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On a Random Topological Characteristic for Inclusions with Nonlinear Fredholm Operators: Applications to Some Classes of Feedback Control Systems

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Abstract: We define and study an oriented random coincidence index for a pair consisting of a nonlinear zero index Fredholm operator f and a nonconvex - valued random multivalued map G which is fundamentally restrictible with respect to f. It is shown how this characteristic can be used for the justification of the existence of random coincidence points. We present an application of developed results to the existence of a random solution for a control system whose dynamics is governed by an implicit integro-differential equation and the feedback is realized by a random differential inclusion.

Keywords: feedback control system, random differential inclusion, random coincidence index, random coincidence point, nonlinear Fredholm operator, random multivalued map

1. INTRODUCTION

Topological methods provide strong and useful tools to deal with problems in the control theory and optimization (among many others, let us mention monographs [3], [7], [8], [9], [15], [16], [19], [22] and references therein). An important place in these applications is occupied by the effective use of various topological characteristics. In the present paper we define a random coincidence index for a pair consisting of a nonlinear Fredholm operator and a random nonconvex-valued multimap and use it for the study of a control system whose dynamics is presented by an implicit integro-differential equation and the feedback is realized by a random differential inclusion.

It is worth noting that the problem of a coincidence of nonlinear Fredholm operators and their perturbations of various types is of a great mathematical interest and finds contentable applications in many fields of contemporary mathematics (see, for example, [5], [27], [29], [32], [33] and references therein). It should be mentioned that the authors of the works [20], [25], [26], [30], [31] introduced topological coincidence index for various classes of multivalued perturbations of nonlinear Fredholm operators and used it to obtain some applications.

In the recent years the topological degree theory was successfully extended to the case of random maps and new random fixed point results and their applications were obtained (see, for example, [1], [11], [24], [28]). In particular, in papers [24], [28] the random coincidence degree theory was developed for some classes of multivalued perturbations of linear Fredholm operators.

The present paper is organized in the following way. In the next section we collect necessary preliminaries. The third section is devoted to the construction of an oriented random coincidence index, the description of its properties and its usage to random

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coincidence points. In the forth section we present an application to the problem of existence of a solution for a feedback control system.

2. PRELIMINARIES

2.1. Nonlinear Fredholm operators

By the symbols E, E' we will denote real Banach spaces. Everywhere, by Y we will denote an open bounded set $U \subset E$ (case (i)) or $U_* \subset E \times [0,1]$ (case (ii)). We recall some notions (see, e.g. [5], [33]).

Definition 2.1:
A C^1-map $f : Y \to E'$ is called a Fredholm operator of index $k \geq 0$ $(f \in \Phi_k C^1(Y))$ if for every $y \in Y$ the Frechet derivative $f'(y)$ is a linear Fredholm operator of index k, that is, $\dim Ker\, f'(y) < \infty$, $\dim Co\ker\, f'(y) < \infty$ *and*

$$\dim Ker\, f'(y) - \dim Co\ker f'(y) = k\,.$$

Definition 2.2:
A map $f : \overline{Y} \to E'$ is proper if $f^{-1}(\mathcal{K})$ is compact for every compact set $\mathcal{K} \subset E'$.

We recall now the notion of oriented Fredholm structure on Y.

An atlas $\{(Y_i, \Psi_i)\}$ on Y is said to be Fredholm if, for each intersecting charts (Y_i, Ψ_i) and (Y_j, Ψ_j) and every $y \in Y_i \cap Y_j$ it is

$$\left(\Psi_j \circ \Psi_i^{-1}\right)'(\Psi_i(y)) \in CG\left(\widetilde{E}\right)\,,$$

where $\widetilde{E}$ is the corresponding model space, and $CG\left(\widetilde{E}\right)$ denotes the collection of all linear invertible operators in $\widetilde{E}$ of the form $I + K$, where I is the identity map and K is a compact linear operator.

The set $CG\left(\widetilde{E}\right)$ is divided into two connected components. The component containing the identity map will be denoted by $CG^+\left(\widetilde{E}\right)$.

Two Fredholm atlases are said to be equivalent if their union is still a Fredholm atlas. The class of equivalent atlases is called a *Fredholm structure.*

A Fredholm structure on U is associated to a $\Phi_0 C^1$-map $f : U \to E'$ if it admits an atlas $\{(Y_i, \Psi_i)\}$ with model space E' for which

$$\left(f \circ \Psi_i^{-1}\right)'(\Psi_i(y)) \in LC(E')$$

at each point $y \in U$, where $LC(E')$ denotes the collection of all linear operators in E' of the form: identity plus a compact map. Let us note that each $\Phi_0 C^1$-map $f : U \to E'$ generates a Fredholm structure on U associated to f.

A Fredholm atlas $\{(Y_i, \Psi_i)\}$ on Y is said to be oriented if for each intersecting charts (Y_i, Ψ_i) and (Y_j, Ψ_j) and every $y \in Y_i \cap Y_j$ it is true that

$$\left(\Psi_j \circ \Psi_i^{-1}\right)'(\Psi_i(y)) \in CG^+(E)\ .$$

Two oriented Fredholm atlases are called orientally equivalent if their union is an oriented Fredholm atlas on Y. The equivalence class with respect to this relation is said to be the oriented Fredholm structure on Y.

2.2. Multivalued maps

We describe now some notions of the theory of multivalued maps that will be used in the sequel (details can be found, e.g. in [2], [4], [11], [13], [15], [22]).

Let X, Z be metric spaces, $K(Z)$ [$C(Z)$] denote the collection of all nonempty compact [resp., closed] subsets of Z.

A multimap $\mathfrak{F} : X \to C(Z)$ is said to be upper semicontinuous (u.s.c.) [lower semicontinuous (l.s.c.)] if for every open [closed] set $V \subset Z$, the set $\mathfrak{F}_+^{-1}(V) = \{x \in X : \mathfrak{F}(x) \subset V\}$ is open [resp. closed] in X.

If a multimap $\mathfrak{F} : X \to C(Z)$ is both u.s.c and l.s.c. it is called continuous.

A multimap $\mathfrak{F} : X \to C(Z)$ is said to be closed if its graph

$$\Gamma_{\mathfrak{F}} = \{(x, z) \in X \times Z \colon z \in \mathfrak{F}(x)\}$$

is a closed subset of $X \times Z$.

To present the class of multimaps which will be considered, we recall some notions.

Definition 2.3 (see, e.g. [21], [4], [11], [12])**:**
A nonempty compact subset A of a metric space Z is said to be aspheric (or UV^∞, or ∞-proximally connected) if for every $\varepsilon > 0$ there exists $\delta, 0 < \delta < \varepsilon$, such that for each $n = 0, 1, 2, \ldots$ every continuous map $g : S^n \to O_\delta(A)$ can be extended to a continuous map $\widetilde{g} : B^{n+1} \to O_\varepsilon(A)$, where $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ and $B^{n+1} = \{x \in \mathbb{R}^{n+1} : \|x\| \leq 1\}$.

Definition 2.4 (see [11])**:**
A u.s.c. multimap $\mathfrak{F} : X \to K(Z)$ is said to be a J-multimap ($\mathfrak{F} \in J(X, Z)$) if every value $\mathfrak{F}(x)$, $x \in X$ is an aspheric set.

