lyapunov's method is used to show the global stability of the disease-free equilibrium if the basic reproduction number $R_0 \leq 1$, while if $R_0 > 1$ and under some conditions of delays, the existence of Hopf bifurcation appears for the endemic equilibrium.

Keywords: Treatment, nonlinear incidence, time delay, global stability, Hopf bifurcation.

1. INTRODUCTION

Mathematical modeling has been used in several areas, it is considered as a decision support tool on any situation. In the epidemiological area, it has contributed to the epidemiological surveillance of the disease because it makes to predict the health consequences of actions as varied as vaccination, quarantine or the distribution of screening tests. Among the models used in epidemiology, there are the SIR models which are compartment models, where S stands for susceptible subpopulation, I is infected subpopulation, and R is recovered subpopulation.

The incidence rate of a disease measures how fast the disease is spreading and it plays an important role in the research of mathematical epidemiology. In many previous epidemic models, the bilinear incidence rate βSI was frequently used [13, 21, 22, 26, 27, 29, 34, 40, 43, 44]. However, there are some advantages for adopting more general forms of incidence rates. For instance, Capasso et al. [14–16] observed that the incidence rate may increase more slowly as I increases. So, they proposed a saturated incidence rate $\frac{\beta SI}{1+\alpha I}$ which was used in [2, 9, 33, 41], where α is the saturation factor that measures the inhibitory effect, βI measures the infection force of the disease and $\frac{1}{1+\alpha I}$ measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infected individuals. This incidence rate seems more reasonable than the bilinear incidence rate βSI, because it includes the behavioral change and crowding

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effect of the infected individuals and prevents the unboundedness of the contact rate by choosing suitable parameters. In the last years, many forms of incidence function have been considered by the researchers in mathematical epidemiology. For example, the first one is the saturated incidence $\frac{\beta SI}{d+S+I}$ [3], where β and d are the positive constants. The second one is the Beddington-Deangelis incidence $\frac{\beta SI}{1+\alpha_1 S+\alpha_2 I}$ [6], where α_1 and α_2 are the positive constants. The effect of saturation factor (refers to α_1 and α_2) stems from epidemic control and the protection mesures. The third one is the standard incidence $\frac{\beta SI}{N}$ [18, 26]. In addition, a recent Hattaf-Yousfi incidence $\frac{\beta SI}{\alpha_0+\alpha_1 S+\alpha_2 I+\alpha_3 SI}$ that includes the three above functions was introduced in [24] and used in [36]. Models with incidence functions of the form $g(I)h(S)$ have been studied in [30, 37]. The most general incidence function $f(S,I)I$ [25] which generalizes the previous incidence functions has been studied by the many authors [4, 10–12].

It is well known that treatment is an important and effective method to prevent and control the spread of various infectious diseases. Therefore, it very important to adopt a suitable treatment function. For instance, Wang and Ruan [40] introduced a constant treatment in SIR model as follows:

$$\begin{cases} r, I > 0, \\ 0, I = 0, \end{cases}$$

which simulated a limited capacity for treatment. Further, Wang [39] considered the following piecewise linear treatment function:

$$\begin{cases} rI, 0 \leq I \leq I_0, \\ rI_0, I > I_0, \end{cases}$$

where I_0 is the infective level at which the health care system reaches capacity. Based on this, Zhang et al. [42] proposed the following saturated treatment function $\frac{rI}{1+kI}$, where r is the maximal medical resources supplied per unit time and k is the saturation factor that measures the effect of the infected being delayed treatment. A very general from of treatment function $T(I)$ was considered by Elazzouzi et al. [19].

Epidemiological models can contain a delay that is either discrete [8, 17] or continuous [7, 10, 35] because the delays appear in differential equations to describe the time lag between the action on the system and the system's response to this action, or because a some threshold must be reached before the system is activated. In [33], Liu proposed an SEIR epidemic model with saturated incidence and saturated treatment function with two discrete delays: the time delay due to the latent period of the disease and the time delay due to the period that the infected individuals use to move into the recovered class.

Motivated by the above works, we propose a mathematical SIR model that generalizes the above models and incorporates the general nonlinear incidence function $f(S,I)I$, general function treatment $T(I)$ and two discrete time delays, the first τ_1 described the time delay due to the latent period of the disease and the second τ_2 is the time delay due to the period that the infected individuals use to move into the recovered class. This model is given by the following nonlinear system:

$$\begin{cases} \frac{dS(t)}{dt} &= A - \mu S(t) + \gamma_1 I(t) - f(S(t), I(t))I(t), \\ \frac{dI(t)}{dt} &= f(S(t-\tau_1), I(t-\tau_1))I(t-\tau_1) - \eta_1 I(t) - \gamma_2 I(t-\tau_2) - T(I(t-\tau_2)), \\ \frac{dR(t)}{dt} &= T(I(t-\tau_2)) + \gamma_2 I(t-\tau_2) - \mu R(t), \end{cases} \tag{1.1}$$

where $\eta_1 = \mu + \gamma_1 + \alpha$. Here A is the constant recruitment rate into the population, $S(t)$ represents the number of individuals who are susceptible to disease, that is, who are not yet infected at time t, $I(t)$ represents the number of infected individuals who are infectious and are able to spread the disease by contact with susceptible individuals at time t, $R(t)$ is the number of individuals who have been infected and temporarily recovered at time t, μ is the natural death rate of the population, τ_1 is the units of time after infection expressing latent period, τ_2, is the time delay due to the period that the infected individuals use to move into the recovered class, γ_1 is the transfer rate from the infected class to the susceptible class, γ_2 is the transfer rate from the infected class to the recovered class, α is the disease-induced death rate, $T(I)$ is the general treatment function and $f(S,I)I$ is the incidence function, i.e., the number of susceptible individuals infected through their contacts with the infectious individuals.

Model (1.1) generalizes several special cases existing in the literature. For example, Abta et al. [1] studied the model (1.1) with $f(S,I)I = \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I}$, $\gamma_1 = 0$, $\tau_2 = 0$ and $T(I) = 0$. In [25], Hattaf et al. studied the model (1.1) with $\gamma_1 = 0$, $\tau_2 = 0$ and $T(I) = 0$. In [40], Wang and Ruan studied the model (1.1) with $f(S,I)I = \beta SI$, $\gamma_1 = 0$, $\tau_1 = \tau_2 = 0$ and $T(I) = r$. In [39], Wang studied the model (1.1) with $f(S,I)I = \beta SI$, $\gamma_1 = 0$, $\tau_1 = \tau_2 = 0$ and

$$T(I) = \begin{cases} rI, 0 \le I \le I_0, \\ rI_0, I > I_0. \end{cases}$$

In [5], Balamuralitharan et al. studied the model (1.1) with $f(S,I)I = \beta SI$, $\gamma_1 = 0$, $\tau_1 = \tau_2 = 0$ and $T(I) = rI$. In this paper, the initial condition for the system (1.1) is:

$$S(\theta) = \varphi_1(\theta), \quad I(\theta) = \varphi_2(\theta), \quad R(\theta) = \varphi_3(\theta), \quad \theta \in [-\tau, 0], \tag{1.2}$$

where $\tau = \max\{\tau_1, \tau_2\}$ and $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T \in \mathbb{C}$ such that $\varphi_i(\theta) \ge 0$ for $-\tau \le \theta \le 0$ and $i = 1,2,3$. The space $\mathbb{C}$ denotes the Banach space $\mathbb{C}([-\tau,0], \mathbb{R}^3_{+0})$ of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^3_{+0}$ with the supremum norm, where $\mathbb{R}_{+0} = \{x \in \mathbb{R} \mid x \ge 0\}$. Also, we assume that $\varphi_i(0) > 0$ for $i = 1,2,3$. On the other hand, the first two equation in system (1.1) do not depend on the third equation, and therefore this equation can be omitted without loss of generality. System (1.1) can be rewritten as

$$\begin{cases} \frac{dS(t)}{dt} &= A - \mu S(t) + \gamma_1 I(t) - f(S(t), I(t))I(t), \\ \frac{dI(t)}{dt} &= f(S(t-\tau_1), I(t-\tau_1))I(t-\tau_1) - \eta_1 I(t) - \gamma_2 I(t-\tau_2) - T(I(t-\tau_2)), \end{cases} \tag{1.3}$$

the incidence function $f(S,I)I : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuously differentiable and locally Lipschitz function on $\mathbb{R}^+ \times \mathbb{R}^+$ and satisfying the following hypotheses (see [11, 25]):

(H_0) $f(0,I) = 0$ for $I \ge 0$;

(H_1) $f(S,I)$ is a strictly monotone increasing function of $S > 0$, for any fixed $I \ge 0$;

(H_2) $f(S,I)$ is a monotone decreasing function of $I \ge 0$, for any fixed $S \ge 0$;

(H_3) $\phi(S,I) = f(S,I))I$ is a monotone increasing function of $I \ge 0$, for any fixed $S \ge 0$.

Moreover, the treatment function $T : \mathbb{R}^+ \to \mathbb{R}^+$ is a locally Lipschitz continuous differentiable function on $\mathbb{R}^+$ satisfying the following hypotheses (see [19])

(T_0) $T(0) = 0$;

(T_1) $\frac{T(I)}{I}$ is a monotone increasing function of $I > 0$.

The rest of the paper is organized as follows. In Section 2, we carry out mathematical analysis about the basic reproduction number and the existence of equilibria of the model (1.1). In Section 3, we study the global stability of the disease-free equilibrium. Section 4 is devoted to the stability and Hopf bifurcation of the endemic equilibrium. An application of our results and some numerical simulations are presented in Section 5. At the end, we present some concluding remarks.

2. MATHEMATICAL ANALYSIS

In this section, we prove the basic results which guarantee the positivity of solutions as well as the existence and uniqueness of the endemic equilibrium for system (1.3) under initial condition (1.2).

Lemma 2.1:
The closed set:

$$\Omega := \left\{ (S, I) \in (\mathbb{R}^+)^2 | S + I \leq \frac{A}{\mu} \right\}$$

is positively invariant with respect to system (1.3).

Proof
Let $N(t) = S(t) + I(t)$, and $(S, I) \in (\mathbb{R}^+)^2$. Then it follows from system (1.3) that

$$\begin{aligned} \frac{dN(t)}{dt} &= A - \mu S(t) - \mu I(t) - \alpha I(t) - \gamma_2 I(t - \tau_2) - f(S(t), I(t))I(t) \\ &+ f(S(t - \tau_1), I(t - \tau_1))I(t - \tau_1) - T(I(t - \tau_2)) \\ &= A - \mu N(t) - \int_{t-\tau_1}^{t} f(S(\sigma), I(\sigma))I(\sigma)d\sigma - T(I(t - \tau_2)) - \alpha I(t) - \gamma_2 I(t - \tau_2). \end{aligned}$$

By the hypotheses, we have

$$T(I(t - \tau_2)) \geq 0 \; and \; f(S(t), I(t))I(t) \geq 0 \; for \; all \; t \geq 0 \; and \; \tau_2 \geq 0.$$

Then

$$\frac{dN(t)}{dt} \leq A - \mu N(t),$$

which implies that $N(t) \leq \frac{A}{\mu}$ when $N(0) \leq \frac{A}{\mu}$. This completes the proof. □

Hence, we discuss system (1.3) in the closed set Ω. Next, we discuss the existence of equilibria for system (1.3).
The basic reproduction number of system (1.3) can be defined by

$$R_0 = \frac{f(\frac{A}{\mu}, 0)}{\eta_1 + \eta_2},$$

where $\eta_2 = \gamma_2 + T^{'}(0)$. Note that the system (1.3) always has a disease-free equilibrium $P_0 = (\frac{A}{\mu}, 0)$. On the other hand, to prove the existence and uniqueness of an endemic equilibrium, we need the following Lemma as in [19].

Lemma 2.2:
Assume that the assumptions (T_0) *and* (T_1)*, are satisfied. Then* $b_2 - a_2 u - T(u) = 0$*, for* $a_2 > 0$ *and* $b_2 > 0$*, has a unique positive solution* u_1*, and* $b_2 - a_2 u - T(u) > 0$*, for all* $u \in [0, u_1)$*, and* $b_2 - a_2 u - T(u) \leq 0$ *for all* $u \in [u_1, \frac{b_2}{a_2}]$.

