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On Codimension 3 Singular Points of First-order Implicit Differential Equations

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Abstract: We investigate local phase portraits of first-order implicit differential equations in a neighborhood of their singular points of codimension 3. Namely, we consider singular points where the lifted field of the equation on a surface is non-singular, but the projection of the surface to the phase plane has a singularity of one of three types: lips, beaks, swallowtail. We also consider generic bifurcations in one-parameter families of such equations.

Keywords: implicit differential equations, vector fields, singularities of mappings, fold, cusp, swallowtail

1. INTRODUCTION

The work in this paper is a part of an ongoing research on understanding singular points of first-order implicit differential equations

$$F(x, y, p) = 0, \;\; p = dy/dx, \tag{1.1}$$

that is, points where F_p vanishes (see, for example, [1] – [19] and the references therein). Here $x \in \mathbb{R}^1$, $y \in \mathbb{R}^1$, and $F : \mathbb{R}^3 \to \mathbb{R}^1$ is a smooth (C^∞) function. Since a single point of the (x, y)-plane may correspond to several different values of p such that $F(x, y, p) = 0$, equation (1.1) determines a multi-valued direction field on the (x, y)-plane. For studying this equation, we shall use the Legendrian lift of the equation to a surface in the 1-jet space, which goes back to Poincaré.

The idea of this method is quite similar to the Riemann surface for multi-valued functions of the complex variable. It is used in the most of modern works devoted to implicit differential equations, including the works mentioned below. Moreover, this method naturally generalizes the case of multi-dimensional systems of implicit equations. The latter leads to an interesting phenomenon: vector fields with non-isolated singular points, see [12, 16]. The recent paper [20] should be especially remarked.

1.1. Lifting of equation

Let $\mathbb{J}^1$ be the 1-jet space of functions $y(x)$, i.e., the space with coordinates (x, y, p). Equation (1.1) defines a surface $\mathscr{F}$ in $\mathbb{J}^1$, which is regular at points where $\nabla F \neq 0$. Further, we shall always assume that this condition holds true, and the tangent plane is defined at every point of

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$\mathscr{F}$. The intersection of the field of *contact* planes $pdx - dy = 0$ with the fields of the tangent planes defines a direction field of $\mathscr{F}$, which is called the lifted field of equation (1.1).

Integral curves of equation (1.1) are obtained by the projection π of trajectories of the lifted field from the surface $\mathscr{F}$ to the (x, y)-plane parallel to the p-direction. The p-direction in the space $\mathbb{J}^1$ we shall call vertical. Thus, a study of the phase portrait of equation (1.1) is reduced to the projection of the phase portrait of the lifted field from the surface $\mathscr{F}$ to the (x, y)-plane.

The projection π has singularity at point of the surface $\mathscr{F}$ where $F_p = 0$. Singularities of the germs of smooth mappings are very well studied, the first result in this direction belongs to H. Whitney. See, for example, [21–23] and [24, 25].

The intersection of the contact planes $pdx - dy = 0$ with the fields of the tangent planes to $\mathscr{F}$ can be written in the Pfaffian form:

$$\begin{aligned} &F_x dx + F_y dy + F_p dp = 0, \\ &pdx - dy = 0. \end{aligned} \tag{1.2}$$

which yields the field

$$\dot{x} = F_p, \ \ \dot{y} = pF_p, \ \ \dot{p} = -G, \ \ G = F_x + pF_y, \tag{1.3}$$

where the dot stands for differentiation by a new parameter t, which plays a role of time. Formula (1.3) presents the lifted field of equations (1.1).

We remark that the lifted field is defined at all points of the surface $\mathscr{F}$ except points where the contact plane coincide with the tangent plane, that is, $F_p = G = 0$. The equality $F_p = 0$ defines a subset of $\mathscr{F}$, which is called the *criminant* of equation (1.1). Points of the criminant are called *singular points* of equation (1.1), they are critical points of the projection π.

The equality $G = 0$ defines a subset of $\mathscr{F}$, which is called the *inflection curve* of equation (1.1), since its projection on the (x, y)-plane is the locus of inflection points of integral curves of (1.1). Generically, the criminant and the inflection curve are curves on the surface $\mathscr{F}$ intersecting at isolated points, which are singular points of the lifted field (1.3).