We recall (see, for example, [6]) that a metric space M is said to be an absolute retract (an AR-space) (respectively, an absolute neighborhood retract (an ANR-space)), if for each homeomorphism h, which maps it to a closed subset of the metric space Z the set $h(M)$ is a retract of Z (respectively, of some its open neighborhood in Z). Notice that the class of ANR-spaces is sufficiently wide: in particular, a compact subset of a finite-dimensional space is an ANR-space if and only if it is locally contractible. In turn, this means that compact polyhedra and compact finite-dimensional manifolds are ANR-spaces. The union of a finite number of closed convex subsets of a normed space is also an ANR-space.

Definition 2.5 ([14])**:**
A nonempty compact space A is said to be an R_δ-set, if it can be presented as the intersection of a decreasing sequence of compact AR-spaces.

Proposition 2.1 (see [11])**:**
Let Z be an ANR-space. In each of the following cases a u.s.c. multimap $\mathfrak{F} : X \to K(Z)$ is a J-multimap:
for each $x \in X$ the value $\mathfrak{F}(x)$ is

a) a convex set;

b) a contractible set;

c) an R_δ-set;

d) an AR-space.

In particular, every continuous map $\sigma : X \to Z$ is a J-multimap.

Definition 2.6:
By $CJ(X, X')$ we will denote the collection of all multimaps $G : X \to K(X')$ of the form $G = \varphi \circ \mathfrak{F}$, where $\mathfrak{F} \in J(X, Z)$ for some metric space Z, $\varphi : Z \to X'$ is a continuous map. The composition $\varphi \circ \mathfrak{F}$ will be called the representation (or decomposition, cfr. [11]) of G. We will denote $G = (\varphi \circ \mathfrak{F}) \in CJ(X, X')$.

It is worth noting that a multimap can admit different representations (see [11]).

2.3. An Oriented Coincidence Index for Compact Triplets

We will start with the following notion.

Definition 2.7:
A map $f : \overline{Y} \to E'$, a multimap $G = (\varphi \circ \mathfrak{F}) \in CJ\left(\overline{Y}, E'\right)$ and the space $\overline{Y}$ form a compact triplet $\left(f, G, \overline{Y}\right)_C$ if the following conditions are satisfied:

- *h1) f is a continuous proper map, $f_{|Y} \in \Phi_k C^1(Y)$ with $k = 0$ in case (i), $k = 1$ in case (ii), and the Fredholm structure on Y generated by f is oriented;*
- *h2) G is compact, i.e., $G\left(\overline{Y}\right)$ is a relatively compact subset of E';*
- *h3) $Coin\,(f, G) \cap \partial Y = \emptyset$, where*

$$Coin\,(f, G) = \{y \in \overline{Y} \colon f(y) \in G(y)\}$$

is the coincidence point set of f and G.

Let us mention that from hypotheses (h1), (h2) it follows that the coincidence point set $Q = Coin\,(f, G)$ is compact.

As earlier, let U be an open bounded subset of a Banach space E. For a compact triplet $\left(f, G, \overline{U}\right)_C$ the oriented coincidence index

$$Ind\left(f, G, \overline{U}\right)_C,$$

the integer-valued topological characteristic with the following basic properties can be defined (see [26]).

Proposition 2.2 (The coincidence point property)**:**
If $Ind\left(f, G, \overline{U}\right)_C \neq 0$, then $\emptyset \neq Coin\,(f, G) \subset U$.

To formulate the topological invariance property of the coincidence index, we will give the following definition.

Definition 2.8:
Two compact triplets

$$\left(f_0, G_0 = (\varphi_0 \circ \mathfrak{F}_0), \overline{U}_0\right)_C \text{ and } \left(f_1, G_1 = (\varphi_1 \circ \mathfrak{F}_1), \overline{U}_1\right)_C$$

are said to be homotopic

$$\left(f_0, G_0 = (\varphi_0 \circ \mathfrak{F}_0), \overline{U}_0\right)_C \sim \left(f_1, G_1 = (\varphi_1 \circ \mathfrak{F}_1), \overline{U}_1\right)_C$$

if there exists a compact triplet $\left(f_*, G_*, \overline{U_*}\right)_C$, where $U_* \subset E \times [0, 1]$ is an open set, such that:

- *a) $U_i = U_* \cap (E \times \{i\})$, $i = 0, 1$;*
- *b) $f_{*|\overline{U_i}} = f_i$, $i = 0, 1$;*
- *c) G_* has the form*

$$G_*(x, \lambda) = \varphi_*(\mathfrak{F}_*(x, \lambda), \lambda)$$

where $\mathfrak{F}_* \in J\left(\overline{U_*}, Z\right)$, $\varphi_* : Z \times [0, 1] \to E'$ is a continuous map, and

$$\mathfrak{F}_* |_{\overline{U}_i} = \mathfrak{F}_i\,, \quad \varphi_* |_{Z \times \{i\}} = \varphi_i\,, \quad i = 0, 1\,.$$

Proposition 2.3 (The homotopy invariance property)**:**
If

$$\left(f_0, G_0, \overline{U}_0\right)_C \sim \left(f_1, G_1, \overline{U}_1\right)_C ,$$

then

$$\left|Ind\ \left(f_0, G_0, \overline{U}_0\right)_C\right| = \left|Ind\ \left(f_1, G_1, \overline{U}_1\right)_C\right| .$$

Remark 2.1:
If the Fredholm operator f is constant under the homotopy, i.e., U_* has the form $U_* = U \times [0,1]$ where $U \subset E$ is an open set and $f_*(x, \lambda) = f(x)$ for all $\lambda \in [0,1]$, where $f \in \Phi_0 C^1(U)$, then

$$Ind\left(f, G_0, \overline{U}\right)_C = Ind\left(f, G_1, \overline{U}\right)_C .$$

Proposition 2.4 (Additive dependence on the domain property)**:**
Let U_0 and U_1 be disjoint open subsets of an open bounded set $U \subset E$ and $\left(f, G, \overline{U}\right)_C$ be a compact triplet such that

$$Coin\,(f, G) \cap \left(\overline{U} \setminus (U_0 \cup U_1)\right) = \emptyset .$$

Then

$$Ind\left(f, G, \overline{U}\right)_C = Ind\left(f, G, \overline{U_0}\right)_C + Ind\left(f, G, \overline{U_1}\right)_C$$

2.4. Random Multivalued Maps and Random Coincidence Points

Let (Ω, Σ) be a measurable space and X, Z separable metric spaces. Let $\mathbb{B}(X)$ be the σ-algebra of all Borel subsets of X and $\Sigma \otimes \mathbb{B}(X)$ denote the minimal σ-algebra containing the sets $A \times B$, where $A \in \Sigma$, $B \in \mathbb{B}(X)$.

Definition 2.9:
A multimap $\Phi\colon \Omega \times X \to C(Z)$ is called random if it is measurable with respect to the σ-algebra $\Sigma \otimes \mathbb{B}(X)$, i.e.,

$$\Phi_+^{-1}(V) \in \Sigma \otimes \mathbb{B}(X)$$

for each open $V \subset Z$.

As example of a random multimap we can consider a Carathéodory type multimap $\Phi\colon \Omega \times X \to C(Z)$, i.e., it is supposed that Φ is such that: 1) $\Phi(\cdot, x)\colon \Omega \to C(Z)$ is measurable w.r.t. Σ for each $x \in X$; 2)$\Phi(\omega, \cdot)\colon X \to C(Z)$ is continuous for each $\omega \in \Omega$ (see, e.g., [13], Proposition 7.9).