Proof
Let L be the function defined on $\mathbb{R}^+$ by

$$L(u) := b_2 - a_2 u - T(u),$$

we have: $L(0) = b_2 > 0$ and $L(\frac{b_2}{a_2}) = -T(\frac{b_2}{a_2}) < 0$ and $L^{'} = -a_2 - T^{'} < 0$.
Since L is continuous and is strictly monotone decreasing function, then the equation $L(u) = 0$, has a unique positive solution in the interval $(0, \frac{b_2}{a_2})$. This completes the proof. □

Theorem 2.1:
Under the hypotheses $(H_1), (H_2), (T_0)$ *and* (T_1) *if* $R_0 > 1$, *then system* (1.3) *admits a unique endemic equilibrium* $P^* = (S^*, I^*)$, *with*

$$S^* = \frac{A}{\mu} - \frac{(\mu + \gamma_2 + \alpha)I^* + T(I^*)}{\mu},$$

and I^* *is the unique solution of the following equation:*

$$f(\frac{A}{\mu} - \frac{(\mu + \gamma_2 + \alpha)I + T(I)}{\mu}, I) = \eta_1 + \gamma_2 + \frac{T(I)}{I}.$$

Proof
For simplicity, we put

$$K(I) = \frac{T(I)}{I}.$$

At an equilibrium point (S, I) of system (1.3), the following equations hold.

$$\begin{cases} A - \mu S + \gamma_1 I - f(S, I)I = 0, \\ f(S, I)I - (\eta_1 + \gamma_2)I - T(I) = 0. \end{cases} \tag{2.4}$$

Substituting the second equation into the first equation of (2.4), we obtain the following system:

$$\begin{cases} S = \frac{A}{\mu} - \frac{(\mu + \gamma_2 + \alpha)I + T(I)}{\mu}, \\ f(S, I)I = (\eta_1 + \gamma_2)I + T(I). \end{cases} \tag{2.5}$$

If $I = 0$, we obtain the disease-free equilibrium point $P_0 = (\frac{A}{\mu}, 0)$. If $I \neq 0$, then using (2.5) we get the following equation

$$f(\frac{A}{\mu} - \frac{(\mu + \gamma_2 + \alpha)I + T(I)}{\mu}, I) = \eta_1 + \gamma_2 + K(I).$$

We have $S = \frac{A}{\mu} - \frac{[(\mu+\gamma_2+\alpha)I+T(I)]}{\mu}$, which implies that $S \leq \frac{A}{\mu}$. By Lemma 2.2, we have $\frac{A}{\mu} - \frac{(\mu+\gamma_2+\alpha)I+T(I)}{\mu} > 0$ if and only if $I \in [0, I_1)$, where I_1 is a unique positive solution of the equation $\frac{A}{\mu} - \frac{[(\mu+\gamma_2+\alpha)I+T(I)]}{\mu} = 0$. Hence, there is no positive equilibrium point if $S > \frac{A}{\mu}$ or $I \geq I_1$. Now, we consider the function g defined on the interval $[0, I_1]$ as follows

$$g(I) := f(\frac{A}{\mu} - \frac{(\mu + \gamma_2 + \alpha)I + T(I)}{\mu}, I) - (\eta_1 + \gamma_2 + K(I)).$$

We have

$$\begin{aligned} \lim_{I \to 0} g(I) &= f(\frac{A}{\mu}, 0) - (\eta_1 + \eta_2) \\ &= (\eta_1 + \eta_2)\Big(\frac{f(\frac{A}{\mu}, 0)}{\eta_1 + \eta_2} - 1\Big) \\ &= (\eta_1 + \eta_2)(R_0 - 1) > 0 \; for \; R_0 > 1, \end{aligned}$$

and

$$g(I_1) = -(\eta_1 + \gamma_2 + K(I_1)) < 0.$$

Furthermore,

$$g^{'}(I) = -\frac{\mu + \gamma_2 + \alpha + T^{'}}{\mu}\frac{\partial f}{\partial S} + \frac{\partial f}{\partial I} - K^{'}.$$

According to the hypotheses (H_1), (H_2) and (T_1), we have $g^{'}(I) < 0$. Hence, there exists a unique endemic equilibrium $P^* = (S^*, I^*)$ with $I^* \in (0, I_1)$ and $S^* > 0$ satisfies the equations $S^* = \frac{A}{\mu} - \frac{(\mu+\gamma_2+\alpha)I^*+T(I^*)}{\mu}$. This completes the proof. □

3. GLOBAL STABILITY OF THE DISEASE-FREE EQUILIBRIUM

In this section, we establish the global stability of the disease-free equilibrium P_0 of system (1.3). We can defined

$$R_c = \frac{f(\frac{A}{\mu}, 0)}{\eta_1}.$$

Note that

$$R_0 < R_c.$$

Theorem 3.1:
Suppose the hypotheses (H_1), (H_2), (T_0) and (T_1) hold.

- *If $R_0 > 1$, then the disease-free equilibrium P_0 of system* (1.3) *is unstable.*
- *If $R_0 \leq 1$ then we distinguish three cases:*
  - *(i) The disease-free equilibrium P_0 of system* (1.3) *is globally asymptotically stable whenever $\tau_1 \geq \tau_2$.*
  - *(ii) When $R_c \leq 1$, the disease-free equilibrium P_0 of system* (1.3) *is globally asymptotically stable for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$, then in particularly for $\tau_2 \geq \tau_1$.*
  - *(iii) When $R_c > 1$, the equation* (3.9) *has a purely imaginary root. If more $h_0^{'}(\omega^*_{22,0}) \neq 0$, then there exists a positive $\tau^*_{22,0}$ where system* (1.3) *undergoes Hopf bifurcation at P_0 when $\tau_2 = \tau^*_{22,0}$. However, the steady state P_0 is locally asymptotically stable when $\tau_2 \in [0, \tau^*_{22,0})$ and unstable when $\tau_2 > \tau^*_{22,0}$, where $\tau^*_{22,0}$ is given by*

$$\tau^*_{22,0} = \frac{1}{\omega^*_{22,0}} \arccos\left(\frac{f(\frac{A}{\mu}, 0)\cos(\omega^*_{22,0}\tau_1) - \eta_1}{\eta_2}\right),$$

and $\omega^*_{22,0}$ is the positive root of equation (3.9)*, and h_0 are defined in equation* (3.13).

Proof

• If $R_0 > 1$, then the characteristic equation at P_0 is given by

$$(\lambda + \mu)(\lambda - f(\frac{A}{\mu}, 0)e^{-\lambda\tau_1} + \eta_1 + \eta_2 e^{-\lambda\tau_2}) = 0. \quad (3.6)$$

Obviously, $\lambda = -\mu$ is eigenvalue for (3.6), and hence, the stability of P_0 is determined by the distribution of the roots of equation

$$\lambda - f(\frac{A}{\mu}, 0)e^{-\lambda\tau_1} + \eta_1 + \eta_2 e^{-\lambda\tau_2} = 0. \quad (3.7)$$

We put

$$\Psi(\lambda) = \lambda - f(\frac{A}{\mu}, 0)e^{-\lambda\tau_1} + \eta_1 + \eta_2 e^{-\lambda\tau_2}.$$

We have

$$\lim_{\lambda \to +\infty} \Psi(\lambda) = +\infty,$$

and

$$\begin{aligned}\Psi(0) &= -f(\frac{A}{\mu}, 0) + \eta_1 + \eta_2 \\ &= (\eta_1 + \eta_2)(1 - \frac{f(\frac{A}{\mu}, 0)}{(\eta_1 + \eta_2)}) \\ &= (\eta_1 + \eta_2)(1 - R_0).\end{aligned}$$

Since the function $\Psi(\lambda)$ is continuous on the interval $[0, +\infty)$, we conclude that the equation $\Psi(\lambda) = 0$ has a positive real root and the disease-free equilibrium is unstable when $R_0 > 1$.

• If $R_0 \leq 1$, then there are three cases:

- First case $\tau_1 > \tau_2$

Consider the following Lyapunov functional

$$\begin{aligned}V_0(t) = &\int_{\frac{A}{\mu}}^{S(t-\tau_1)} (1 - \frac{f(\frac{A}{\mu}, 0)}{f(\sigma, 0)})d\sigma + I \\ &+ \eta_1 \int_{t-\tau_2}^{t} I(\xi)d\xi + \int_{t-\tau_1}^{t-\tau_2} \frac{f(\frac{A}{\mu}, 0)}{f(S(\sigma), 0)} f(S(\sigma), I(\sigma))I(\sigma)d\sigma.\end{aligned}$$

We will show that $\frac{dV_0(t)}{dt} \leq 0$ for all $t \geq 0$. We have

$$\begin{aligned}\frac{dV_0(t)}{dt} &= (1 - \frac{f(\frac{A}{\mu}, 0)}{f(S(t-\tau_1), 0)})(A - \mu S(t-\tau_1) + \gamma_1 I(t-\tau_1) \\ &- f(S(t-\tau_1), I(t-\tau_1))I(t-\tau_1)) \\ &+ f(S(t-\tau_1), I(t-\tau_1))I(t-\tau_1) - \eta_1 I(t) - \gamma_2 I(t-\tau_2) - T(I(t-\tau_2)) \\ &+ \eta_1 I(t) - \eta_1 I(t-\tau_2) \\ &+ \frac{f(\frac{A}{\mu}, 0)}{f(S(t-\tau_2), 0)} f(S(t-\tau_2), I(t-\tau_2))I(t-\tau_2)) \\ &- \frac{f(\frac{A}{\mu}, 0)}{f(S(t-\tau_1), 0)} f(S(t-\tau_1), I(t-\tau_1))I(t-\tau_1)) \\ &= \mu(1 - \frac{f(\frac{A}{\mu}, 0)}{f(S(t-\tau_1), 0)})(\frac{A}{\mu} - S(t-\tau_1)) + \gamma_1 I(t-\tau_1)(1 - \frac{f(\frac{A}{\mu}, 0)}{f(S(t-\tau_1), 0)}) \\ &+ \frac{f(\frac{A}{(\mu)}, 0)}{f(S(t-\tau_2), 0)} f(S(t-\tau_2), I(t-\tau_2))I(t-\tau_2)) \\ &- (\eta_1 + \gamma_2)I(t-\tau_2) - T(I(t-\tau_2)).\end{aligned}$$

Then we have

$$\begin{aligned}
\frac{dV_0(t)}{dt} &= \mu(1-\frac{f(\frac{A}{\mu},0)}{f(S(t-\tau_1),0)})(\frac{A}{\mu}-S(t-\tau_1))+\gamma_1 I(t-\tau_1)(1-\frac{f(\frac{A}{\mu},0)}{f(S(t-\tau_1),0)})\\
&+\Big((\eta_1+\gamma_2)I(t-\tau_2)+T(I(t-\tau_2))\Big)\\
&\times\Big(\frac{f(\frac{A}{\mu},0)f(S(t-\tau_2),I(t-\tau_2))I(t-\tau_2)}{f(S(t-\tau_2),0)[(\eta_1+\gamma_2)I(t-\tau_2)+T(I(t-\tau_2))]}-1\Big)\\
&= \mu(1-\frac{f(\frac{A}{\mu},0)}{f(S(t-\tau_1),0)})(\frac{A}{\mu}-S(t-\tau_1))+\gamma_1 I(t-\tau_1)(1-\frac{f(\frac{A}{\mu},0)}{f(S(t-\tau_1),0)})\\
&+\Big((\eta_1+\gamma_2)I(t-\tau_2)+T(I(t-\tau_2))\Big)\\
&\times\Big(R_0\frac{\eta_1+\eta_2}{\eta_1+\gamma_2+K(I(t-\tau_2))}\frac{f(S(t-\tau_2),I(t-\tau_2))}{f(S(t-\tau_2),0)}-1\Big).
\end{aligned}$$