1.2. Classification of singular points

Remind that we excluded from consideration points where $\nabla F = 0$ and we call a point of $\mathscr{F}$ a singular points of equations (1.1), if $F_p = 0$. Here we shall use terminology from [16]:

Definition 1.1:
A singular point of equation (1.1) *is called proper, if* $G \neq 0$ *and improper otherwise.*

For every proper singular point there exists a unique integral curve of the field (1.3). The projection of this integral curve form $\mathscr{F}$ to the (x, y)-plane has a singularity. For simplicity, assume that the proper singular point is the origin of the space $\mathbb{J}^1$. If the germ of F at this point has a finite multiplicity by p, that is,

$$\frac{\partial F}{\partial p}(0) = 0, \ \ldots, \ \frac{\partial^{n-1} F}{\partial p^{n-1}}(0) = 0, \ \ \frac{\partial^n F}{\partial p^n}(0) \neq 0 \tag{1.4}$$

with an integer $n \geq 2$, then the germ of the corresponding integral curve of equation (1.1) has the form

$$x = t^n \varphi(t), \ \ y = t^{n+1} \psi(t), \ \ \varphi(0)\psi(0) \neq 0, \tag{1.5}$$

with smooth functions φ, ψ, see [3]. From the geometric viewpoint, here there are two different cases: if n is even or odd. In Fig. 1.1 we present integral curves (1.5) with $n = 2$ (left) and $n = 3$ (right).

A more interesting and complicated problem is a study of local phase portraits of equation (1.1) in a neighborhood of its singular point. In a neighborhood of a proper singular point the

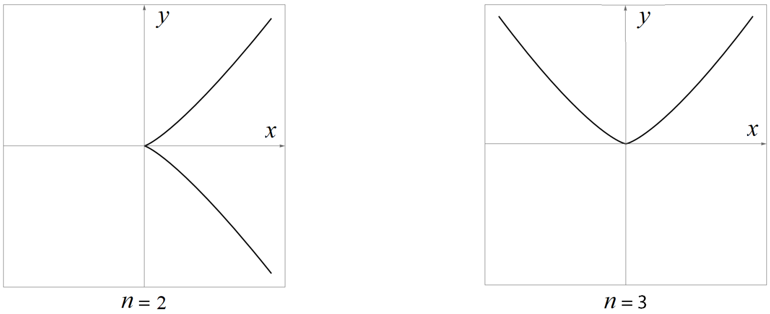

Fig. 1.1. Two types of integral curves passing trough a proper singular point

phase portrait of the lifted field on $\mathscr{F}$ is diffeomorphic to a family of parallel lines (the flow box theorem), whence the problem is reduced to the study of its projection.

The phase portraits of equation (1.1) near a proper singular point where the mapping π has a fold or a cusp (pleat), are well studied; see [9, 10] or survey in [16]. For the convenience of the reader, we shall briefly describe the related results below. The phase portraits of equation (1.1) near an improper singular point where the mapping π has a fold (codimension 2) are studied in [9, 10]. The phase portraits of equation (1.1) near an improper singular point where the mapping π has a pleat (codimension 3) are studied in [7].

The cases mentioned above cover singularities of equation (1.1) of codimensions 1 and 2, but not all singularities of codimension 3, see the table below. The columns correspond to all types of singularities of the projection

$$\pi : \mathscr{F} \to \mathbb{R}^2$$

up to codimension 3 called *fold*, *pleat*, *lips*, *beaks*, *swallowtail*.

The first row presents the corresponding left-right normal forms (see, for example, [24, 25]). The second (third) row contains proper (respectively, improper) singular points of equation (1.1) with the corresponding singularity of the mapping π. The dash stands in the cases if the codimension of singularity is greater than 3. In other cases we write **new** if the given type of singularity is not studied before, otherwise we give the references to related sources. It is worth observing that the left-right normal forms mentioned above are obtained by independent C^∞-diffeomorphisms (changes of variables) in the image and preimage. Therefore, they cannot be directly applied to simplification of equation (1.1), since a differential equation is connected with another group of transformations.

	fold	pleat	lips	beaks	swallowtail
$(u, v) \to (z, w)$	$z = u^2$ $w = v$	$z = u^3 + uv$ $w = v$	$z = u^3 + uv^2$ $w = v$	$z = u^3 - uv^2$ $w = v$	$z = u^4 + uv$ $w = v$
$G \neq 0$	codim $= 1$ [1,2]	codim $= 2$ [2, 9, 10]	codim $= 3$ **new**	codim $= 3$ **new**	codim $= 3$ **new**
$G = 0$	codim $= 2$ [2, 9, 10]	codim $= 3$ [7]	codim $= 4$ –	codim $= 4$ –	codim $= 4$ –

The paper presents a study of three cases marked as **new** in the table above. For convenience of the reader, we start with a brief description of results for proper singular points with the fold and pleat of π. After that, we consider proper singular points with the swallowtail, lips and beaks of π, which are not considered before. Together with [7], the obtained results cover all singularities of equation (1.1) of codimensions 1, 2, 3.