Definition 2.10:
For a multimap $\Phi\colon \Omega \times X \to C(Z)$ and a map $\psi\colon X \to Z$, a measurable map $\xi\colon \Omega \to X$ is called a random coincidence point of ψ and Φ if it satisfies the following inclusion

$$\psi(\xi(\omega)) \in \Phi(\omega, \xi(\omega))$$

for each $\omega \in \Omega$.

The following assertion on a random coincidence point holds true.

Proposition 2.5:
Let (Ω, Σ) be a complete measurable space (see, e.g., [13], Definition 1.29); a space X is complete; $\Phi\colon \Omega \times X \to C(Z)$ a random multimap; a map $\psi\colon X \to Z$ be measurable w.r.t. $\mathbb{B}(X)$. If for each $\omega \in \Omega$ the coincidence point set

$$Coin_\omega(\psi, \Phi) = \{x \in X\colon \psi(x) \in \Phi(\omega, x)\}$$

is nonempty, then ψ and Φ have a random coincidence point.

Proof
The measurable map $\psi\colon X \to Z$ can be naturally extended to the random map $\widetilde{\psi}\colon \Omega \times X \to Z$ if we set $\widetilde{\psi}(\omega, x) = \psi(x)$ for all $\omega \in \Omega$. In fact, for each open $V \subset Z$ we will have then $\widetilde{\psi}^{-1}(V) = \Omega \times \psi^{-1}(V) \in \Sigma \times \mathbb{B}(X)$.

Notice that the function $\mu\colon \Omega \times X \to \mathbb{R}$,

$$\mu(\omega, x) = dist(\psi(x), \Phi(\omega, x)) = dist(\widetilde{\psi}(\omega, x), \Phi(\omega, x))$$

is random (see [10]).

Define the multimap $\mathcal{F}\colon \Omega \multimap X$ as

$$\mathcal{F}(\omega) = Coin_\omega(\psi, \Phi).$$

Then for the graph $\Gamma_\mathcal{F}$ of the multimap $\mathcal{F}$ we have $\Gamma_\mathcal{F} = \mu^{-1}(0) \in \Sigma \times \mathbb{B}(X)$ and applying to $\mathcal{F}$ the Aumann selection theorem for a multifunction with a measurable graph (see, e.g., [13], Theorem 2.2.14) we conclude that $\mathcal{F}$ admits a measurable selection $\xi\colon \Omega \to X$ which is the desirable random coincidence point. □

3. A RANDOM COINCIDENCE INDEX FOR FUNDAMENTALLY RESTRICTIBLE QUADRIPLES

3.1. A Completely Fundamentally Restrictible Triplet

At first we recall some notions (see, e.g. [15]). Let again $Y = U \subset E$, or $U_* \subset E \times [0, 1]$ be open bounded sets, $f : \overline{Y} \to E'$ a map; $G : \overline{Y} \to K(E')$ a multimap.

Definition 3.1:
A convex, closed subset $T \subset E'$ is said to be fundamental for a triplet $\left(f, G, \overline{Y}\right)$ if:

(i) $G\left(f^{-1}(T)\right) \subseteq T$;

(ii) for any point $y \in \overline{Y}$, the inclusion $f(y) \in \overline{co}\left(G(y) \cup T\right)$ implies $f(y) \in T$.

The entire space E' and $\overline{co}G\left(\overline{Y}\right)$ are natural examples of fundamental sets for $\left(f, G, \overline{Y}\right)$. It is easy to verify the following properties of a fundamental set.

Proposition 3.1:
The following holds:

a) The set $Coin\,(f, G)$ is included in $f^{-1}(T)$ for each fundamental set T of $\left(f, G, \overline{Y}\right)$;

b) Let T be a fundamental set of $\left(f, G, \overline{Y}\right)$, and $P \subset T$, then the set $\widetilde{T} = \overline{co}\left(G\left(f^{-1}(T)\right) \cup P\right)$ is also fundamental;

c) Let $\{T_\alpha\}$ be a system of fundamental sets of $\left(f, G, \overline{Y}\right)$. The set $T = \cap_\alpha T_\alpha$ is also fundamental.

Definition 3.2:
We will say that a triple $\left(f, G, \overline{Y}\right)$ is completely fundamentally restrictible if it possesses a nonempty, compact fundamental set T.

The fact that a triple $\left(f, G, \overline{Y}\right)$ is completely fundamentally restrictible will be denoted as $\left(f, G, \overline{Y}\right)_F$. It is clear that each compact triple $\left(f, G, \overline{Y}\right)_C$ is completely fundamentally restrictible since we can take $\overline{co}G\left(\overline{Y}\right)$ as a compact fundamental set. To obtain more valuable example of a completely fundamentally restrictible triplet, we will need the following notions.

Denote by $P(E')$ the collection of all nonempty subsets of a Banach space $E^{'}$. Let $(\mathcal{A}, \geq)$ be a partially ordered set.

Definition 3.3:
A map $\beta : P(E') \to \mathcal{A}$ is called a measure of noncompactness (MNC) in $E^{'}$ if

$$\beta(\overline{co}\, D) = \beta(D) \quad \text{for every } D \in P(E').$$

A MNC β is called:

(i) monotone, if $D_0,\ D_1 \in P(E')$, $D_0 \subseteq D_1$ implies $\beta(D_0) \leq \beta(D_1)$;

(ii) nonsingular, if $\beta(\{a\} \cup D) = \beta(D)$ for every $a \in E^{'}$, $D \in P(E')$;

(iii) real, if $A = \overline{\mathbb{R}_+} = [0, +\infty]$ with the natural ordering, and $\beta(D) < +\infty$ for every bounded set $D \in P(E')$.

Among the known examples of MNC satisfying all the above properties we can consider the Hausdorff MNC

$$\chi(D) = \inf\{\varepsilon > 0 : D \text{ has a finite } \varepsilon\text{-net}\}\ .$$

and the Kuratowski MNC

$$\alpha(D) = \inf\{d > 0 :\ D \text{ has a finite partition with sets of diameter less than } d\}\ .$$

Definition 3.4:
Maps f, G and the space $\overline{Y}$ form a β-condensing triplet, $\left(f, G, \overline{Y}\right)_\beta$ if they satisfy conditions (h1) and (h3) in Definition 2.7 and

h2$_\beta$) a multimap $G = \varphi \circ \mathfrak{F} \in CJ\left(\overline{Y}, E^{'}\right)$ is β-condensing w.r.t. f, i.e.,

$$\beta(G(\mathcal{D})) \ngeq \beta(f(\mathcal{D}))$$

for every $\mathcal{D} \subseteq \overline{Y}$ such that $G(\mathcal{D})$ is not relatively compact.

Proposition 3.2:
Each β-condensing triplet $\left(f, G, \overline{U}\right)_\beta$, where β is a monotone, nonsingular MNC is completely fundamentally restrictible.

Proof
Consider the collection $\{T_\alpha\}$ of all fundamental sets of $\left(f, G, \overline{U}\right)_\beta$ containing an arbitrary point $a \in E^{'}$. This collection is nonempty since it contains $E^{'}$. Then, taking $T = \cap_\alpha T_\alpha \neq \emptyset$ we obviously have

$$T = \overline{co}\left(G\left(f^{-1}(T)\right) \cup \{a\}\right)$$

and hence

$$\beta\left(f\left(f^{-1}(T)\right)\right) \leq \beta(T) = \beta\left(G\left(f^{-1}(T)\right)\right)\ ,$$

so $G\left(f^{-1}(T)\right)$ is relatively compact and T is compact. □

3.2. A Random Coincidence Index

Let (Ω, Σ) be a complete measurable space.