Furthermore, it follows from the hypotheses (H_2), (T_0) and (T_1) that

$$\frac{f(S(t-\tau_2),I(t-\tau_2))}{f(S(t-\tau_2),0)}\leq 1,$$

and

$$\begin{aligned}
\frac{\eta_1+\eta_2}{\eta_1+\gamma_2+K(I(t-\tau_2))} &= \frac{\eta_1+\gamma_2+\lim_{I\to 0}\frac{T(I(t-\tau_2))}{I(t-\tau_2)}}{\eta_1+\gamma_2+\frac{T(I(t-\tau_2))}{I(t-\tau_2)}}\\
&\leq 1.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{dV_0(t)}{dt} &\leq \mu(1-\frac{f(\frac{A}{\mu},0)}{f(S(t-\tau_1),0)})(\frac{A}{\mu}-S(t-\tau_1))+\gamma_1 I(t-\tau_1)(1-\frac{f(\frac{A}{\mu},0)}{f(S(t-\tau_1),0)})\\
&+[(\eta_1+\gamma_2)I(t-\tau_2)+T(I(t-\tau_2))](R_0-1).
\end{aligned}$$

By the hypothesis (H_1), we obtain that

$$(1-\frac{f(\frac{A}{\mu},0)}{f(S(t-\tau_1),0)})(\frac{A}{\mu}-S(t-\tau_1))\leq 0,$$

and

$$I(t-\tau_1)(1-\frac{f(\frac{A}{\mu},0)}{f(S(t-\tau_1),0)})\leq 0,$$

where equality holds if and only if $S=\frac{A}{\mu}$.
Since $R_0\leq 1$, ensures that $\frac{dV_0(t)}{dt}\leq 0$, for all $t\geq 0$, Thus, the disease-free equilibrium P_0 is stable and $\frac{dV_0(t)}{dt}=0$ holds if and only if $S=\frac{A}{\mu}$ and $[(\eta_1+\gamma_2)I(t-\tau_2)+T(I(t-\tau_2))]\Big(R_0\frac{\eta_1+\eta_2}{\eta_1+\gamma_2+K(I(t-\tau_2))}\frac{f(S(t-\tau_2),I(t-\tau_2))}{f(S(t-\tau_2),0)}-1\Big)=0$. We discuss two cases:

- If $R_0 < 1$, then it follows from $S = \frac{A}{\mu}$ and Lemma 2.1 that $I = 0$.
- If $R_0 = 1$, then it follows from $S = \frac{A}{\mu}$ and Lemma 2.1 that $I = 0$.

By the above discussions, we deduce that $\{P_0\}$ is the largest invariant set in $\left\{(S, I)|\frac{dV_0(t)}{dt} = 0\right\}$. From the Lyapunov-LaSalle theorem [31, 32], we conclude that P_0 is globally asymptotically stable.

- Second case $\tau_1 = \tau_2 = \tau$

Consider the following Lyapunov functional

$$W_0(t) = \int_{\frac{A}{\mu}}^{S(t-\tau)} (1 - \frac{f(\frac{A}{\mu}, 0)}{f(\sigma, 0)})d\sigma + I + \eta_1 \int_{t-\tau}^{t} I(\xi)d\xi.$$

We will show that $\frac{dW_0(t)}{dt} \leq 0$ for all $t \geq 0$. We have

$$\begin{aligned}
\frac{dW_0(t)}{dt} &= (1 - \frac{f(\frac{A}{\mu}, 0)}{f(S(t-\tau), 0)})(A - \mu S(t-\tau) + \gamma_1 I(t-\tau) \\
&- f(S(t-\tau), I(t-\tau))I(t-\tau)) \\
&+ f(S(t-\tau), I(t-\tau))I(t-\tau) - \eta_1 I(t) - T(I(t-\tau)) + \eta_1 I(t) \\
&- (\eta_1 + \gamma_2)I(t-\tau) \\
&= \mu(1 - \frac{f(\frac{A}{\mu}, 0)}{f(S(t-\tau), 0)})(\frac{A}{\mu} - S(t-\tau)) + \gamma_1 I(t-\tau)(1 - \frac{f(\frac{A}{\mu}, 0)}{f(S(t-\tau), 0)}) \\
&+ \frac{f(\frac{A}{\mu}, 0)}{f(S(t-\tau), 0)} f(S(t-\tau), I(t-\tau))I(t-\tau)) \\
&- (\eta_1 + \gamma_2)I(t-\tau) - T(I(t-\tau)).
\end{aligned}$$

Then we have

$$\begin{aligned}
\frac{dW_0(t)}{dt} &= \mu(1 - \frac{f(\frac{A}{\mu}, 0)}{f(S(t-\tau), 0)})(\frac{A}{\mu} - S(t-\tau)) + \gamma_1 I(t-\tau)(1 - \frac{f(\frac{A}{\mu}, 0)}{f(S(t-\tau), 0)}) \\
&+ \Big((\eta_1 + \gamma_2)I(t-\tau) + T(I(t-\tau))\Big) \\
&\times \Big(\frac{f(\frac{A}{\mu}, 0) f(S(t-\tau), I(t-\tau))}{f(S(t-\tau), 0)(\eta_1 + \gamma_2 + K(I(t-\tau)))} - 1\Big) \\
&= \mu(1 - \frac{f(\frac{A}{\mu}, 0)}{f(S(t-\tau), 0)})(\frac{A}{\mu} - S(t-\tau)) + \gamma_1 I(t-\tau)(1 - \frac{f(\frac{A}{\mu}, 0)}{f(S(t-\tau), 0)}) \\
&+ \Big((\eta_1 + \gamma_2)I(t-\tau) + T(I(t-\tau))\Big) \\
&\times \Big(R_0 \frac{\eta_1 + \eta_2}{\eta_1 + \gamma_2 + K(I(t-\tau))} \frac{f(S(t-\tau), I(t-\tau))}{f(S(t-\tau), 0)} - 1\Big).
\end{aligned}$$

Furthermore, it follows from the hypotheses (H_2), (T_0) and (T_1), we have

$$\frac{f(S(t-\tau), I(t-\tau))}{f(S(t-\tau), 0)} \leq 1,$$

and

$$\frac{\eta_1+\eta_2}{(\eta_1+\gamma_2)+K(I(t-\tau))} = \frac{\eta_1+\gamma_2+\lim_{I\to 0}\frac{T(I(t-\tau))}{I(t-\tau)}}{\eta_1+\gamma_2+\frac{T(I(t-\tau))}{I(t-\tau)}} \leq 1.$$

Then

$$\frac{dW_0(t)}{dt} \leq \mu(1-\frac{f(\frac{A}{\mu},0)}{f(S(t-\tau),0)})(\frac{A}{\mu}-S(t-\tau))+\gamma_1 I(t-\tau)(1-\frac{f(\frac{A}{\mu},0)}{f(S(t-\tau),0)}) + [(\eta_1+\gamma_2)I(t-\tau)+T(I(t-\tau))](R_0-1).$$

By the hypothesis (H_1), we get that

$$(1-\frac{f(\frac{A}{\mu},0)}{f(S(t-\tau),0)})(\frac{A}{\mu}-S(t-\tau)) \leq 0,$$

and

$$I(t-\tau)(1-\frac{f(\frac{A}{\mu},0)}{f(S(t-\tau),0)}) \leq 0,$$

where equality holds if and only if $S=\frac{A}{\mu}$.
Since $R_0 \leq 1$, we deduce that $\frac{dW_0(t)}{dt} \leq 0$ for all $t \geq 0$. Thus, the disease-free equilibrium P_0 is stable and $\frac{dW_0(t)}{dt}=0$, holds if and only if $S=\frac{A}{\mu}$,
and $[(\eta_1+\gamma_2)I(t-\tau)+T(I(t-\tau))]\Big(R_0\frac{\eta_1+\eta_2}{\eta_1+\gamma_2+K(I(t-\tau))}\frac{f(S(t-\tau),I(t-\tau))}{f(S(t-\tau),0)}-1\Big)=0$. We discuss two cases:

- If $R_0<1$, then it follows from $S=\frac{A}{\mu}$ and Lemma 2.1 that $I=0$.
- If $R_0=1$, then it follows from $S=\frac{A}{\mu}$ and Lemma 2.1 that $I=0$.

Therefore, $\{P_0\}$ is the largest invariant set in $\Big\{(S,I)|\frac{dW_0(t)}{dt}=0\Big\}$. From the Lyapunov-LaSalle theorem [31, 32], we conclude that P_0 is globally asymptotically stable.

- Third case $\tau_2>\tau_1$

Suppose that $R_c \leq 1$. Consider the following Lyapunov functional

$$U_0(t)=\int_{\frac{A}{\mu}}^{S(t)}(1-\frac{f(\frac{A}{\mu},0)}{f(\sigma,0)})d\sigma+I+\int_{t-\tau_1}^{t}f(S(\xi),I(\xi))I(\xi)d\xi.$$

We will show that $\frac{dU_0(t)}{dt} \leq 0$ for all $t \geq 0$. We have

$$\begin{aligned}\frac{dU_0(t)}{dt} &= (1 - \frac{f(\frac{A}{\mu},0)}{f(S(t),0)})(A - \mu S(t) + \gamma_1 I(t) - f(S(t),I(t))I(t)) \\ &+ f(S(t-\tau),I(t-\tau))I(t-\tau) - \eta_1 I(t) - \gamma_2 I(t-\tau_2) - T(I(t-\tau_2)) \\ &+ f(S(t),I(t))I(t) - f(S(t-\tau),I(t-\tau))I(t-\tau) \\ &= \mu(1 - \frac{f(\frac{A}{\mu},0)}{f(S(t),0)})(\frac{A}{\mu} - S(t)) + \gamma_1 I(t)(1 - \frac{f(\frac{A}{\mu},0)}{f(S(t),0)}) \\ &+ \frac{f(\frac{A}{\mu},0)}{f(S(t),0)} f(S(t),I(t))I(t)) - \eta_1 I(t) - \gamma_2 I(t-\tau_2) - T(I(t-\tau_2)).\end{aligned}$$

Then we have

$$\begin{aligned}\frac{dU_0(t)}{dt} &= \mu(1 - \frac{f(\frac{A}{\mu},0)}{f(S(t),0)})(\frac{A}{\mu} - S(t)) + \gamma_1 I(t)(1 - \frac{f(\frac{A}{\mu},0)}{f(S(t),0)}) \\ &+ \eta_1 I(t)\Big(\frac{f(\frac{A}{\mu},0) f(S(t),I(t))}{f(S(t),0)\eta_1} - 1\Big) - \gamma_2 I(t-\tau_2) - T(I(t-\tau_2)).\end{aligned}$$

Hence,

$$\begin{aligned}\frac{dU_0(t)}{dt} &= \mu(1 - \frac{f(\frac{A}{\mu},0)}{f(S(t),0)})(\frac{A}{\mu} - S(t)) + \gamma_1 I(t)(1 - \frac{f(\frac{A}{\mu},0)}{f(S(t),0)}) \\ &+ \eta_1 I(t)\Big(R_c \frac{f(S(t),I(t))}{f(S(t),0)} - 1\Big) - \gamma_2 I(t-\tau_2) - T(I(t-\tau_2)).\end{aligned} \tag{3.8}$$

Furthermore, it follows from the hypotheses (H_2) that

$$\frac{f(S(t),I(t))}{f(S(t),0)} \leq 1,$$

and by the hypotheses, we have

$$I(t-\tau_2) \geq 0 \; and \; T(I(t-\tau_2)) \geq 0.$$

Then

$$\begin{aligned}\frac{dU_0(t)}{dt} &\leq \mu(1 - \frac{f(\frac{A}{\mu},0)}{f(S(t),0)})(\frac{A}{\mu} - S(t)) + \gamma_1 I(t)(1 - \frac{f(\frac{A}{\mu},0)}{f(S(t),0)}) \\ &+ \eta_1 I(t)\Big(R_c - 1\Big).\end{aligned}$$

By the hypothesis (H_1), we obtain that

$$(1 - \frac{f(\frac{A}{\mu},0)}{f(S(t),0)})(\frac{A}{\mu} - S(t)) \leq 0,$$

and

$$I(t-\tau)(1-\frac{f(\frac{A}{\mu},0)}{f(S(t),0)})\leq 0,$$

where equality holds if and only if $S=\frac{A}{\mu}$.
Since $R_c\leq 1,$ ensures that $\frac{dU_0(t)}{dt}\leq 0$ for all $t\geq 0$, Thus, the disease-free equilibrium P_0 is stable and by the equation (3.8) $\frac{dU_0(t)}{dt}=0$, holds if and only if $S=\frac{A}{\mu}$ and $I=0$.
By the above discussions, we deduce that $\{P_0\}$ is the largest invariant set in $\left\{(S,I)|\frac{dU_0(t)}{dt}=0\right\}$. From the Lyapunov-LaSalle theorem [31, 32], we conclude that P_0 is globally asymptotically stable for all $\tau_1\geq 0\ and\ \tau_2\geq 0$. Then P_0 is globally asymptotically stable for all $\tau_2>\tau_1$.