2. PRELIMINARY RESULTS

Without loss of generality, we shall assume that the considered singular point of equation (1.1) is the origin of the space $\mathbb{J}^1$. Then the condition $G(0) \neq 0$ is equaivalent to $F_x(0) \neq 0$, and the germ of $\mathscr{F}$ is the graph of a smooth function $x = f(y,p)$, where $f(0) = 0$ and $f_p(0) = 0$. Therefore, we shall further assume that $F(x,y,p) = f(y,p) - x$. By the Hadamard lemma, we have

$$F(x,y,p) = f(y,p) - x = f_0(y) + f_1(y)p + f_2(y,p)p^2 - x,$$

where f_i are smooth functions, $f_0(0) = 0$ (follows from $F(0) = 0$) and $f_1(0) = 0$ (from $F_p(0) = 0$). After the change of variables $\tilde{x} = f_0(y) - x$, we get $f_0(y) \equiv 0$, and we brought equation (1.1) to the form

$$F(x,y,p) = f(y,p) - x = \alpha p^2 + pyg(y) + p^3 h(y,p) - x = 0, \tag{2.6}$$

where g, h are smooth functions, $\alpha = F_{pp}(0)/2$ is a real number.

3. FOLD AND PLEAT

From now on, we shall deal with equation (2.6). If $\alpha \neq 0$, that is,

$$F_{pp}(0) \neq 0,$$

then the germ of π has a fold. If $\alpha = 0$, but $g(0) \neq 0$ and $h(0) \neq 0$, that is,

$$F_{pp}(0) = 0, \quad F_{ppp}(0) \neq 0, \quad F_{py}(0) \neq 0,$$

then the germ of π has a pleat (cusp). In Fig. 3.2, we present the surface $\mathscr{F}$ and the projections of integral curves of the lifted field to the (x,y)-plane for the both cases.

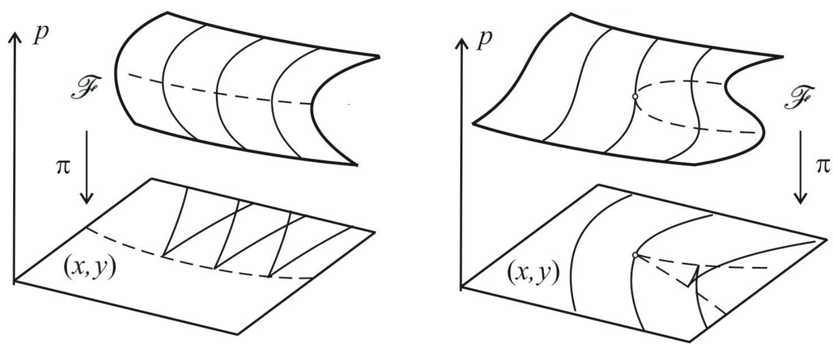

Fig. 3.2. Two singularities of the mapping $\pi : \mathscr{F} \to \mathbb{R}^2$. Fold (on the left) and pleat (on the right).

Theorem 3.1:
The germ of equation (1.1) *at every proper singular point where* π *has a fold, is equivalent to the simple equation*

$$p^2 - x = 0. \tag{3.7}$$

In a neighborhood of such singular point, the phase portrait of equation (1.1) is a family of semicubic parabolas, whose cusps fill the discriminant curve of the equations, the projection of the criminant to the (x,y)-plane; Fig. 3.2 (left). Theorem (3.1) states that an appropriate local diffeomorphism of the (x,y)-plane brings this family to the standard form $y = \frac{2}{3}x^{3/2} + \mathrm{const}$, which corresponds to equation (3.7). The proof can be found in [1].

Consider the germ of equation (2.6) at a proper singular point where π has a pleat; see Fig. 3.2 (right). Using topological arguments, in [9] it is proved that the classification of the germs of equations (2.6) with pleat has functional invariants. Moreover, functional invariants do not disappear even if changes of variables are homeomorphisms.

However, the geometric classification of possible phase portraits is quite simple. Consider integral curves of the corresponding lifted field on the surface $\mathscr{F}$, which has the form

$$\dot{y} = pf_p(y, p), \quad \dot{p} = 1 - pf_y(y, p). \tag{3.8}$$

Let us present f in the form

$$f(y, p) = apy + bp^3 + py^2 h_1(y, p) + p^4 h_2(y, p), \ \ ab \neq 0. \tag{3.9}$$

Then, substituting (3.9) into (3.8), one can see that the criminant of equation (2.6) is given by the equation $y = kp^2 + o(p^2)$ with $k = -3b/a$. In a neighborhood of the origin, the criminant is similar to the parabola $y = kp^2$, which is tangent to the direction (3.8) of zero. This point is the cusp of the discriminant curve, which has the form

$$x = f(kp^2 + o(p^2), p) = -2bp^3 + o(p^3), \ \ y = kp^2 + o(p^2).$$

This yields the following result:

Theorem 3.2:
There exist only two geometrically different types of phase portraits of the field (3.8) *corresponding to different signs of* k*. These phase portraits and their projections to the* (x, y)*-plane are presented in Fig. 3.3 (up and down, respectively).*

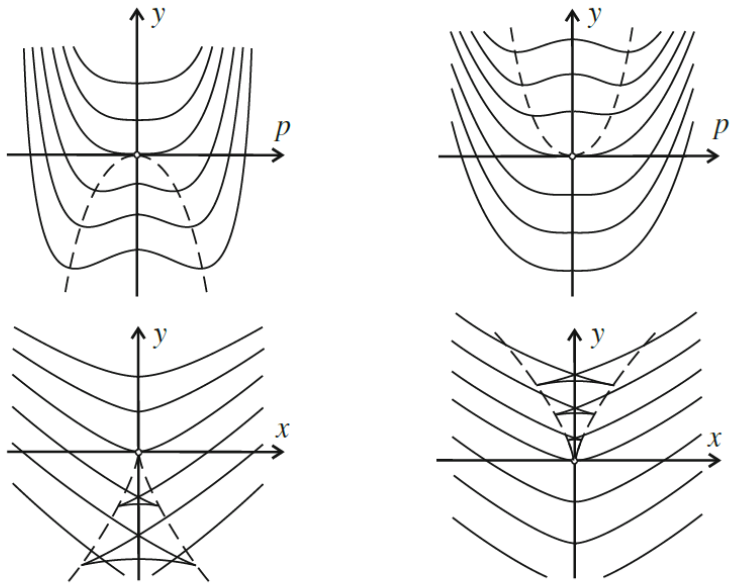

Fig. 3.3. Down: two geometrically different types of phase portraits of equation (2.6) at a proper singular point where π has a pleat. Up: the corresponding lifted fields. In the both cases, there exists a narrow domain on the (x, y)-plane, where integral curves constitute a 3-web.

4. SINGULARITIES OF CODIMENSION 3

In this section, we present the main results of the paper: a study of the local phase portraits of equation (1.1) at its proper singular points, where the mapping π has a singularity of one of three following types: swallowtail, lips, beaks.

4.1. Swallowtail

Consider the germ of equations (1.1) at its proper singular points, where the mapping π has a swallowtail (not to be confused with a surface with the same name). This means that

$$F_p(0) = F_{pp}(0) = 0, \;\; F_{py}(0) \neq 0, \;\; F_{ppp}(0) = 0, \;\; F_{pppp}(0) \neq 0. \tag{4.10}$$

Under conditions (4.10), the germ of equation (2.6) has the form

$$\begin{aligned} &F(x, y, p) = f(y, p) - x = 0, \\ &f(y, p) = apy + bp^4 + py^2h_1(y, p) + p^2yh_2(y, p) + p^5h_3(y, p), \;\; ab \neq 0. \end{aligned} \tag{4.11}$$

The surface of such equation is presented in Fig. 4.4, together with the criminant of the equation and its discriminant curve. The lifted field on $\mathscr{F}$ has the form

$$\dot{y} = pf_p(y, p), \quad \dot{p} = 1 - pf_y(y, p). \tag{4.12}$$

The above formulas show that the criminant of (4.11) has the form $y = kp^3 + o(p^3)$, where $k = -4b/a$. In a neighborhood of the origin, the criminant is similar to the cubic parabola $y = kp^3$, which is tangent to the field (4.12) at the origin. The tangency point correspond to a singular point of the discriminant curve:

$$x = f(kp^3 + o(p^3), p) = -3bp^4 + o(p^4), \;\; y = kp^3 + o(p^3). \tag{4.13}$$

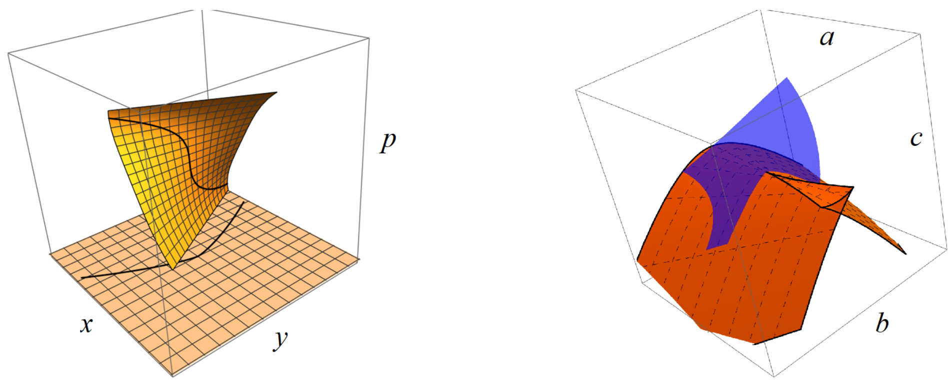

Fig. 4.4. Left: the surface $\mathscr{F}$ of equations (1.1) whose mapping π has a swallowtail; the bold curve on $\mathscr{F}$ is the criminant of the equation, and the bold curve on the plane is its discriminant curve. Right: intersection of the surface S (blue) with the surface *swallowtail* (red) in the space of the coefficients.