Definition 3.5:
A map $f : \overline{Y} \to E'$, a multimap $G\colon \Omega \times \overline{Y} \to K(E')$ and the spaces Ω and $\overline{Y}$ form a random fundamentally restrictible quadriple $\left(f, G, \Omega, \overline{Y}\right)_F$ if the following conditions are satisfied:

$g1)$ *f is a continuous proper map, $f_{|Y} \in \Phi_k C^1(Y)$ with $k = 0$ in case (i), $k = 1$ in case (ii), and the Fredholm structure on Y generated by f is oriented;*

$g2)$ *G is a random multimap;*

$g3)$ *for each $\omega \in \Omega$ the multimap $G_\omega = G(\omega, \cdot) = (\varphi_\omega \circ \mathfrak{F}_\omega) \in CJ\left(\overline{Y}, E'\right)$;*

$g4)$ *for each $\omega \in \Omega$ the triplet $(f, G_\omega, \overline{Y})$ is completely fundamentally restrictible and $Coin\,(f, G_\omega) \cap \partial Y = \emptyset$.*

Now our purpose is to define an oriented random coincidence index for a random fundamentally restrictible quadriple $\left(f, G, \Omega, \overline{U}\right)_F$. To this end, define for each $\omega \in \Omega$ a coincidence index of a completely fundamentally restrictible triplet $\left(f, G_\omega, \overline{U}\right)_F$.

Let T^ω be a nonempty compact fundamental set of a triplet $\left(f, G_\omega, \overline{U}\right)_F$ and $\rho : E' \to T^\omega$ be any retraction. Consider the multimap $\widetilde{G}_\omega = \rho \circ \varphi_\omega \circ \mathfrak{F}_\omega \in CJ\left(\overline{U}, E'\right)$. From Proposition 3.1(a) it follows that

$$Coin\left(f, \widetilde{G}_\omega\right) = Coin\,(f, G_\omega)\ . \tag{3.1}$$

Hence, $f, \widetilde{G}_\omega$, and $\overline{U}$ form a compact triplet $\left(f, \widetilde{G}_\omega, \overline{U}\right)_C$. We will say that $\left(f, \widetilde{G}_\omega, \overline{U}\right)_C$ is a compact approximation of the triplet $\left(f, G_\omega, \overline{U}\right)_F$.

Definition 3.6:
The oriented coincidence index of a completely fundamentally restrictrictible triplet $\left(f, G_\omega, \overline{U}\right)_F$ is defined by the equality

$$Ind\,\left(f, G_\omega, \overline{U}\right)_F := Ind\,\left(f, \widetilde{G}_\omega, \overline{U}\right)_C,$$

where $\left(f, \widetilde{G}_\omega, \overline{U}\right)_C$ is a compact approximation of $\left(f, G_\omega, \overline{U}\right)_F$.

To prove the consistency of the above definition, consider two nonempty, compact fundamental sets T_0 and T_1 of the triplet $\left(f, G_\omega = \varphi_\omega \circ \mathfrak{F}_\omega, \overline{U}\right)_F$ with retractions $\rho_0 : E' \to T_0$ and $\rho_1 : E' \to T_1$ respectively.

If $T_0 \cap T_1 = \emptyset$, then by Proposition 3.1 (a), (c), $Coin\left(f, \widetilde{G}_{\omega 0}\right) = Coin\left(f, \widetilde{G}_{\omega 1}\right) = Coin\left(f, \widetilde{G}_\omega\right) = \emptyset$, where $\widetilde{G}_{\omega i} = \rho_i \circ \varphi_\omega \circ \mathfrak{F}_\omega$, $i = 0, 1$. Hence, by Proposition 2.2, $Ind\,\left(f, \widetilde{G}_{\omega 0}, \overline{U}\right)_C = Ind\,\left(f, \widetilde{G}_{\omega 1}, \overline{U}\right)_C = 0$.

Otherwise, we can assume, w.l.o.g., that $T_0 \subseteq T_1$. In this case, consider the map $\overline{\varphi}_\omega : Z_\omega \times [0,1] \to E'$, given by $\overline{\varphi}_\omega(z, \lambda) = \rho_1 \circ (\lambda \varphi_\omega(z) + (1-\lambda)\,\rho_0 \circ \varphi - \omega(z))$ and the multimap $\overline{G}_\omega \in CJ\left(\overline{U} \times [0,1], E'\right)$, $\overline{G}_\omega(x, \lambda) = \overline{\varphi}_\omega(\mathfrak{F}_\omega(x), \lambda)$.

The compact triplet $\left(f, \overline{G}_\omega, \overline{U} \times [0,1]\right)_C$ realizes the homotopy

$$\left(f, \widetilde{G}_{\omega 0}, \overline{U}\right)_C \sim \left(f, \widetilde{G}_{\omega 1}, \overline{U}\right)_C .$$

Indeed, the only fact that we need to verify is that

$$Coin\ \left(\overline{f}, \overline{G}_\omega\right) \cap \left(\partial U \times [0,1]\right) = \emptyset,$$

where $\overline{f}\,(x,\lambda) \equiv f\,(x)$ is the natural extension.

To the contrary, suppose that there exists $(x,\lambda) \in \partial U \times [0,1]$ such that

$$f\,(x) = \rho_1 \circ \left(\lambda \varphi_\omega\,(z) + (1-\lambda)\,\rho_0 \circ \varphi_\omega\,(z)\right)$$

for some $z \in \mathfrak{F}_\omega\,(x)$. But in this case, $x \in f^{-1}\,(T_1)$ and hence $\varphi_\omega\,(z) \in T_1$. Since also $\rho_0 \circ \varphi_\omega\,(z) \in T_1$ we have

$$\lambda \varphi_\omega\,(z) + (1-\lambda)\,\rho_0 \circ \varphi_\omega\,(z) \in T_1$$

and so

$$f\,(x) = \lambda \varphi_\omega\,(z) + (1-\lambda)\,\rho_0 \circ \varphi_\omega\,(z) \in \overline{co}\,(G_\omega\,(x) \cup T_0)$$

and we obtain that $f\,(x) \in T_0$ and $x \in f^{-1}\,(T_0)$, implying $\varphi_\omega\,(z) \in T_0$ and $\rho_0 \circ \varphi_\omega\,(z) = \varphi_\omega\,(z)$. We conclude that $f\,(x) = \varphi_\omega\,(z) \in G_\omega\,(x)$ giving the contradiction.

Definition 3.7:
For a given random fundamentally restrictible quadriple $\left(f, G, \Omega, \overline{U}\right)_F$ the oriented random coincidence index is defined as the following collection of integers:

$$Ind\left(f, G, \Omega, \overline{U}\right)_F = \{Ind\left(f, G_\omega, \overline{U}\right)_F : \omega \in \Omega\}.$$

By definition we set $Ind\left(f, G, \Omega, \overline{U}\right)_F \neq 0$ under the condition that $Ind\left(f, G_\omega, \overline{U}\right)_F \neq 0$ for all $\omega \in \Omega$.

From Proposition 2.5 we obtain the following random coincidence point property.