Now, we suppose that $R_0<1$ and $R_c>1$ The characteristic equation at P_0 is given by the equation (3.6), obviously, $\lambda=-\mu$, is eigenvalue for (3.6), and hence, the stability of P_0 is determined by the distribution of the roots of equation

$$\lambda-f(\frac{A}{\mu},0)e^{-\lambda\tau_1}+\eta_1+\eta_2e^{-\lambda\tau_2}=0. \tag{3.9}$$

Equation (3.9) has a purely imaginary root $i\omega_{22}$, with $\omega_{22}>0$ if and only if

$$f(\frac{A}{\mu},0)\cos(\omega_{22}\tau_1)-\eta_1=\eta_2\cos(\omega_{22}\tau_2), \tag{3.10}$$

$$\omega_{22}+f(\frac{A}{\mu},0)\sin(\omega_{22}\tau_1)=\eta_2\sin(\omega_{22}\tau_2). \tag{3.11}$$

Squaring and adding the squares together, we obtain

$$\begin{aligned}&\omega_{22}^2+\eta_1^2+f(\frac{A}{\mu},0)^2-\eta_2^2\\&\quad-2\eta_1f(\frac{A}{\mu},0)\cos(\omega_{22}\tau_1)+2\omega_{22}f(\frac{A}{\mu},0)\sin(\omega_{22}\tau_1)=0.\end{aligned} \tag{3.12}$$

We put

$$\begin{aligned}h_0(\omega_{22})=&\ \omega_{22}^2+\eta_1^2+f(\frac{A}{\mu},0)^2-\eta_2^2\\&-2\eta_1f(\frac{A}{\mu},0)\cos(\omega_{22}\tau_1)+2\omega_{22}f(\frac{A}{\mu},0)\sin(\omega_{22}\tau_1).\end{aligned} \tag{3.13}$$

On the other hand, we have

$$-1\leq\cos(\omega_{22}\tau_1)\leq 1\ and\ -1\leq\sin(\omega_{22}\tau_1)\leq 1.$$

Then we have

$$g_{01}(\omega_{22})\leq h_0(\omega_{22})\leq g_{02}(\omega_{22}),$$

where

$$g_{01}(\omega_{22})=\omega_{22}^2+\eta_1^2+f(\frac{A}{\mu},0)^2-\eta_2^2-2\eta_1f(\frac{A}{\mu},0)-2\omega_{22}f(\frac{A}{\mu},0),$$

and

$$g_{02}(\omega_{22})=\omega_{22}^2+\eta_1^2+f(\frac{A}{\mu},0)^2-\eta_2^2+2\eta_1f(\frac{A}{\mu},0)+2\omega_{22}f(\frac{A}{\mu},0).$$

We have

$$\lim_{\omega_{22}\to+\infty} g_{01}(\omega_{22}) = \lim_{\omega_{22}\to+\infty} g_{02}(\omega_{22}) = +\infty.$$

Then

$$\lim_{\omega_{22}\to+\infty} h_0(\omega_{22}) = +\infty,$$

and

$$\begin{aligned}
h_0(0) &= \eta_1^2 + f(\frac{A}{\mu},0)^2 - \eta_2^2 - 2\eta_1 f(\frac{A}{\mu},0) \\
&= (f(\frac{A}{\mu},0) - \eta_1)^2 - \eta_2^2 \\
&= \Big(f(\frac{A}{\mu},0) - \eta_1 - \eta_2\Big)\Big(f(\frac{A}{\mu},0) - \eta_1 + \eta_2\Big) \\
&= \Big(\eta_1 + \eta_2\Big)\Big(R_0 - 1\Big)\Big(\eta_1(R_c - 1) + \eta_2\Big) \\
&\le 0.
\end{aligned}$$

Since $h_0(\omega_{22})$ is continuous in $[0, +\infty)$, then the equation (3.9) has at least one positive root. We assume that equation (3.9) admits a finite family of solution $\omega^*_{22,o}$, with $o = 1, ..., m$ and $m \in \mathbb{N}$.
By the equation (3.10), we have

$$\tau^f_{2,o} = \frac{1}{\omega^*_{22,o}}\Big(\arccos\big(\frac{f(\frac{A}{\mu},0)\cos(\omega^*_{22,o}\tau_1) - \eta_1}{\eta_2}\big) + 2\Pi f\Big), f = 0, 1, ...; o = 1, 2, ..., m.$$

Then $\pm i\omega^*_{22,o}$ is a pair of purely imaginary root of equation (3.9), with $\tau_2 = \tau^f_{2,o},\ f = 1, 2, ...;\ o = 1, 2, ..., m$. Clearly,

$$\lim_{f\to\infty} \tau^f_{2,o} = \infty,\ o = 1, 2, ..., m.$$

Thus, we can define

$$\tau^*_{22,0} = \tau^{f_0}_{2,o_0} = \min_{f=0,1...,o=1,2,...,m}(\tau^f_{2,o}),\ \omega^*_{22,0} = \omega^*_{2,o_0}.$$

Lets show that $i\omega^*_{22,0}$ is simple, consider the branche of characteristic roots $\lambda(\tau_2) = x_1(\tau_2) + iy_1(\tau_2)$, of equation (3.9) bifurcating from $i\omega^*_{22,0}$ at $\tau_2 = \tau^*_{22,0}$. By derivation (3.9) with respect to the delay τ_2, we obtain

$$\frac{d\lambda}{d\tau_2}\Big(1 + \tau_1 f(\frac{A}{\mu},0)e^{-\lambda\tau_1} - \tau_2\eta_2 e^{-\lambda\tau_2}\Big) = \lambda\eta_2 e^{-\lambda\tau_2}. \tag{3.14}$$

If we suppose, by contradiction, that $i\omega^*_{22,0}$ is not simple, the right hand side (3.14) gives

$$i\omega^*_{22,0}\eta_2 = 0,$$

and leads a contradiction with the fact that $\omega^*_{22,0} > 0$, and $\eta_2 > 0$.
Next, we need to grantee the transversality condition of the Hopf bifurcation theorem (see [23]). Clearly, $\lambda(\tau_2) = x_1(\tau_2) + iy_1(\tau_2)$, is a root of equation (3.9) if and only if

$$x_1 + \eta_1 - e^{-x_1\tau_1} f(\frac{A}{\mu},0)\cos(y_1\tau_1) = -e^{-x_1\tau_2}\eta_2\cos(y_1\tau_2), \tag{3.15}$$

$$y_1 + e^{-x_1\tau_1} f(\frac{A}{\mu}, 0) \sin(y_1\tau_1) = e^{-x_1\tau_2}\eta_2 \sin(y_1\tau_2). \tag{3.16}$$

Let $x_1(\tau^*_{22,0})$ and $y_1(\tau^*_{22,0})$ satisfying $x_1(\tau^*_{22,0}) = 0$, and $y_1(\tau^*_{22,0}) = \omega^*_{22,0}$. By differentiating equations (3.15) and (3.16) with respect to τ_2 and then set $\tau_2 = \tau^*_{22,0}$. Doing this, we get

$$\begin{aligned} &G_{1,0}\frac{dx_1(\tau^*_{22,0})}{d\tau_2} + G_{2,0}\frac{dy_1(\tau^*_{22,0})}{d\tau_2} = h_{1,0}, \\ &- G_{2,0}\frac{dx_1(\tau^*_{22,0})}{d\tau_2} + G_{1,0}\frac{dy_1(\tau^*_{22,0})}{d\tau_2} = h_{2,0}, \end{aligned} \tag{3.17}$$

where

$$G_{1,0} = 1 + \tau_1 f(\frac{A}{\mu}, 0) \cos(\omega^*_{22,0}\tau_1) - \tau_2\eta_2 \cos(\omega^*_{22,0}\tau_2),$$

$$G_{2,0} = \tau_1 f(\frac{A}{\mu}, 0) \sin(\omega^*_{22,0}\tau_1) - \tau_2\eta_2 \sin(\omega^*_{22,0}\tau_2),$$

$$h_{1,0} = \omega^*_{22,0}\eta_2 \sin(\omega^*_{22,0}\tau_2),$$

and

$$h_{2,0} = \omega^*_{22,0}\eta_2 \cos(\omega^*_{22,0}\tau_2).$$

Calculating $\frac{dx_1(\tau^*_{22,0})}{d\tau_2}$, we get

$$\frac{dx_1(\tau^*_{22,0})}{d\tau_2} = \frac{G_{1,0}h_{1,0} - G_{2,0}h_{2,0}}{G^2_{1,0} + G^2_{2,0}}.$$

Therefore, according to the equation (3.10) and (3.11) we have

$$\frac{dx_1(\tau^*_{22,0})}{d\tau_2} = \frac{\omega^*_{22,0}h'(\omega^*_{22,0})}{2(G^2_{1,0} + G^2_{2,0})} \neq 0.$$

This proves the Theorem. □

4. STABILITY AND HOPF BIFURCATION OF THE ENDEMIC EQUILIBRIUM

In the next, we will study the local stability of the positive equilibrium P^* with respect to the time delay.

We set $x = S - S^*$, and $y = I - I^*$. Then the linearized system of equations around the equilibrium point P^* is given as follows:

$$\begin{aligned} \frac{dx}{dt} &= (-\mu - \frac{\partial f(S^*, I^*)}{\partial S}I^*)x(t) - (\frac{\partial f(S^*, I^*)}{\partial I}I^* + f(S^*, I^*) - \gamma_1)y(t), \\ \frac{dy}{dt} &= \frac{\partial f(S^*, I^*)}{\partial S}I^*x(t-\tau_1) + (\frac{\partial f(S^*, I^*)}{\partial I}I^* + f(S^*, I^*))y(t-\tau_1) - \eta_1 y(t) \\ &\quad -(\gamma_2 + T'(I^*))y(t-\tau_2). \end{aligned}$$

For simplicity, we put $\eta_3 = \gamma_2 + T'(I^*)$, and $\eta_4 = \mu + I^*\frac{\partial f(S^*,I^*)}{\partial S}$. Hence, the characteristic equation at P^* is given by

$$\Delta(\lambda, \tau_1, \tau_2) = \begin{vmatrix} \lambda + \eta_4 & I^*\frac{\partial f(S^*,I^*)}{\partial I} + f(S^*, I^*) - \gamma_1 \\ -I^*\frac{\partial f(S^*,I^*)}{\partial S}e^{-\lambda\tau_1} & \lambda + \eta_1 - (I^*\frac{\partial f(S^*,I^*)}{\partial I} + f(S^*, I^*))e^{-\lambda\tau_1} + \eta_3 e^{-\lambda\tau_2} \end{vmatrix} = 0.$$

By a simple computation, we get

$$\Delta(\lambda,\tau_1,\tau_2) = P(\lambda) + Q(\lambda)e^{-\lambda\tau_1} + J(\lambda)e^{-\lambda\tau_2} = 0, \tag{4.18}$$

where

$$P(\lambda) = \lambda^2 + A_1\lambda + B,$$
$$Q(\lambda) = C\lambda + D,$$
$$J(\lambda) = E\lambda + F,$$
$$A_1 = \eta_1 + \eta_4,$$
$$B = \eta_1\eta_4,$$
$$C = -(\frac{\partial f(S^*,I^*)}{\partial I}I^* + f(S^*,I^*)),$$

$$\begin{aligned} D &= -\eta_4(\frac{\partial f(S^*,I^*)}{\partial I}I^* + f(S^*,I^*)) \\ &\quad + \frac{\partial f(S^*,I^*)}{\partial S}I^*(\frac{\partial f(S^*,I^*)}{\partial I}I^* + f(S^*,I^*)) - \gamma_1\frac{\partial f(S^*,I^*)}{\partial S}I^* \\ &= -\mu(\frac{\partial f(S^*,I^*)}{\partial I}I^* + f(S^*,I^*)) - \gamma_1\frac{\partial f(S^*,I^*)}{\partial S}I^*, \end{aligned}$$

$$E = \eta_3,$$

and

$$F = \eta_4\eta_3.$$

By the hypotheses (H_2) and (T_1), we easily deduce that (see [20, 28])

$$T^{'}(I^*) - \frac{T(I^*)}{I^*} \geq 0, \tag{4.19}$$

and

$$f(S^*,I^*) - \frac{\partial\phi(S^*,I^*)}{\partial I} \geq 0. \tag{4.20}$$

Several cases arise.