Lemma 4.1:
Locally, the discriminant curve of equation (4.11) *splits the* (x, y)*-plane into two open domains, in which equation* (4.11) *has two real roots* p *or zero real roots* p*, respectively. At point of the discriminant curve (except the origin) it has one double real root* p*, while at the origin the multiplicity of* p *is equal to 4.*

Proof
By the division theorem [22, 23], the germ of equation (2.6) satisfying conditions (4.10) is equivalent to

$$\widetilde{F}(x, y, p) = p^4 + \sum_{i=0}^{3} a_i(x, y)p^i = 0, \;\; p = dy/dx, \tag{4.14}$$

where $a_i(x, y)$ are smooth functions, $a_i(0) = 0$, and da_0, da_1 at zero are linearly independent. Using an appropriate change of variables (x, y), one can kill the monomial p^3, that is, bring

(4.14) to the form

$$p^4 + a(x,y)p^2 + b(x,y)p + c(x,y) = 0, \;\; p = dy/dx, \tag{4.15}$$

with smooth functions $a, b, c,\ a(0) = b(0) = c(0)$. Form the independence of $da_0,\ da_1$ in (4.14) it follows the independence of db, dc in (4.15).

Then the statement of the lemma can be interpreted in the following way. In the space with cartesian coordinates a, b, c consider the discriminant surface of the polynomial

$$M(p) = p^4 + ap^2 + bp + c, \tag{4.16}$$

that is, a surface that consists of points (a, b, c) that the polynomial $M(p)$ has multiple roots. This surface W defined by the equations $M(p) = M'(p) = 0$ is called the swallowtail, it is presented in Fig. 4.4 (right) in red. It separates the whole space into three open domains U_4, U_2, U_0, where $M(p)$ has 4, 2, 0 real roots, respectively.

In the (a, b, c)-space equation (4.15) defines a surface S

$$a = a(x,y), \;\; b = b(x,y), \;\; b = c(x,y), \tag{4.17}$$

it is presented in Fig. 4.4 (right) in blue. Since the differentials of the functions b, c at zero are linearly independent, in a neighborhood of the origin the surface S is regular and it intersects the swallowtail W as presented in Fig. 4.4 (right). This proves all statements of the lemma. □

Theorem 4.1:
There exist only two geometrically different types of local phase portraits of equation (4.11) *corresponding to different signs of* k*, which is equivalent to those of* ab*. These phase portraits are presented in Fig. 4.5.*

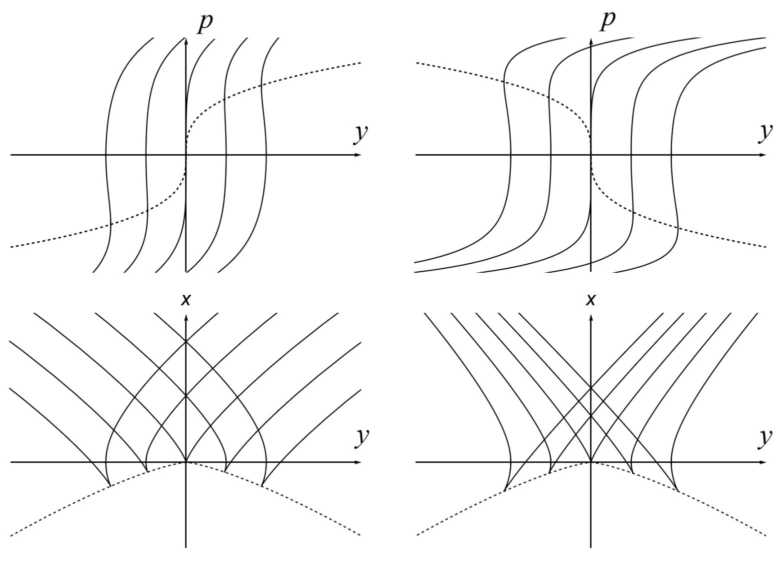

Fig. 4.5. Down: two different types of local phase portraits of equation (1.1), whose mapping π has a swallowtail, which correspond to different signs of ab in (4.11). Up: the corresponding phase portraits of the lifted field. The dotted lines present the criminant of the equation (up) and its discriminant curve (down).