Theorem 3.1:
If $Ind\left(f, G, \Omega, \overline{U}\right)_F \neq 0$ then the quadriple $\left(f, G, \Omega, \overline{U}\right)_F$ has a random coincidence point, i.e., there exists a measurable map $\xi\colon \Omega \to U$ such that

$$f(\xi(\omega)) \in G(\omega.\xi(\omega)), \quad \forall \omega \in \Omega.$$

Definition 3.8:
Two random fundamentally restrictible quadriples $\left(f_0, G_0 = (\varphi_0 \circ \mathfrak{F}_0)\,, \Omega, \overline{U}_0\right)_F$ and $\left(f_1, G_1 = (\varphi_1 \circ \mathfrak{F}_1)\,, \Omega, \overline{U}_1\right)_F$ are said to be homotopic

$$\left(f_0, G_0 = (\varphi_0 \circ \mathfrak{F}_0)\,, \Omega, \overline{U}_0\right)_F \sim \left(f_1, G_1 = (\varphi_1 \circ \mathfrak{F}_1)\,, \Omega, \overline{U}_1\right)_F$$

if there exists a random completely fundamentally restrictible quadriple $\left(f_*, G_*, \Omega, \overline{U_*}\right)_F$, where $U_* \subset E \times [0,1]$ is an open set, such that:

a) $U_i = U_* \cap (E \times \{i\})$, $i = 0, 1$ *;*

b) $f_* \mid_{\overline{U_i}} = f_i$, $i = 0, 1$ *;*

c) for each $\omega \in \Omega$, the multimap $G_{*\omega} = G_*(\omega, \cdot, \cdot)$ has the form

$$G_{*\omega}(x, \lambda) = \varphi_{*\omega}(\mathfrak{F}_{*\omega}(x, \lambda), \lambda)$$

where $\mathfrak{F}_{*\omega} \in J(\overline{U_*}, Z_\omega)$, $\varphi_{*\omega} : Z_\omega \times [0,1] \to E'$ is a continuous map, and

$$\mathfrak{F}_{*\omega}|_{\overline{U}_i} = \mathfrak{F}_{i\omega}, \quad \varphi_{*\omega}|_{Z_\omega \times \{i\}} = \varphi_{i\omega}, \quad i = 0, 1.$$

We get the following homotopy invariance property.

Theorem 3.2:
If

$$\left(f_0, G_0, \Omega, \overline{U}_0\right)_F \sim \left(f_1, G_1, \Omega, \overline{U}_1\right)_F,$$

then

$$\left|Ind\left(f_0, G_0, \Omega, \overline{U}_0\right)_F\right| = \left|Ind\left(f_1, G_1, \Omega, \overline{U}_1\right)_F\right|$$

in the sense that

$$\left|Ind((f_0, G_{0\omega}, \overline{U}_0)_F\right| = \left|Ind\left(f_1, G_{1\omega}, \overline{U}_1\right)_F\right|$$

for all $\omega \in \Omega$.

Proof
For a given $\omega \in \Omega$, let T_*^ω be a nonempty compact fundamental set of the triplet $\left(f_*, G_{*\omega} = (\varphi_{*\omega} \circ \mathfrak{F}_{*\omega}), \overline{U_*}\right)_F$. It is easy to see that T_*^ω is fundamental also for the triplets $\left(f_i, G_{i\omega}, \overline{U}_i\right)_F$, $i = 0, 1$.
Let $\rho_{*\omega} : E' \to T_*^\omega$ be any retraction, and $\left(f_*, \widetilde{G}_{*\omega} = \rho_{*\omega} \circ \varphi_{*\omega} \circ \mathfrak{F}_{*\omega}, \overline{U}_*\right)_C$ the corresponding compact approximation of $\left(f_*, G_{*\omega}, \overline{U}_*\right)_F$. Then $\left(f_*, \widetilde{G}_{*\omega}, \overline{U}_*\right)_C$ realizes a compact homotopy connecting the triplets $\left(f_i, \rho_{*\omega} \circ \varphi_{i\omega} \circ \mathfrak{F}_{i\omega}, \overline{U}_i\right)_C$, $i = 0, 1$ which are compact approximations of $\left(f_i, G_{i\omega}, \overline{U}_i\right)_F$, $i = 0, 1$ respectively.
By Proposition 2.3 we have

$$\left|Ind\ \left(f_0, \rho_{*\omega} \circ \varphi_{0\omega} \circ \mathfrak{F}_{0\omega}, \overline{U}_0\right)_C\right| = \left|Ind\ \left(f_1, \rho_{*\omega} \circ \varphi_{1\omega} \circ \mathfrak{F}_{1\omega}, \overline{U}_1\right)_C\right|$$

giving the desired. □

Remark 3.1:
In connection with Remark 2.1 if the operator f is constant under the homotopy, i.e., U_* has the form $U_* = U \times [0,1]$, where $U \subset E$ is an open set and $f_*(x, \lambda) = f(x)$ for all $\lambda \in [0,1]$ with $f \in \Phi_0 C^1(U)$ then

$$Ind\left(f, G_0, \Omega, \overline{U}\right)_F = Ind\left(f, G_1, \Omega, \overline{U}\right)_F$$

Let us formulate the additive dependence on domain property for random fundamentally restrictible quadriples which follows from Proposition 2.4.

Theorem 3.3:
Let U_0 and U_1 be disjoint open subsets of an open bounded set $U \subset E$ and $\left(f, G, \Omega, \overline{U}\right)_F$ be a fundamentally restrictible quadriple such that

$$Coin\,(f, G_\omega) \cap \left(\overline{U} \setminus (U_0 \cup U_1)\right) = \emptyset$$

for each $\omega \in \Omega$. *Then*

$$Ind\left(f, G, \Omega, \overline{U}\right)_F = Ind\left(f, G, \Omega, \overline{U_0}\right)_F + Ind\left(f, G, \Omega, \overline{U_1}\right)_F$$

in the sense that

$$Ind\left(f, G_\omega, \overline{U}\right)_F = Ind\left(f, G_\omega, \overline{U_0}\right)_F + Ind\left(f, G_\omega, \overline{U_1}\right)_F$$

for each $\omega \in \Omega$.

Analogs of the above notions and assertions for the condensing case can be formulated in the natural way. Let β be a monotone, nonsingular MNC in E'.

Definition 3.9:
A map $f : \overline{Y} \to E'$, *a multimap* $G\colon \Omega \times \overline{Y} \to K(E')$ *and the spaces* Ω *and* $\overline{Y}$ *form a random* β*-condensing quadriple* $\left(f, G, \Omega, \overline{Y}\right)_\beta$ *if they satisfy conditions* $(g1) - (g3)$ *of Definition 3.5 and the following condition*

$g4_\beta)$ *for each* $\omega \in \Omega$ *the triplet* $(f, G_\omega, \overline{Y})$ *is* β*-condensing and* $Coin\,(f, G_\omega) \cap \partial Y = \emptyset$.

For a given $\omega \in \Omega$, the oriented coincidence index of a β-condensing triplet $(f, G_\omega, \overline{U})_\beta$ can be defined via its compact approximation as in Definition 3.6 and the oriented random coincidence index of a random β-condensing quadriple $(f, G, \Omega, \overline{U})_\beta$ also is defined in accordance with Definition 3.7:

$$Ind\left(f, G, \Omega, \overline{U}\right)_\beta = \{Ind\left(f, G_\omega, \overline{U}\right)_\beta : \omega \in \Omega\}.$$

The notion of homotopy for random β-condensing quadriples follows Definition 3.8 with the substitution of "completely fundamentally restrictible" with "β-condensing" and the corresponding homotopy invariance property also holds true.