4.1. Case $\tau_1 = \tau_2 = 0$

Theorem 4.1:
If $R_0 > 1$ and $\tau_1 = \tau_2 = 0$, then the endemic equilibrium P^* is locally asymptotically stable.

Proof
When $\tau_1 = \tau_2 = 0$, the characteristic equation (4.18) reads as

$$\lambda^2 + (A_1 + C + E)\lambda + (B + D + F) = 0$$

By using the second equation of system (1.3), we have

$$f(S^*,I^*) = (\eta_1 + \gamma_2) + \frac{T(I^*)}{I^*}. \tag{4.21}$$

Then

$$\begin{aligned} A_1 + C + E &= \eta_1 + \eta_4 - (\frac{\partial f(S^*, I^*)}{\partial I} I^* + f(S^*, I^*)) + \eta_3 \\ &= \eta_4 - \frac{\partial f(S^*, I^*)}{\partial I} I^* + (T^{'}(I^*) - \frac{T(I^*)}{I^*}), \end{aligned}$$

and

$$\begin{aligned} B + D + F &= \eta_1\eta_4 - \mu(\frac{\partial f(S^*, I^*)}{\partial I} I^* + f(S^*, I^*)) - \gamma_1 \frac{\partial f(S^*, I^*)}{\partial S} I^* \\ &+ \eta_4\eta_3 \\ &= \mu(T^{'}(I^*) - \frac{T(I^*)}{I^*}) + \frac{\partial f(S^*, I^*)}{\partial S} I^*[(\mu + \alpha) + \eta_3] - \mu \frac{\partial f(S^*, I^*)}{\partial I} I^*. \end{aligned}$$

By the hypotheses (H_1), (H_2), (T_1), and equation (4.19), we have $A_1 + C + E > 0$ and $B + D + F > 0$. Hence, according to the Routh-Hurwitz criterion, the endemic equilibrium P^* is locally asymptotically stable. This completes the proof. □

4.2. Case $\tau_1 > 0$ *and* $\tau_2 = 0$

Theorem 4.2:
If $R_0 > 1$, $\tau_1 > 0$ and $\tau_2 = 0$, then the endemic equilibrium P^* is locally asymptotically stable.

Proof
When $\tau_1 > 0$ and $\tau_2 = 0$ the characteristic equation (4.18) becomes

$$\Delta(\lambda, \tau_1) = \lambda^2 + (A_1 + E)\lambda + B + F + (\lambda C + D)e^{-\lambda\tau_1} = 0. \tag{4.22}$$

Equation (4.22) has a purely imaginary root $i\omega_1$, with $\omega_1 > 0$ if and only if

$$\omega_1^2 - (B + F) = D\cos(\omega_1\tau_1) + \omega_1 C \sin(\omega_1\tau_1),$$

and

$$\omega_1(A_1 + E) = D\sin(\omega_1\tau_1) - \omega_1 C\cos(\omega_1\tau_1).$$

Squaring and adding the squares together, we obtain

$$\omega_1^4 + c\omega_1^2 + d = 0, \tag{4.23}$$

with $c = (A_1 + E)^2 - 2(B + F) - C^2$ and $d = (B + F)^2 - D^2$.
Letting $z_1 = \omega_1^2$, then equation (4.23) becomes the following equation

$$z_1^2 + cz_1 + d = 0.$$

On the other hand, we have

$$\begin{aligned} d &= (B + F)^2 - D^2 \\ &= (B + F - D)(B + F + D). \end{aligned}$$

Using the case $\tau_1 = \tau_2 = 0$, we have $(B + F + D) > 0$ and by using the hypotheses (H_1), (H_3) and (T_1), we have $(B + F - D) > 0$ then we have $d > 0$. Now, we will prove $c > 0$.

Indeed $c = (A_1 + E)^2 - 2(B + F) - C^2$, we have

$$(A_1 + E)^2 = (\eta_1 + \eta_3 + \eta_4)^2.$$

By using the equation (4.21), find that

$$\begin{aligned}(A_1 + E)^2 &= \Big(f(S^*, I^*) + \eta_4 + T^{'}(I^*) - \frac{T(I^*)}{I^*}\Big)^2 \\ &= f(S^*, I^*)^2 + \eta_4^2 + 2f(S^*, I^*)\eta_4 \\ &+ (T^{'}(I^*) - \frac{T(I^*)}{I^*})^2 + 2(T^{'}(I^*) - \frac{T(I^*)}{I^*})f(S^*, I^*) + 2(T^{'}(I^*) - \frac{T(I^*)}{I^*})\eta_4,\end{aligned}$$

and

$$\begin{aligned}-2(B + F) &= -2\eta_4(\eta_1 + \eta_3) \\ &= -2f(S^*, I^*)\eta_4 - 2\eta_4(T^{'}(I^*) - \frac{T(I^*)}{I^*}),\end{aligned}$$

and

$$\begin{aligned}-C^2 &= -(\frac{\partial f(S^*, I^*)}{\partial I} I^* + f(S^*, I^*))^2 \\ &= -(\frac{\partial \phi(S^*, I^*)}{\partial I})^2.\end{aligned}$$

Then we have

$$\begin{aligned}c = &\Big(f(S^*, I^*) - \frac{\partial \phi(S(t), I(t))}{\partial I}\Big)\Big(f(S^*, I^*) + \frac{\partial \phi(S(t), I(t))}{\partial I}\Big) + \eta_4^2 \\ &+ (T^{'}(I^*) - \frac{T(I^*)}{I^*})^2 + 2(T^{'}(I^*) - \frac{T(I^*)}{I^*})f(S^*, I^*).\end{aligned}$$

By the hypothesis (H_3) and equations (4.19) and (4.20), we have $c > 0$. Then the equation (4.23) has non positive solution.
Consequently, using the case $\tau_1 = \tau_2 = 0$, the endemic equilibrium P^* is locally asymptotically stable . This completes the proof. □

4.3. Case $\tau_1 = \tau_2 = \tau$

Consider the assumption:

$$\eta_1 - \gamma_2 + \frac{\mu \frac{\partial \phi(S^*, I^*)}{\partial I} + \gamma_1 \frac{\partial f(S^*, I^*)}{\partial S} I^*}{\eta_4} < T^{'}(I^*). \tag{4.24}$$

Theorem 4.3:
If $R_0 > 1$ and (4.24) *holds, then there exists a positive τ_0 where system* (1.3) *undergoes Hopf bifurcation at P^* when $\tau = \tau_0$. However, the steady state P^* is locally asymptotically stable*

when $\tau \in [0, \tau_0)$ and unstable when $\tau > \tau_0$. Here τ_0 is given by

$$\tau_0 = \frac{1}{\omega_0} \arccos\left(\frac{(D+F)(\omega_0^2 - B) - \omega_0^2 A_1 (C+E)}{\omega_0^2 (C+E)^2 + (D+F)^2}\right),$$

and

$$\omega_0 = \sqrt{\frac{-x + \sqrt{(x^2 - 4y)}}{2}},$$

where x and y are defined in equation (4.28).

Proof
When $\tau_1 = \tau_2 = \tau$, the characteristic equation (4.18) becones

$$\Delta(\lambda, \tau) = \lambda^2 + A_1 \lambda + B + [\lambda(C+E) + (D+F)]e^{-\lambda\tau} = 0. \tag{4.25}$$

Equation (4.25) has a purely imaginary root $i\omega$, with $\omega > 0$

$$\Delta(i\omega, \tau) = 0,$$

if and only if

$$\omega^2 - B = \omega(C+E)\sin(\omega\tau) + (D+F)\cos(\omega\tau), \tag{4.26}$$

$$\omega A_1 = (D+F)\sin(\omega\tau) - \omega(C+E)\cos(\omega\tau). \tag{4.27}$$

Squaring and adding the squares together, we obtain

$$\omega^4 + x\omega^2 + y = 0, \tag{4.28}$$

with

$$x = A_1^2 - 2B - (C+E)^2,$$

and

$$y = B^2 - (D+F)^2.$$

Letting $z = \omega^2$, equation (4.28), becomes the following equation

$$z^2 + xz + y = 0. \tag{4.29}$$

On the other hand, we have

$$\begin{aligned} y &= B^2 - (D+F)^2 \\ &= \Big(B + D + F\Big)\Big(B - D - F\Big). \end{aligned}$$

Using the case $\tau_1 = \tau_2 = 0$, we have $B + F + D > 0$, and the assumption (4.24) implies that $B - D - F < 0$, then we have $y < 0$.
Consequently, the equation (4.29) has a unique solution positive

$$z_0 = \frac{-x + \sqrt{(x^2 - 4y)}}{2}.$$

Then the equation (4.28) has a unique positive solution

$$\omega_0 = \sqrt{\frac{-x + \sqrt{(x^2 - 4y)}}{2}}.$$

On the other hand, the equations (4.26) and (4.27) imply that

$$\tau_0 = \frac{1}{\omega_0} \arccos\left(\frac{(D+F)(\omega_0^2 - B) - \omega_0^2 A_1(C+E)}{\omega_0^2(C+E)^2 + (D+F)^2}\right).$$

Lets show that $i\omega_0$ is simple, consider the branche of characteristic roots $\lambda(\tau) = k(\tau) + ij(\tau)$, of equation (4.25) bifurcating from $i\omega_0$ at $\tau = \tau_0$. By derivation (4.25) with respect to the delay τ, we obtain

$$\frac{d\lambda}{d\tau}\Big(2\lambda + A_1 + e^{-\lambda\tau}([C+E] - [\lambda(C+E) + (D+F)]\tau)\Big) = \lambda[\lambda(C+E) + (D+F)]e^{-\lambda\tau}. \tag{4.30}$$

If we suppose, by contradiction, that $i\omega_0$ is not simple, the right hand side (4.30) gives

$$i\omega_0(C+E) + (D+F) = 0.$$

On the other hand, the equation (4.24) implies that $B - D - F < 0$, and by the hypotheses (H_1), (H_3) and (T_1), we have $B > 0, -D > 0$ and $F > 0$ then we have $-D < F$. This contradicts $-D = F$.

Next, we need to guarantee the transversality condition of the Hopf bifurcation theorem (see [23]). Clearly, $\lambda(\tau) = k(\tau) + ij(\tau)$ is a root of equation (4.25) if and only if

$$k^2 - j^2 + A_1 k + B = -e^{-k\tau}\Big([k(C+E) + (D+F)]\cos(j\tau) + j(C+E)\sin(j\tau)\Big), \tag{4.31}$$

and

$$2kj + jA_1 = -e^{-k\tau}\Big(j(C+E)\cos(j\tau) - [k(C+E) + (D+F)]\sin(j\tau)\Big). \tag{4.32}$$

Let $k(\tau_0)$ and $j(\tau_0)$, satisfying $k(\tau_0) = 0$, and $j(\tau_0) = \omega_0$. By differentiating equations (4.31) and (4.32) with respect to τ and then set $\tau = \tau_0$. Doing this, we get

$$G_1 \frac{dk(\tau_0)}{d\tau} + G_2 \frac{dj(\tau_0)}{d\tau} = h_1,$$

$$-G_2 \frac{dk(\tau_0)}{d\tau} + G_1 \frac{dj(\tau_0)}{d\tau} = h_2,$$

where

$$\begin{aligned} G_1 &= A_1 + [(C+E) - \tau_0(D+F)]\cos(\omega_0\tau_0) - \tau_0\omega_0(C+E)\sin(\omega_0\tau_0), \\ G_2 &= -2\omega_0 + \tau_0\omega_0(C+E)\cos(\omega_0\tau_0) + [(C+E) - \tau_0(D+F)]\sin(\omega_0\tau_0), \\ h_1 &= \omega_0(D+F)\sin(\omega_0\tau_0) - \omega_0^2(C+E)\cos(\omega_0\tau_0), \end{aligned}$$

and

$$h_2 = \omega_0^2(C+E)\sin(\omega_0\tau_0) + \omega_0(D+F)\cos(\omega_0\tau_0).$$

Also, we have

$$\frac{dk(\tau_0)}{d\tau} = \frac{G_1 h_1 - G_2 h_2}{G_1^2 + G_2^2},$$

Therefore, according to the equations (4.26) and (4.27) we have

$$\frac{dk(\tau_0)}{d\tau} = \frac{\omega_0^2\sqrt{x^2 - 4y}}{G_1^2 + G_2^2}.$$

This completes the proof. □

4.4. Case $\tau_2 > 0$ *and* $\tau_1 = 0$

Theorem 4.4:
If $R_0 > 1$ *and* (4.24) *holds, then there exists a positive* $\tau_{2,0}$ *where system* (1.3) *undergoes Hopf bifurcation at* P^* *when* $\tau_2 = \tau_{2,0}$*. However, the steady state* P^* *is locally asymptotically stable when* $\tau_2 \in [0, \tau_{2,0})$ *and unstable when* $\tau_2 > \tau_{2,0}$*. Here* $\tau_{2,0}$ *is given by*

$$\tau_{2,0} = \frac{1}{\omega_{2,0}} \arccos\left[\frac{\omega_{2,0}^2(F - E[A_1 + C]) - (B + D)F}{\omega_{2,0}^2 E^2 + F^2}\right],$$

and

$$\omega_{2,0} = \sqrt{\frac{-p + \sqrt{(p^2 - 4q)}}{2}},$$

where p *and* q *are defined in equation* (4.36).