Proof
Consider local phase portraits of the lifted field (4.12) for equation (4.11). In a neighborhood of the origin integral curves of the field (4.12) are similar to straight lines parallel to the p-axis, and the criminant is similar to the cubic parabola $y = kp^3$. This proves the statement of the theorem. □

4.2. *Lips and Beaks*

Consider the germ of equations (1.1) at its proper singular points, where the mapping π has a singularity of the type *lips* or *beaks*. Then, after an appropriate linear change $y \mapsto \alpha y$ equation (2.6) can be brought to the form

$$\begin{aligned} F(x,y,p) &= f(y,p) - x = 0, \\ f(y,p) &= p^3 + h_1(y,p)yp^2 + h_2(y,p)y^2p + h_3(y,p)p^4, \end{aligned} \tag{4.18}$$

where the Hessian of the function $f_p(y,p)$ at the origin is not zero:

$$H[f_p](0) = 4(3h_2(0) - h_1^2(0)) \neq 0. \tag{4.19}$$

The mapping π has a singularity of the type *lips* (respectively, *beaks*) if $H[f_p] > 0$ (respectively, $H[f_p] < 0$). The corresponding surfaces of the equation (1.1) are presented in Fig. 4.6.

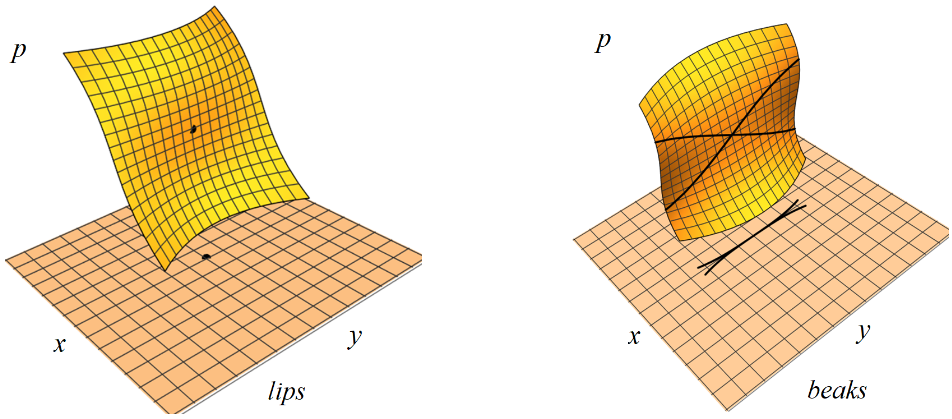

Fig. 4.6. The surface $\mathscr{F}$ of equations (1.1) whose mapping π has a singularity of the type *lips* (left) or *beaks* (right). The bold curve on $\mathscr{F}$ is the criminant and the bold curve on the plane is its discriminant curve.

By the division theorem, the germ of the function F from (4.18) has the form

$$F(x,y,p) = \Phi(x,y,p)\big(p^3 + a(x,y)p^2 + b(x,y)p + c(x,y)\big),$$

where a, b, c, Φ are smooth functions, $a(0) = b(0) = c(0) = 0$, and $\Phi(0) \neq 0$. Therefore, the germ of equation (4.18) is equivalent to

$$p^3 + a(x,y)p^2 + b(x,y)p + c(x,y) = 0, \quad p = dy/dx. \tag{4.20}$$

Lemma 4.2:
In (4.20), the coefficient c has the form $c(x,y) = x\tilde{c}(x,y)$ with a smooth function $\tilde{c}$. The condition $G \neq 0$ for equation (1.1) is equivalent to $\tilde{c}(0,0) \neq 0$.

Proof
Formula (4.18) shows that the surface $\mathscr{F}$ given by the equation $x = f(y,p)$ contains the y-axis ($x = p = 0$). This means that the substitution $x = p = 0$ into (4.20) gives a certain identity by y. This yields $c(x,y)\big|_{x=0} \equiv 0$, and the statement follows from the Hadamard lemma. □

Lemma 4.3:
In (4.20), the equality $b_y(0) = 0$ holds true.

Proof
Substitution $x = f(y,p)$ with the function f from (4.18) brought equation (4.20) to a certain identity by y, p. The left hand side of this identity contains the monomial $b_y(0)yp$, while the right hand side is identically zero. This proves the statement. $\square$

Lemma 4.4:
Hessian of the function f_p at zero is equal to $H[f_p](0) = 4\Phi^2(0)\Delta$, where

$$\Delta = \tfrac{3}{2}b_{yy}(0) - a_y^2(0),$$

whence the type of singularity of π is determined by the sign of Δ: lips if $\Delta > 0$ and beaks if $\Delta < 0$.