Remark 3.2:
Let us mention that in case of constant f *and* $\overline{U}$:

$$\begin{aligned} U_* &= U \times [0,1] \\ f_*(x,\lambda) &\equiv f(x)\,, \qquad \forall \lambda \in [0,1]\,, \end{aligned}$$

the condition of β*-condensivity for a triplet* $\left(f, G_{*\omega}, \overline{U} \times [0,1]\right)_\beta$ *may be weakened: for the existence of a nonempty, compact fundamental set* T^ω *it is sufficient to demand that*

$$\beta\left(G_{*\omega}\left(\mathcal{D} \times [0,1]\right)\right) \ngeq \beta\left(f\left(\mathcal{D}\right)\right)$$

for every $\mathcal{D} \subseteq \overline{U}$ *such that* $G_{*\omega}\left(\mathcal{D} \times [0,1]\right)$ *is not relatively compact.*
In fact, it is enough to notice that in this case $f_*^{-1}\left(T\right) = f^{-1}\left(T\right) \times [0,1]$ *and to follow the line of reasoning of Proposition 3.2.*
Taking into consideration the corresponding property of compact triplets, we can precise the above property of homotopy invariance.
If $\left(f, G_{*\omega}, \overline{U} \times [0,1]\right)_\beta$ *is a* β*-condensing triplet, where* $G_{*\omega}$ *has the form (c) of Definition 2.8, then*

$$Ind\,\left(f, G_{0\omega}, \overline{U}\right)_\beta = Ind\,\left(f, G_{1\omega}, \overline{U}\right)_\beta$$

where $G_{i\omega} = G_{*\omega}\left(\cdot, \{i\}\right)$, $i = 0, 1$.

As an example of the application of a random coincidence point property, let us consider the following assertion.

Theorem 3.4:
Let $f \in \Phi_0 C^1(E, E')$ be odd; $G\colon \Omega \times E \to K(E')$ a random multimap such that for each $\omega \in \Omega$, $G_\omega \in CJ(E, E')$ and β-condensing w.r.t. f on bounded subsets of E, i.e. $\beta(G_\omega(\mathcal{D})) \ngeq \beta(f(\mathcal{D}))$ for every bounded set $\mathcal{D} \subset E$ such that $G(\mathcal{D})$ is not relatively compact.
If the set of solutions of two-parameter family of operator inclusions

$$f(x) \in \lambda G_\omega(x) \tag{3.2}$$

is a priori bounded, then there exists a random coincidence point of f and G.

Proof
From the condition it follows that there exists a ball $\mathcal{B} \subset E$ centered at the origin whose boundary $\partial\mathcal{B}$ does not contain solutions of (3.2).
Let $\varphi_\omega \circ \mathfrak{F}_\omega$ be a representation of G_ω. If $G_{*\omega} : \overline{\mathcal{B}} \times [0,1] \to K(E')$ has the form

$$\begin{aligned} G_{*\omega}(x,\lambda) &= \varphi_{*\omega}(\mathfrak{F}_\omega(x), \lambda), \qquad (x,\lambda) \in \overline{\mathcal{B}} \times [0,1], \\ \varphi_{*\omega}(z,\lambda) &= \lambda\varphi_\omega(z) \end{aligned}$$

then $f, G_{*\omega}$ and $\overline{\mathcal{B}} \times [0,1]$ form a β-condensing triplet $\left(f, G_{*\omega}, \overline{\mathcal{B}} \times [0,1]\right)_\beta$.
In fact, suppose that $\beta(G_{*\omega}(\mathcal{D})) \geq \beta(f(\mathcal{D}))$ for some $\mathcal{D} \subset \overline{\mathcal{B}}$. Since $G_{*\omega}(\mathcal{D} \times [0,1]) = \overline{co}(G_\omega(\mathcal{D}) \cup \{0\})$ we have that $\beta(G_\omega(\mathcal{D})) \geq \beta(f(\mathcal{D}))$ implying that $G_\omega(\mathcal{D})$ and hence $G_{*\omega}(\mathcal{D} \times [0,1])$ is relatively compact.
So the triplet $\left(f, G_{*\omega}, \overline{\mathcal{B}} \times [0,1]\right)_\beta$ induces a homotopy connecting the triplets $\left(f, G_\omega, \overline{\mathcal{B}}\right)_\beta$ and $\left(f, 0, \overline{\mathcal{B}}\right)_\beta$. Since the triplet $\left(f, 0, \overline{\mathcal{B}}\right)_\beta$ is finite dimensional, from the odd condition on f and the odd field property of the Brouwer degree, it follows that $\left(f, 0, \overline{\mathcal{B}}\right)_\beta$ is an odd number.
Then, from the equality $Ind\left(f, G_\omega, \overline{\mathcal{B}}\right)_\beta = Ind\left(f, 0, \overline{\mathcal{B}}\right)_\beta$ it follows that $Ind\left(f, G_\omega, \overline{\mathcal{B}}\right)_\beta \neq 0$ for all $\omega \in \Omega$ and hence $Ind\left(f, G, \Omega, \overline{\mathcal{B}}\right)_\beta \neq 0$ and we can apply the coincidence point property. □

4. AN APPLICATION: EXISTENCE OF A SOLUTION FOR A RANDOM FEEDBACK CONTROL SYSTEM

Let (Ω, Σ, μ) be a σ-finite measure space. Consider a random feedback control system governed by the following relations:

$$A(t, x(t), x'(t)) = B\left(t, x(t), x'(t), \int_0^t y_\omega(s)\,ds\right), \quad t \in [0,a]; \tag{4.3}$$

$$y_\omega(\tau) \in C(\omega, \tau, x(\tau)), \quad \text{a.e. } \tau \in [0,a],\ \forall\omega \in \Omega; \tag{4.4}$$

$$x(0) = x_0, \tag{4.5}$$

where $\omega \in \Omega,\ A : [0,a] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,\ B : [0,a] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are continuous maps; $C : \Omega \times [0,a] \times \mathbb{R}^n \multimap \mathbb{R}^m$ is a multimap, and $x_0 \in \mathbb{R}^n$.

By a random solution of problem (4.3)-(4.5) we mean a pair (x_ω, y_ω) consisting of a trajectory function $x_\omega(\cdot)$ such that $\omega \in \Omega \to x_\omega \in C^1([0,a]; \mathbb{R}^n)$ is a measurable map and the function x_ω satisfies for each $\omega \in \Omega$ relations (4.3)-(4.5), whereas the control function $y_\omega(\cdot)$ is such that $\omega \in \Omega \to y_\omega \in L^1([0,a]; \mathbb{R}^m)$ is a measurable map and the function y_ω satisfies for each $\omega \in \Omega$ relations (4.3)-(4.4).

Our aim is to show that, under appropriate conditions, the problem of finding of a random solution to systrm (4.3)-(4.5) can be reduced to the study of a condensing quadriple of the above mentioned form (see Section 3).

Consider the following condition:

(A) For each $(t,u,v) \in [0,a] \times \mathbb{R}^n \times \mathbb{R}^n$ there exist continuous partial derivatives $A'_u(t,u,v)$, $A'_v(t,u,v)$ and moreover, $\det A'_v(t,u,v) \neq 0$.

The following assertion holds true (see [26], Proposition 5.1).