Proof
When $\tau_2 > 0$ and $\tau_1 = 0$, the characteristic equation (4.18) becomes

$$\Delta(\lambda, \tau_2) = \lambda^2 + (A_1 + C)\lambda + B + D + (\lambda E + F)e^{-\lambda\tau_2} = 0. \tag{4.33}$$

Equation (4.33), has a purely imaginary root $i\omega_2$, with $\omega_2 > 0$,

$$\Delta(i\omega_2, \tau_2) = 0,$$

if and only if

$$\omega_2^2 - (B + D) = \omega_2 E \sin(\omega_2\tau_2) + F\cos(\omega_2\tau_2), \tag{4.34}$$

$$\omega_2(A_1 + C) = F\sin(\omega_2\tau_2) - \omega_2 E\cos(\omega_2\tau_2). \tag{4.35}$$

Squaring and adding the squares together, we get

$$\omega_2^4 + p\omega_2^2 + q = 0, \tag{4.36}$$

with

$$p = (A_1 + C)^2 - 2(B + D) - E^2,$$

and

$$q = (B + D)^2 - F^2.$$

Letting $z_2 = \omega_2^2$, equation (4.36) becomes the following equation

$$z_2^2 + pz_2 + q = 0. \tag{4.37}$$

On the other hand, we have

$$\begin{aligned} q &= (B + D)^2 - F^2 \\ &= \Big(B + D + F\Big)\Big(B + D - F\Big). \end{aligned}$$

Using the case $\tau_1 = \tau_2 = 0$, we have $B + D + F > 0$, and the equation (4.24) implies that $B - D < F$, and by the hypotheses (H_1), (H_2), and (T_1), we have $B > 0$, $D < 0$ and $F > 0$, then we have $B + D < F$, then, we have $q < 0$.
Consequently, the equation (4.37) has a unique solution positive

$$z_{2,0} = \frac{-p + \sqrt{(p^2 - 4q)}}{2}.$$

Then the equation (4.36) has a unique positive solution

$$\omega_{2,0} = \sqrt{\frac{-p + \sqrt{(p^2 - 4q)}}{2}}.$$

On the other hand, the equations (4.34) and (4.35) imply that

$$\tau_{2,0} = \frac{1}{\omega_{2,0}} \arccos\,[\frac{\omega_{2,0}^2(F - E[A_1 + C]) - (B + D)F}{\omega_{2,0}^2 E^2 + F^2}].$$

Lets show that $i\omega_{2,0}$ is simple, consider the branche of characteristic roots $\lambda(\tau_2) = u(\tau_2) + iv(\tau_2)$, of equation (4.33) bifurcating from $i\omega_{2,0}$ at $\tau_2 = \tau_{2,0}$. By derivation (4.33) with respect to the delay τ_2, we obtain

$$\frac{d\lambda}{d\tau_2}\Big(2\lambda + A_1 + C + e^{-\lambda\tau_2}(E - [\lambda E + F]\tau_2)\Big) = \lambda(\lambda E + F)e^{-\lambda\tau_2}. \tag{4.38}$$

If we suppose, by contradiction, that $i\omega_{2,0}$ is not simple, the right hand side (4.38) gives

$$i\omega_{2,0}E + F = 0,$$

and leads a contradiction with the fact that $\omega_{2,0} > 0$, $E > 0$ and $F > 0$.
Next we need to guarantee the transversality condition of the Hopf bifurcation theorem (see [23]). Clearly, $\lambda(\tau_2) = u(\tau_2) + iv(\tau_2)$, is a root of equation (4.33), if and only if

$$u^2 - v^2 + (A_1 + C)u + B + D = -e^{-u\tau_2}\Big([uE + F]\cos(v\tau_2) + vE\sin(v\tau_2)\Big), \tag{4.39}$$

$$2uv + v(A_1 + C) = -e^{-u\tau_2}\Big(vE\cos(v\tau_2) - (uE + F)\sin(v\tau_2)\Big). \tag{4.40}$$

Let $u(\tau_{2,0})$ and $v(\tau_{2,0})$, satisfying $u(\tau_{2,0}) = 0$, and $v(\tau_{2,0}) = \omega_{2,0}$. By differentiating equations (4.39) and (4.40) with respect to τ_2 and then set $\tau_2 = \tau_{2,0}$. Doing this, we get

$$G_3\frac{du(\tau_{2,0})}{d\tau_2} + G_4\frac{dv(\tau_{2,0})}{d\tau_2} = h_3,$$

$$-G_4\frac{du(\tau_{2,0})}{d\tau_2} + G_3\frac{dv(\tau_{2,0})}{d\tau_2} = h_4,$$

where

$$G_3 = A_1 + C + (E - \tau_{2,0}F)\cos(\omega_{2,0}\tau_{2,0}) - \tau_{2,0}\omega_{2,0}E\sin(\omega_{2,0}\tau_{2,0}),$$
$$G_4 = -2\omega_{2,0} + \tau_{2,0}\omega_{2,0}E\cos(\omega_{2,0}\tau_{2,0}) + (E - \tau_{2,0}F)\sin(\omega_{2,0}\tau_{2,0}),$$
$$h_3 = \omega_{2,0}F(0)\sin(\omega_{2,0}\tau_{2,0}) - \omega_{2,0}^2E\cos(\omega_{2,0}\tau_{2,0}),$$

and

$$h_4 = \omega_{2,0}^2E\sin(\omega_{2,0}\tau_{2,0}) + \omega_{2,0}F(0)\cos(\omega_{2,0}\tau_{2,0}).$$

Further, we have

$$\frac{du(\tau_{2,0})}{d\tau_2} = \frac{G_3h_3 - G_4h_4}{G_3^2 + G_4^2}.$$

Therefore, according to the equations (4.34) and (4.35) we have

$$\frac{du(\tau_{2,0})}{d\tau_2} = \frac{\omega_{2,0}^2\sqrt{p^2 - 4q}}{G_3^2 + G_4^2}.$$

This completes the proof. □

4.5. Case $\tau_1 > 0$ *and* $\tau_2 > 0$

Theorem 4.5:
If $R_0 > 1$, $G_5h_5 - G_6h_6 \neq 0$ *and* (4.24) *holds, then there exists a positive* $\tau_{2,0}^*$ *where system* (1.3) *undergoes Hopf bifurcation at* P^* *when* $\tau_2 = \tau_{2,0}^*$*. However, the steady state* P^* *is locally asymptotically stable when* $\tau_2 \in [0, \tau_{2,0}^*)$ *and unstable when* $\tau_2 > \tau_{2,0}^*$*. Here* $\tau_{2,0}^*$ *is given by*

$$\tau_{2,0}^* = \frac{1}{\omega_{2,0}^*} \arccos[\frac{(F - EA_1)(\omega_{2,0}^*)^2 - FB + \omega_{2,0}^*(ED - CF)\sin(\omega_{2,0}^*\tau_1)}{(\omega_{2,0}^*E)^2 + F^2} - \frac{(DF + EC(\omega_{2,0}^*)^2)\cos(\omega_{2,0}^*\tau_1)}{(\omega_{2,0}^*E)^2 + F^2}].$$

where $\omega_{2,0}^*$ *is the positive root of the equation* (4.43)*, and* G_5, G_6, h_5, h_6 *as defined in* (4.48).

Proof
Equation (4.18) has a purely imaginary root $i\omega_2^*$, with $\omega_2^* > 0$,

$$\Delta(i\omega_2^*, \tau_1, \tau_2) = 0,$$

if and only if

$$(\omega_2^*)^2 - B - \omega_2^*C\sin(\omega_2^*\tau_1) - D\cos(\omega_2^*\tau_1) = E\omega_2^*\sin(\omega_2^*\tau_2) + F\cos(\omega_2^*\tau_2), \tag{4.41}$$

and

$$A_1\omega_2^* + \omega_2^*C\cos(\omega_2^*\tau_1) - D\sin(\omega_2^*\tau_1) = F\sin(\omega_2^*\tau_2) - E\omega_2^*\cos(\omega_2^*\tau_2). \tag{4.42}$$

Squaring and adding the squares together, we obtain

$$(\omega_2^*)^4 + (\omega_2^*)^2(C^2 + A_1^2 - E^2) + B^2 + D^2 - F^2 + 2\Big([-(\omega_2^*)^3C + \omega_2^*(BC - A_1D)]\sin(\omega_2^*\tau_1) + [(\omega_2^*)^2(A_1C - D) + BD]\cos(\omega_2^*\tau_1)\Big) = 0. \tag{4.43}$$

We define

$$h(\omega_2^*) = (\omega_2^*)^4 + (\omega_2^*)^2(C^2 + A_1^2 - E^2) + B^2 + D^2 - F^2 + 2\Big([-(\omega_2^*)^3C + \omega_2^*(BC - A_1D)]\sin(\omega_2^*\tau_1) + [(\omega_2^*)^2(A_1C - D) + BD]\cos(\omega_2^*\tau_1)\Big). \tag{4.44}$$

On the other hand, we have

$$-1 \leq \sin(\omega_2^*\tau_1) \leq 1 \; and \; -1 \leq \cos(\omega_2^*\tau_1) \leq 1,$$

and by the hypotheses (H_1), (H_3) and (T_1), we have

$$BD < 0, A_1C < 0, -D > 0, -C > 0, BC < 0 \; and \; -A_1D > 0.$$

Then

$$g_2(\omega_2^*) \leq h(\omega_2^*) \leq g_1(\omega_2^*),$$

where

$$g_1(\omega_2^*) = (\omega_2^*)^4 - 2(\omega_2^*)^3C + (\omega_2^*)^2[C^2 + A_1^2 - E^2 - 2A_1C - 2D] - \omega_2^*(2BD + 2A_1D) + B^2 + D^2 - F^2 - 2BD,$$

and

$$g_2(\omega_2^*) = (\omega_2^*)^4 + 2(\omega_2^*)^3 C + (\omega_2^*)^2[C^2 + A_1^2 - E^2 + 2A_1C + 2D] + \omega_2^*(2BD + 2A_1D) + B^2 + D^2 - F^2 + 2BD.$$

We have

$$\lim_{\omega_2^* \to +\infty} g_1(\omega_2^*) = \lim_{\omega_2^* \to +\infty} g_2(\omega_2^*) = +\infty.$$

Then we have

$$\lim_{\omega_2^* \to +\infty} h(\omega_2^*) = +\infty,$$

and

$$\begin{aligned} h(0) &= B^2 + D^2 + 2BD - F^2 \\ &= (B+D)^2 - F^2 \\ &= \Big(B + D + F\Big)\Big(B + D - F\Big). \end{aligned}$$