Proof
From (4.18) we have:

$$\begin{aligned} f_p(y,p) = F_p(x,y,p) = \Phi_p(x,y,p)(p^3 + a(x,y)p^2 + b(x,y)p + c(x,y)) + \\ + \Phi(x,y,p)(3p^2 + 2a(x,y)p + b(x,y)), \end{aligned}$$

where x is replaced with $f(y,p)$. Let us calculate the quadratic term of the obtained function on y, p. From Lemmas 4.2 and 4.3 it follows that the 2-jet of the function

$$p^3 + a(f(y,p),y)p^2 + b(f(y,p),y)p + c(f(y,p),y)$$

is zero and the 1-jet of

$$3p^2 + 2a(f(y,p),y)p + b(f(y,p),y) \tag{4.21}$$

is zero as well. Therefore, the quadratic part of the germ $f_p(y,p)$ coincides with the quadratic part of (4.21) multiplied by $\Phi(0)$. Taking into account 4.3, a simple calculation shows that the quadratic part of (4.21) is

$$3p^2 + 2a_y(0)py + \tfrac{1}{2}b_{yy}y^3.$$

This yields the statement of the lemma. $\square$

Theorem 4.2:
The local phase portrait of equation (1.1) *at proper singular point whose mapping* π *has a singularity lips is presented in Fig. 4.7. Here the discriminant set of the equation is a single point, and integral curves are regular at all points except for this point.*

Theorem 4.3:
The local phase portrait of equation (1.1) *at proper singular point whose mapping* π *has a singularity lips is presented in Fig. 4.8. Here the discriminant set of the equation is a pair of two curves tangent at a single point, which separate the* (x,y)*-plane into four open domains with different stricture of integral curves: 1-webs or 3-webs.*

Proof
To prove Theorems 4.2 and 4.3, consider the left hand side of equation (4.20) as a polynomial on the variable p with coefficients smoothly depending on x, y:

$$M(p) = p^3 + a(x,y)p^2 + b(x,y)p + c(x,y). \tag{4.22}$$

In the space with cartesian coordinates a, b, c consider the discriminant surface of the polynomial $M(p) = p^3 + ap^2 + bp + c$, that is, a surface that consists of points (a,b,c) that the polynomial $M(p)$ has multiple roots. This surface W defined by the equations $M(p) = M'(p) = 0$ is called the cuspidal edge, it is presented in Fig. 4.9 in orange. It

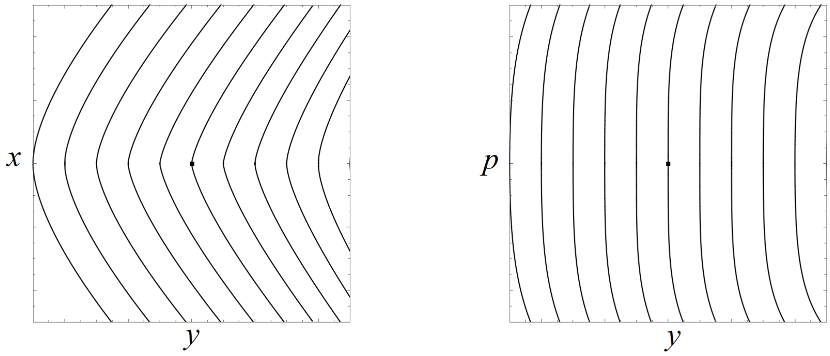

Fig. 4.7. Left: local phase portrait of equation (1.1) whose mapping π has *lips*. Right: the corresponding phase portrait of the lifted field. Here the criminant and the discriminant set are isolated points.

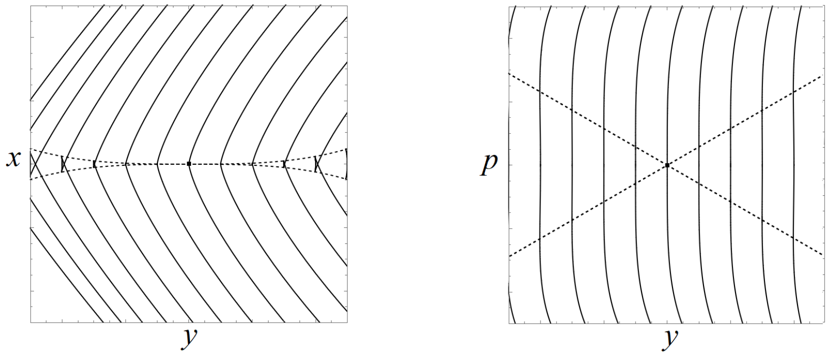

Fig. 4.8. Left: local phase portrait of equation (1.1) whose mapping π has *beaks*. Right: the corresponding phase portrait of the lifted field. The criminant and the discriminant set of the equation are depicted with dotted lines.

separates the whole space into two open domains U_1, U_3, where $M(p)$ has 1, 3 real roots, respectively. The domain U_3 is more narrow.