Proposition 4.1:
Under condition (A) a map $f : C^1([0,a];\mathbb{R}^n) \to C([0,a];\mathbb{R}^n) \times \mathbb{R}^n$ defined as

$$f(x)(t) = (A(t,x(t),x'(t)), x(0))$$

is a Fredholm map of index zero, whose restriction to each closed bounded set $D \subset C^1([0,a];\mathbb{R}^n)$ is proper.

Now we will describe the assumptions on the map B and the multimap C.

Denoting by the symbol $Kv(\mathbb{R}^m)$ the collection of all nonempty compact convex subsets of $\mathbb{R}^m$, we suppose that the multimap $C : \Omega \times [0,a] \times \mathbb{R}^n \to Kv(\mathbb{R}^m)$ satisfies the following conditions:

(C1) C is a random multimap, i.e., it is measurable w.r.t. the σ-algebra $\Sigma \otimes \mathcal{L} \otimes \mathbb{B}(\mathbb{R}^n)$, where $\mathcal{L}$ is the σ-algebra of Lebesgue subsets of $[0,a]$;

(C2) for all $(\omega,t) \in \Omega \times [0,a]$ the multimap $C(\omega,t,\cdot) : \mathbb{R}^n \to Kv(\mathbb{R}^m)$ is upper semicontinuous;

(C3) for each $r > 0$ there exists a function $\gamma_r \colon \Omega \times [0,a] \to \mathbb{R}_+$ such that: (i) $\gamma_r(\omega,\cdot) \in L^1(0,a)$ for each $\omega \in \Omega$; (ii) the function $\gamma_r(\cdot,t) : \Omega \to \mathbb{R}_+$ is measurable for a.e. $t \in [0,a]$ and the following estimate holds for all $\omega \in \Omega$ and $u \in \mathbb{R}^n$, $\|u\| \le r$:

$$\|C(\omega,t,u)\| := \sup\{\|c\| : c \in C(\omega,t,u)\} \le \gamma_r(\omega,t) \ \text{ a.e. } t \in [0,a].$$

Notice that from condition $(C1)$ it follows that for each $(\omega,u) \in \Omega \times \mathbb{R}^n$ the multifunction

$$C(\omega,\cdot,u) \colon [0,a] \to Kv(\mathbb{R}^m)$$

is Lebesgue measurable.

It is known (see, e.g. [8], [11], [13], [15], [22]) that under given conditions for each $\omega \in \Omega$ the superposition multioperator $\mathcal{P}(\omega,\cdot) \colon C([0,a];\mathbb{R}^n) \multimap L^1([0,a];\mathbb{R}^m)$:

$$\mathcal{P}(\omega,x) = \{\psi \in L^1([0,a];\mathbb{R}^m) : \psi(t) \in C(\omega,t,x(t)) \text{ a.e. } t \in [0,a]$$

is well defined. Moreover, if we will consider the composition $\Pi \colon \Omega \times C([0,a];\mathbb{R}^n) \multimap C([0,a];\mathbb{R}^m)$

$$\Pi(\omega,x) = j \circ \mathcal{P}(\omega,x),$$

where $j \colon L^1([0,a];\mathbb{R}^m) \to C([0,a];\mathbb{R}^m)$ is the integral operator $j(\psi)(t) = \int_0^t \psi(s)ds$, then $\Pi(\omega,\cdot) \colon C([0,a];\mathbb{R}^n) \multimap C([0,a];\mathbb{R}^m)$ is a closed multioperator for each $\omega \in \Omega$. Further, since the embedding $C^1([0,a];\mathbb{R}^n) \hookrightarrow C([0,a];\mathbb{R}^n)$ is completely continuous, the restriction of $\Pi(\omega,\cdot)$ to $C^1([0,a];\mathbb{R}^n)$ is locally compact and hence upper semicontinuous (see, e.g. [22], Theorem 1.2.32). It is easy to see also that the multimap Π has compact convex values and hence this restriction is a J-multimap.

At last, from [23], Theorem 3 it follows that Π is a random multimap.

Now, we will assume that the maps A and B satisfy the following Lipschitz type condition:

(AB) there exists a constant q, $0 \leq q < 1$ such that

$$|B(t,u,v,w) - B(t,u,\overline{v},w)| \leq q\,|A(t,u,v) - A(t,u,\overline{v})|$$

for all $t \in [0,a]$, $u, v, \overline{v} \in \mathbb{R}^n$, $w \in \mathbb{R}^m$.

Consider the continuous map $\xi : C^1([0,a];\mathbb{R}^n) \times C([0,a];\mathbb{R}^m) \to C([0,a];\mathbb{R}^n)$ defined as

$$\xi(x,z)(t) = B(t, x(t), x'(t), z(t))$$

and the multimap $\widetilde{\mathfrak{F}} : \Omega \times C^1([0,a];\mathbb{R}^n) \to K(C^1([0,a];\mathbb{R}^n) \times C([0,a];\mathbb{R}^m))$, $\widetilde{\mathfrak{F}}(\omega,x) = \{x\} \times \Pi(\omega,x)$.

Consider the composition $\widetilde{G} = \xi \circ \widetilde{\mathfrak{F}}$: $\Omega \times C^1([0,a];\mathbb{R}^n) \to K(C([0,a];\mathbb{R}^n))$. It is easy to see that the multimap $\widetilde{\mathfrak{F}}$ and hence $\widetilde{G}$ are random. Further, from [22], Theorem 1.3.17 it follows that for each $\omega \in \Omega$ the multimap $\widetilde{\mathfrak{F}}_\omega = \widetilde{\mathfrak{F}}(\omega,\cdot)$ is u.s.c. and hence it is a J-multimap, and therefore the composition $\widetilde{G}_\omega = \xi \circ \widetilde{\mathfrak{F}}_\omega : C^1([0,a];\mathbb{R}^n) \to K(C([0,a];\mathbb{R}^n))$ is a CJ-multimap. It is clear that the set $\widetilde{G}_\omega(x)$ consists of all functions of the form $B(t, x(t), x'(t), \pi_\omega(t))$ where $\pi_\omega \in \Pi(\omega, x)$.

Define now the multimap $G : \Omega \times C^1([0,a];\mathbb{R}^n) \to K(C([0,a];\mathbb{R}^n) \times \mathbb{R}^n)$ by

$$G(\omega,x) = \widetilde{G}(\omega,x) \times \{x_0\}.$$

It is easy to see that G is a random multimap and $G_\omega = G(\omega,\cdot)$ is a CJ-multimap for each $\omega \in \Omega$.

Now, in accordance with Proposition 2.5 the existence of a trajectory for system (4.3)-(4.5) is equivalent to the existence of a coincidence point $x_\omega \in C^1([0,a];\mathbb{R}^n)$ for the pair (f, G_ω) for each $\omega \in \Omega$.

If $U \subset C^1([0,a];\mathbb{R}^n)$ is an open bounded set, then to show, for a given $\omega \in \Omega$ that $\left(f, G_\omega, \overline{U}\right)$ form a condensing triplet w.r.t. the Kuratowski MNC, it is sufficient to prove the following statement.

Proposition 4.2:
The triplet $\left(\widetilde{f}, \widetilde{G}_\omega, \overline{U}\right)$, *where* $\widetilde{f} : C^1([0,a];\mathbb{R}^n) \to C([0,a];\mathbb{R}^n)$, *is defined as*

$$\widetilde{f}(x)(t) = A(t, x(t), x'(t))$$

is α*-condensing w.r.t. the Kuratowski MNC* α *in the space* $C([0,a];\mathbb{R}^n)$ *for each* $\omega \in \Omega$.