By the case $\tau_1 = \tau_2 = 0$, we have $(B + D + F) > 0$, and by the case $(\tau_2 > 0 \; and \; \tau_1 = 0)$ if the equation (4.24) is satisfied then we have $(B + D - F) < 0$. Therefore, $h(0) < 0$.
Since h is continuous in $[0, +\infty)$, then the equation (4.43) has at least one positive root.
We assume that equation (4.43) admits a finite family of solution $\omega_{2,i}^*$ with $i = 1, 2, ..., n$. $n \in \mathbb{N}$.
By the equations (4.41) and (4.42), we have

$$\tau_i^l = \frac{1}{\omega_{2,i}^*}\Big(\arccos\big[\frac{(F - EA_1)(\omega_{2,i}^*)^2 - FB + \omega_{2,i}^*(ED - CF)\sin(\omega_{2,i}^*\tau_1)}{(\omega_{2,0}^* E)^2 + F^2} - \frac{(DF + EC(\omega_{2,i}^*)^2)\cos(\omega_{2,i}^*\tau_1)}{(\omega_{2,0}^* E)^2 + F^2}\big] + 2\Pi l\Big), l = 0, 1, ...; i = 1, 2, ..., n.$$

Then $\pm i\omega_{2,i}^*$ is a pair of purely imaginary roots of equations (4.18), with $\tau_2 = \tau_i^l, \; l = 0, 1, ...; i = 1, 2, ..., n$. Clearly,

$$\lim_{l \to \infty} \tau_i^l = \infty, \; i = 1, 2, ..., n.$$

Thus, we can define

$$\tau_{2,0}^* = \tau_{i_0}^{l_0} = \min_{l=0,1...,i=1,2,...,n} (\tau_i^l), \; \omega_{2,0}^* = \omega_{2,i_0}^*.$$

Lets show that $i\omega_{2,0}^*$ is simple, consider the branche of characteristic roots $\lambda(\tau_2) = u_1(\tau_2) + iv_1(\tau_2)$, of equation (4.18) bifurcating from $i\omega_{2,0}^*$ at $\tau_2 = \tau_{2,0}^*$. By derivation (4.18) with respect to the delay τ_2, we obtain

$$\frac{d\lambda}{d\tau_2}\Big(2\lambda + A_1 + [C - \tau_1(\lambda C + D)]e^{-\lambda\tau_1} + [E - \tau_2(\lambda E + F)]e^{-\lambda\tau_2}\Big) = \lambda[\lambda E + F]e^{-\lambda\tau_2}. \tag{4.45}$$

If we suppose, by contradiction, that $i\omega_{2,0}^*$, is not simple, the right hand side (4.45) gives

$$i\omega_{2,0}^* E + F = 0,$$

and leads a contradiction with the fact that $\omega^*_{2,0} > 0$, $E > 0$ and $F > 0$.
Next we need to guarantee the transversality condition of the Hopf bifurcation theorem (see [23]). Clearly $\lambda(\tau_2) = u_1(\tau_2) + iv_1(\tau_2)$, is a root of equation (4.18) if and only if

$$\begin{aligned} u_1^2 - v_1^2 + A_1u_1 + e^{-u_1\tau_1}\Big(Cu_1\cos(v_1\tau_1) + Cv_1\sin(v_1\tau_1) + D\cos(v_1\tau_1)\Big) \\ = -e^{-u_1\tau_2}\Big(Eu_1\cos(v_1\tau_2) + Ev_1\sin(v_1\tau_2) + F\cos(v_1\tau_2)\Big), \end{aligned} \tag{4.46}$$

and

$$\begin{aligned} 2u_1v_1 + A_1v_1 + e^{-u_1\tau_1}\Big(Cv_1\cos(v_1\tau_1) - Cu_1\sin(v_1\tau_1) - D\sin(v_1\tau_1)\Big) \\ = -e^{-u_1\tau_2}\Big(Ev_1\cos(v_1\tau_2) - Eu_1\sin(v_1\tau_2) - F\sin(v_1\tau_2)\Big). \end{aligned} \tag{4.47}$$

Let $u_1(\tau^*_{2,0})$ and $v_1(\tau^*_{2,0})$, satisfying $u_1(\tau^*_{2,0}) = 0$, and $v_1(\tau^*_{2,0}) = \omega^*_{2,0}$. By differentiating equations (4.46) and (4.47) with respect to τ_2, and then set $\tau_2 = \tau^*_{2,0}$. Doing this, we get

$$\begin{aligned} G_5\frac{du_1(\tau^*_{2,0})}{d\tau_2} + G_6\frac{dv_1(\tau^*_{2,0})}{d\tau_2} = h_5, \\ -G_6\frac{du_1(\tau^*_{2,0})}{d\tau_2} + G_5\frac{dv_1(\tau^*_{2,0})}{d\tau_2} = h_6, \end{aligned} \tag{4.48}$$

where

$$\begin{aligned} G_5 = A_1 - \tau_1C\omega^*_{2,0}\sin(\omega^*_{2,0}\tau_1) + (C - \tau_1D)\cos(\omega^*_{2,0}\tau_1) - \tau^*_{2,0}E\omega^*_{2,0}\sin(\omega^*_{2,0}\tau^*_{2,0}) \\ + (E - \tau^*_{2,0}F)\cos(\omega^*_{2,0}\tau^*_{2,0}), \end{aligned}$$

$$\begin{aligned} G_6 = -2\omega^*_{2,0} + (C - \tau_1D)\sin(\omega^*_{2,0}\tau_1) + \tau_1C\omega^*_{2,0}\cos(\omega^*_{2,0}\tau_1) + (E - \tau^*_{2,0}F)\sin(\omega^*_{2,0}\tau^*_{2,0}) \\ + \tau^*_{2,0}E\omega^*_{2,0}\cos(\omega^*_{2,0}\tau^*_{2,0}), \end{aligned}$$

$$h_5 = F\omega^*_{2,0}\sin(\omega^*_{2,0}\tau^*_{2,0}) - E(\omega^*_{2,0})^2\cos(\omega^*_{2,0}\tau^*_{2,0}),$$

and

$$h_6 = E(\omega^*_{2,0})^2\sin(\omega^*_{2,0}\tau^*_{2,0}) + F\omega^*_{2,0}\cos(\omega^*_{2,0}\tau^*_{2,0}).$$

Calculating $\frac{du_1(\tau^*_{2,0})}{d\tau_2}$, we get

$$\frac{du_1(\tau^*_{2,0})}{d\tau_2} = \frac{G_5h_5 - G_6h_6}{G_5^2 + G_6^2}.$$

This completes the proof. □

5. NUMERICAL SIMULATION

In this section, we shall give some simulations to illustrate the previous results. For the following, let's consider the saturated incidence rate function:

$$f(S, I)I = \frac{\beta SI}{1 + \alpha_1 I},$$

and the treatment function

$$T(I) = rI.$$

We take the parameters of the system (1.3) as follows:

$$A = 8, \quad \beta = 0.955, \quad \mu = 0.5, \quad \gamma_2 = 0.935, \quad \alpha = 0.03$$
$$\gamma_1 = 0.3, \quad r = 0.9, \; and \quad \alpha_1 = 0.5.$$

Some parameters are for the spread of tuberculosis disease in Turkey from 2005 to 2015 (see, [38]). The rest of the parameters are the hypothetical set of parameter values.

5.1. *Case* $\tau_1 = \tau_2 = 0$

By applying Theorem 4.1, the endemic equilibrium P^* is locally asymptotically stable when $R_0 > 1$ and $\tau_1 = \tau_2 = 0$. Figure 5.1 illustrates this result.

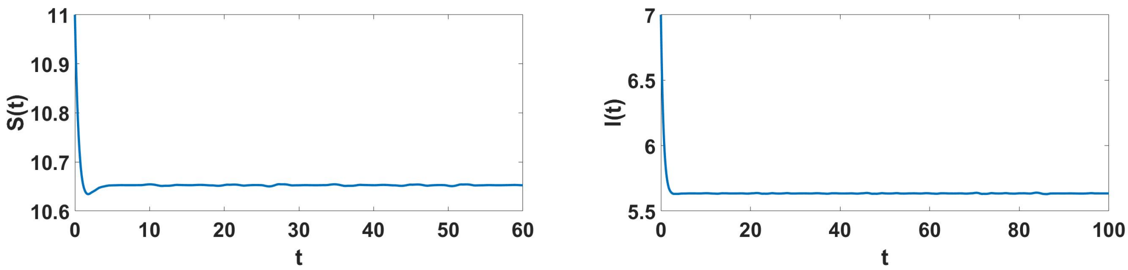


Fig. 5.1. Dynamics of system (1.3) when $\tau_1 = \tau_2 = 0$ and $R_0 = 5.7336 > 1$.

5.2. *Case* $\tau_1 > 0$ *and* $\tau_2 = 0$

According to Theorem 4.2, the endemic equilibrium P^* is locally asymptotically stable if $R_0 > 1$, $\tau_1 > 0$ and $\tau_2 = 0$. Figure 5.2 demonstrates this result.

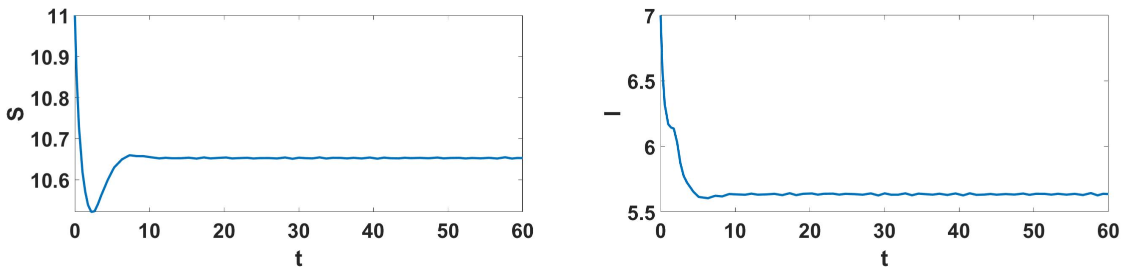
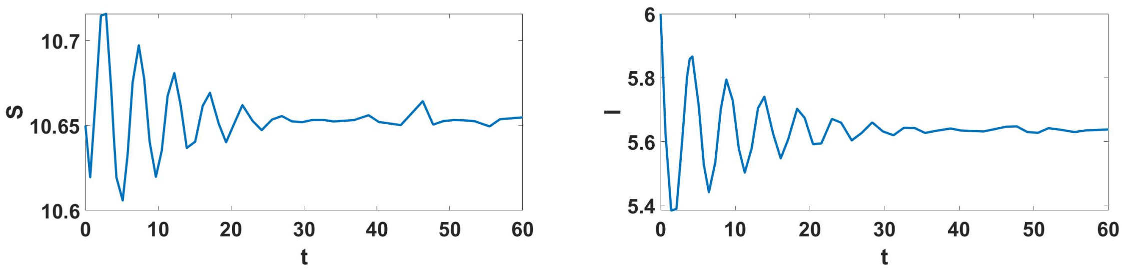


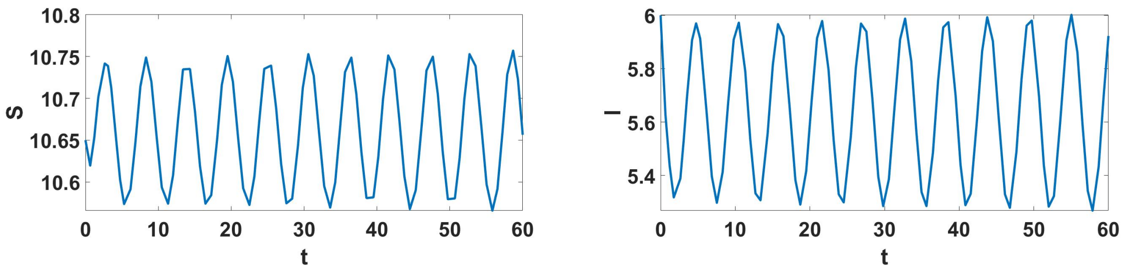
Fig. 5.2. Dynamics of system (1.3) when $\tau_1 = 1.7678$, $\tau_2 = 0$ and $R_0 = 5.7336 > 1$.

5.3. *Case* $\tau_1 = \tau_2 = \tau$

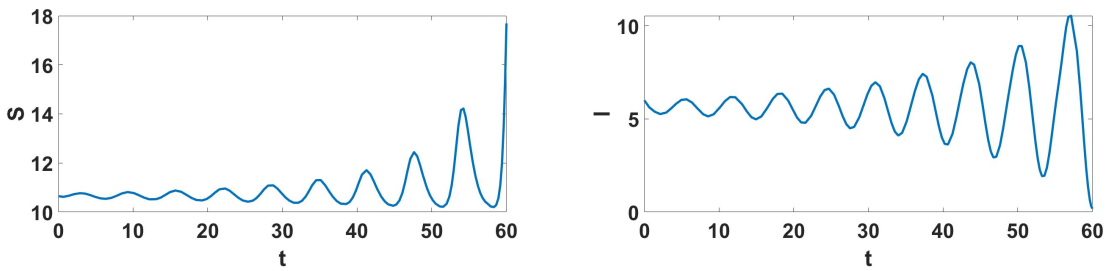
From Theorem 4.3, the following figures show that if the delay is below $\tau_0 = 1.7678$, the endemic equilibrium P^* is locally asymptotically stable (see, Figure 5.3a). When the value of the delay τ increases we lose the stability of the endemic equilibrium (see, Figure 5.3b) and P^* becomes unstable for $\tau > \tau_0 = 1.7678$ (see, Figure 5.3c), and vice versa when the delay is decreasing the model converges rapidly to P^*.



(a) Dynamics of system (1.3) when $\tau_1 = \tau_2 = 1.4178$ and $R_0 = 5.7336 > 1$.



(b) For $\tau_1 = \tau_2 = 1.7678$, Hopf bifurcation occurs
and periodic solutions appear for model (1.3) with $R_0 = 5.7336 > 1$.

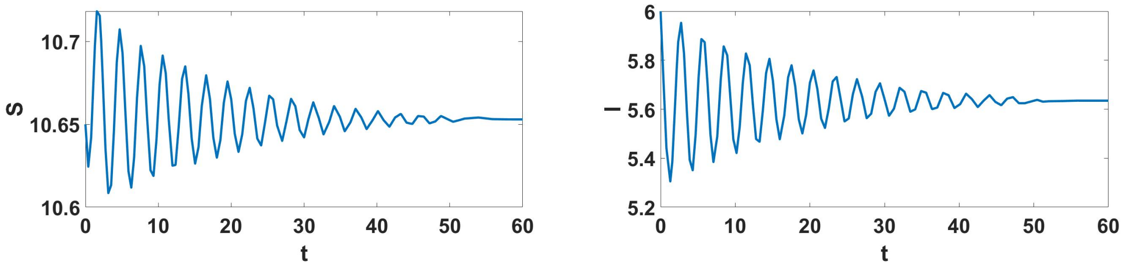


(c) For $\tau_1 = \tau_2 = 2.1178$, all solutions (S, I) of model (1.3) are unstable with $R_0 = 5.7336 > 1$.

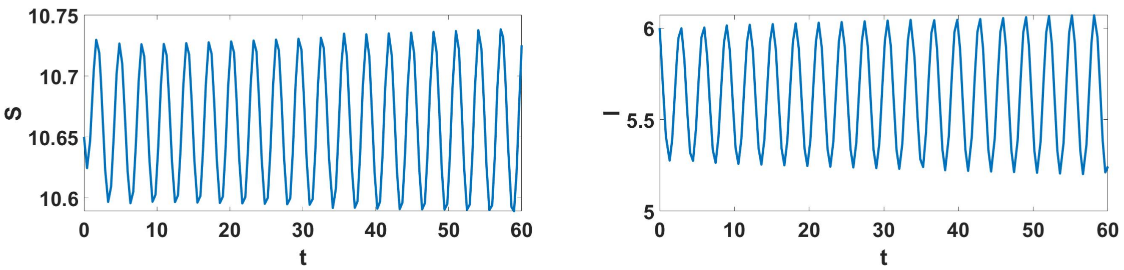
Fig. 5.3. Case $\tau_1 = \tau_2 = \tau$

5.4. *Case* $\tau_1 = 0$ *and* $\tau_2 > 0$

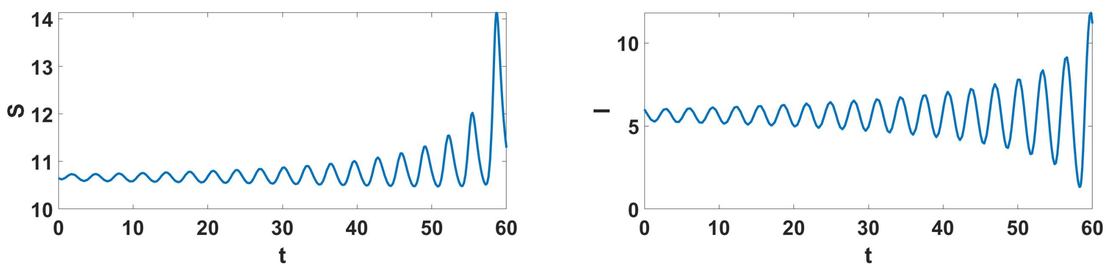
As in Theorem 4.4, the following figures show that if the delay is below $\tau_{2,0} = 0.8357$, the endemic equilibrium P^* is locally asymptotically stable (see, Figure 5.4a). When the value of the delay τ_2 increases we lose the stability of the endemic equilibrium (see, Figure 5.4b) and P^* becomes unstable for $\tau_2 > \tau_{2,0} = 0.8357$ (see, Figure 5.4c), and vice versa when the delay is decreasing the model converges rapidly to P^*.



(a) Dynamics of system (1.3) when $\tau_1 = 0$, $\tau_2 = 0.7857$ and $R_0 = 5.7336 > 1$.



(b) For $\tau_1 = 0$ and $\tau_2 = 0.8357$, Hopf bifurcation occurs
and periodic solutions appear for model (1.3) with $R_0 = 5.7336 > 1$.

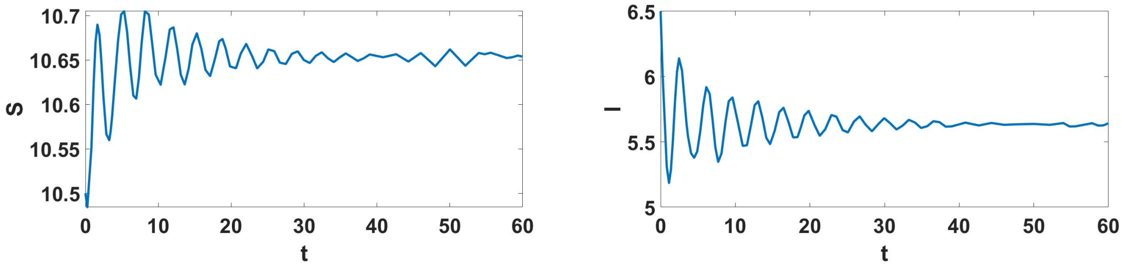


(c) For $\tau_1 = 0$ and $\tau_2 = 0.8651$, all solutions (S, I) of model (1.3) are unstable with $R_0 = 5.7336 > 1$.

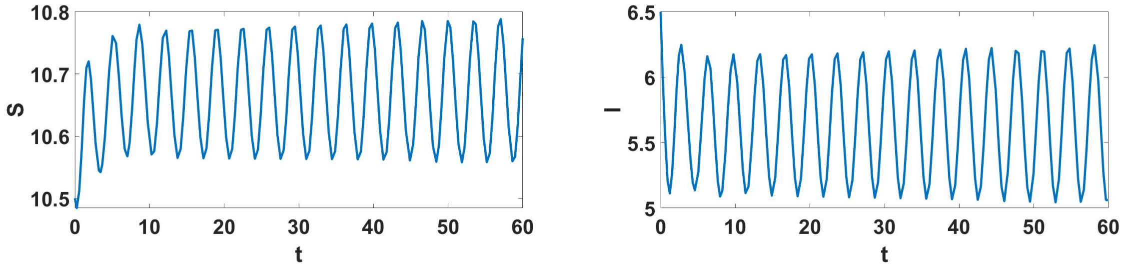
Fig. 5.4. Case $\tau_1 = 0$ and $\tau_2 > 0$

5.5. *Case* $\tau_1 > 0$ *and* $\tau_2 > 0$

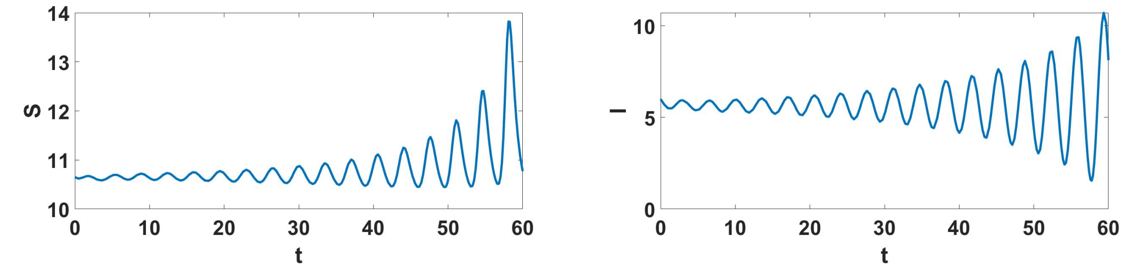
Based on Theorem 4.5, the following figures show that if the delay τ_2 is below $\tau^*_{2,0} = 0.9249$ with τ_1 fixed by taking $\tau_1 = 3.6688$, the endemic equilibrium P^* is locally asymptotically stable (see, Figure 5.5a). When the value of the delay τ_2 increases we lose the stability of the endemic equilibrium (see, Figure 5.5b) and P^* becomes unstable for $\tau_2 > \tau^*_{2,0} = 0.9249$ (see ,Figure 5.5c), and vice versa when the delay is decreasing the model converges rapidly to P^*.



(a) Dynamics of system (1.3) when $\tau_1 = 3.6688$, $\tau_2 = 0.8228$ and $R_0 = 5.7336 > 1$.



(b) For $\tau_1 = 3.6688$ and $\tau_2 = 0.9249$, Hopf bifurcation occurs and periodic solutions appear for model (1.3) with $R_0 = 5.7336 > 1$.



(c) For $\tau_1 = 3.6688$ and $\tau_2 = 0.9976$, all solutions (S, I) of model (1.3) are unstable with $R_0 = 5.7336 > 1$.

Fig. 5.5. Case $\tau_1 > 0$ and $\tau_2 > 0$

6. CONCLUDING REMARKS

In this work, we have proposed and analyzed a delayed SIR model with generalized incidence and treatment functions. Our analysis proved that the two delays τ_1 and τ_2 have a very big influence on the stability of the equilibrium points. In fact,

- When $R_0 \leq 1$, we have:
  - If $\tau_1 \geq \tau_2$, then the disease-free equilibrium is globally asymptotically stable for all R_c, this shows that the disease disappears. This biologically mean that when the latent period is greater than the healing period and the basic reproduction number is

less than or equal to one, the infected individuals cured before becoming infectious and the disease will disappear.

- If $R_c \leq 1$, then the disease-free equilibrium is globally asymptotically stable independently of delays.
- If $R_c > 1$ and τ_1 is held fixed, then there exists a positive constant $\tau^*_{22,0}$ such that when $\tau_2 \in [0, \tau^*_{22,0})$, the disease-free equilibrium is locally asymptotically stable. When $\tau_2 = \tau^*_{22,0}$ a Hopf bifurcation occurs and when $\tau_2 > \tau^*_{22,0}$ the disease-free equilibrium is unstable.

• When $R_0 > 1$, the disease-free equilibrium is unstable and the proposed model admits an endemic equilibrium P^*. In this case, we have:

- If $\tau_2 = 0$ and $\tau_1 \geq 0$, then the endemic equilibrium P^* is locally asymptotically stable.
- If $\tau_2 > 0$, then there exists a positive constant $\widetilde{\tau}$ such that when $\tau_2 \in [0, \widetilde{\tau})$, the endemic equilibrium is locally asymptotically stable. When $\tau_2 = \widetilde{\tau}$ a Hopf bifurcation occurs and the host populations and the disease coexists within an oscillatory mode, and when $\tau_2 > \widetilde{\tau}$ the endemic equilibrium is unstable and the disease persists.

From the aforementioned, we conclude that the longer the recovery period is for the infected individuals, the higher the spreading risk becomes and vice versa.

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