Proof
Take any subset $\mathcal{D} \subset \overline{U}$, and let $\alpha\left(\widetilde{f}(\mathcal{D})\right) = d$. From the definition of Kuratowski MNC it follows that taking an arbitrary $\varepsilon > 0$ we may find a partition of the set $\widetilde{f}(\mathcal{D})$ into subsets $\widetilde{f}(\mathcal{D}_i)$, $i = 1, \cdots, s$ such that $diam\left(\widetilde{f}(\mathcal{D}_i)\right) \leq d + \varepsilon$. Since the embedding $C^1([0,a];\mathbb{R}^n) \hookrightarrow C([0,a];\mathbb{R}^n)$ is completely continuous, the image $\mathcal{D}_C$ of $\mathcal{D}$ under this embedding is relatively compact. It is known (see, e.g. [13], [15], [22]) that a u.s.c. compact-valued multimap sends compact sets to compact sets, then we can conclude that the set $\Pi_\omega(\mathcal{D})$ is relatively compact. It means that taking a fixed $\delta > 0$ and any $\mathcal{D}_i$ we may divide the sets $\mathcal{D}_{iC}$ and $\Pi_\omega(\mathcal{D})$ into a finite number of subsets $\mathcal{D}_{ijC}$, $j = 1, \cdots, p_i$, and balls $\mathcal{B}_{ik}(z_{ik})$, $k = 1, \cdots, r_i$, centered at $z_{ik} \in C([0,a];\mathbb{R}^m)$ respectively, such that for each $t \in [0,a]$;

$u_1(\cdot), u_2(\cdot) \in \mathcal{D}_{ijC}$, $v \in \mathbb{R}^n$; $w_1(\cdot), w_2(\cdot) \in \mathcal{B}_{ik}(z_{ik})$ we have that

$$|A(t, u_1(t), v) - A(t, u_2(t), v)| < \delta \tag{4.6}$$

$$|B(t, u_1(t), v, w_1(t)) - B(t, u_2(t), v, w_2(t))| < \delta. \tag{4.7}$$

Now, the set $\widetilde{G}_\omega(\mathcal{D})$ is covered by a finite numbers of sets Γ_{ijk}, $i = 1, \cdots, s$; $j = 1, \cdots, p_i$; $k = 1, \cdots, r_i$ of the form

$$\Gamma_{ijk} = \{B(\cdot, x(\cdot), x'(\cdot), y(\cdot)) : x \in \mathcal{D}_{ijC}, y \in \mathcal{B}_{ik}(z_{ik})\}.$$

Let us estimate the diameters of these sets. Taking arbitrary $x_1, x_2 \in \mathcal{D}_{ijC}$ and $y_1, y_2 \in \mathcal{B}_{ik}(z_{ik})$ and applying (4.6), (4.7) and condition (AB), for any $t \in [0, a]$ we have

$$\begin{aligned}
&|B(t, x_1(t), x_1'(t), y_1(t)) - B(t, x_2(t), x_2'(t), y_2(t))| \\
&< |B(t, x_1(t), x_1'(t), z_{ik}(t)) - B(t, x_2(t), x_2'(t), z_{ik}(t))| + 2\delta \\
&\le |B(t, x_1(t), x_1'(t), z_{ik}(t)) - B(t, x_1(t), x_2'(t), z_{ik}(t))| + \\
&+ |B(t, x_1(t), x_2'(t), z_{ik}(t)) - B(t, x_2(t), x_2'(t), z_{ik}(t))| + 2\delta \\
&\le q\,|A(t, x_1(t), x_1'(t)) - A(t, x_1(t), x_2'(t))| + 3\delta \\
&\le q\,|A(t, x_1(t), x_1'(t)) - A(t, x_2(t), x_2'(t))| + \\
&+ q\,|A(t, x_2(t), x_2'(t)) - A(t, x_1(t), x_2'(t))| + 3\delta \\
&< q(d + \varepsilon) + q\delta + 3\delta.
\end{aligned}$$

Now, if $q = 0$ it means, by the arbitrariness of the choice of $\delta > 0$ that $\alpha\left(\widetilde{G}(\mathcal{D})\right) = 0$ and then the triplet $\left(\widetilde{f}, \widetilde{G}, \overline{U}\right)$ and therefore $\left(f, G, \overline{U}\right)$ is compact. Otherwise, let us take $\varepsilon > 0$ and $\delta > 0$ so small that

$$q\varepsilon + (q + 3)\delta < (1 - q)d.$$

Then, $q(d + \varepsilon) + q\delta + 3\delta = \mu d$ where $0 < \mu < 1$ and, hence $diam\Gamma_{ijk} \le \mu d$, implying that

$$\alpha\left(\widetilde{G}(\mathcal{D})\right) \le \mu\alpha\left(\widetilde{f}(\mathcal{D})\right).$$

□

Now, if we suppose that the trajectory function $x_\omega(\cdot)$ for system (4.3)-(4.5) is found already, the existence of the corresponding control $y_\omega(\cdot)$ follows from the next reasonings.

For a given trajectory x_ω, let a measurable function $h\colon \Omega \to C([0, a]; \mathbb{R}^n)$ be given as

$$h(\omega)(t) = A(t, x_\omega(t), x_\omega'(t)).$$

Further, consider the random map $g\colon \Omega \times L^1([0, a]; \mathbb{R}^m)$,

$$g(\omega, y) = B(t, x_\omega(t), x_\omega'(t), \int_0^t y(s)ds)$$

and a multimap $V\colon \Omega \multimap L^1([0, a]; \mathbb{R}^m)$, $V(\omega) = \mathcal{P}(\omega, x_\omega)$. Applying again Theorem 3 from [23] we conclude that the multimap V is measurable.

Since x_ω is a trajectory, we get that

$$h(\omega) \in g(\omega, V(\omega)), \quad \forall \omega \in \Omega.$$

From a version of the Filippov implicit function lemma (see Proposition 2.2.25 of [13]) we conclude that there exists a measurable function $v\colon \Omega \to L^1([0,a];\mathbb{R}^m)$ such that

$$h(\omega) = g(\omega, v(\omega)), \quad \forall \omega \in \Omega.$$

It is clear that we can set $y_\omega(\cdot) = v(\omega)$ as the desired control function.

The above arguments yield that the random coincidence index theory, developed in the previous sections, can be applied to the study of the solvability of problem (4.3)-(4.5). For example, the direct application of Theorem 3.4 yields the following assertion.

Proposition 4.3:
Under above conditions, suppose that the map A is odd: $A(t,-u,-v) = A(t,u,v)$ for all $t \in [0,a]$; $u,v \in \mathbb{R}^n$ and the set of functions $x \in C^1([0,a];\mathbb{R}^n)$ satisfying the two-parameter family of relations

$$A(t, x(t), x'(t)) = \lambda B\left(t, x(t), x'(t), \int_0^t y_\omega(s)\, ds\right), \ t \in [0,a], \ \lambda \in [0,1]; \tag{4.8}$$

$$y_\omega(\tau) \in C(\omega, \tau, x(\tau)), \quad \text{a.e. } s \in [0,a], \ \forall \omega \in \Omega; \tag{4.9}$$

$$x(0) = x_0, \tag{4.10}$$

is a priori bounded. Then, there exists a random solution of problem (4.3)–(4.5).

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