In the (a, b, c)-space equation (4.15) defines a surface S defined by (4.17). The proof of Theorems 4.2, 4.3 is based on the intersection of the surfaces S and W. Using lemmas 4.2 – 4.4, it is not hard to show that the mutual position of S and W is exactly as it is presented in Fig. 4.9 for lips and beaks. Here the surfaces S and W are depicted in blue and orange, respectively. In the case $\Delta < 0$ (lips) the surface S belongs to the open domain U_1 except for a point, which corresponds to the unique singular point of the mapping π. In the case $\Delta > 0$ (beaks) the surface S is cutted by W into four parts, two of which belong to U_1 and two belong to U_3. This proves the theorems. $\square$

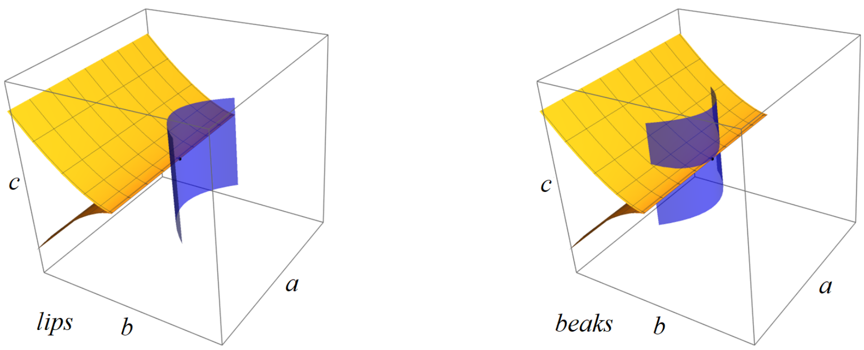

Fig. 4.9. The mutual position of surfaces W and S for singularities *lips* (left) and *beaks* (right).

5. BIFURCATIONS ON ONE-PARAMETER FAMILIES

5.1. Bifurcations of the swallowtail

Consider a generic one-parameter perturbation of equation (4.11):

$$p^4 + a(x,y,\mu)p^2 + b(x,y,\mu)p + c(x,y,\mu) = 0, \;\; p = dy/dx, \tag{5.23}$$

where μ is a small parameter. In the space with cartesian coordinates a, b, c consider the family of surfaces S_μ defined by the coefficients $a(x,y,\mu)$, $b(x,y,\mu)$, $c(x,y,\mu)$ of equation (5.23). In the same space consider the discriminant surface W of the polynomial $M(p) = p^4 + ap^2 + bp + c$, which is the swallowtail. The bifurcation of the phase portrait of equation (5.23) is determined by the intersection of the family S_μ with W.

Assume that

$$\frac{\partial a}{\partial \mu}(0,0,0) \neq 0. \tag{5.24}$$

Then the function $a(0,0,\mu)$ changes its sign when the parameters μ passes through zero.

Without loss of generality we shall further assume that the sign of the function $a(0,0,\mu)$ coincides with the sign of μ. Then the mutual position of S_μ and W is presented in Fig. 5.10, and the corresponding bifurcation of the phase portrait of equation (5.23) is presented in Fig. 5.11. It is worth observing that for all $\mu < 0$ the exists a narrow domain on the (x,y)-plane where integral curves constitute a 4-web. As $\mu \to -0$ this domain tends to a point, and for $\mu \geq 0$ integral curves have no quadruple intersections.

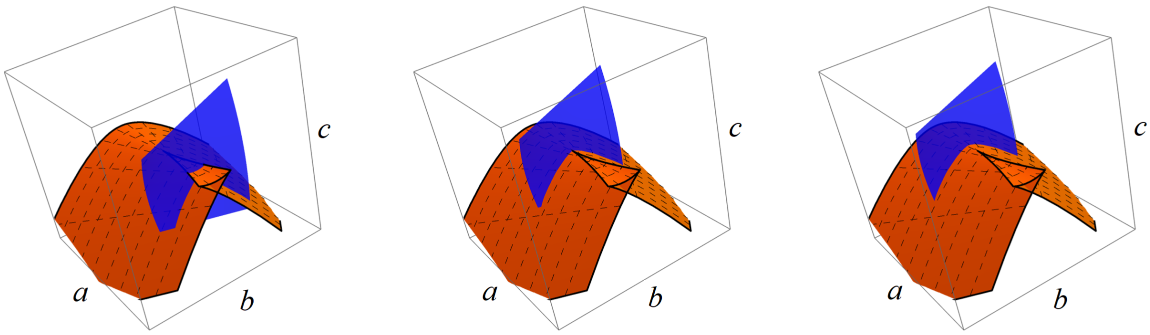

Fig. 5.10. The surfaces S_μ and W for a generic family (5.23) depicted in blue and red respectively. From the left to right: $\mu < 0$, $\mu = 0$, $\mu > 0$.

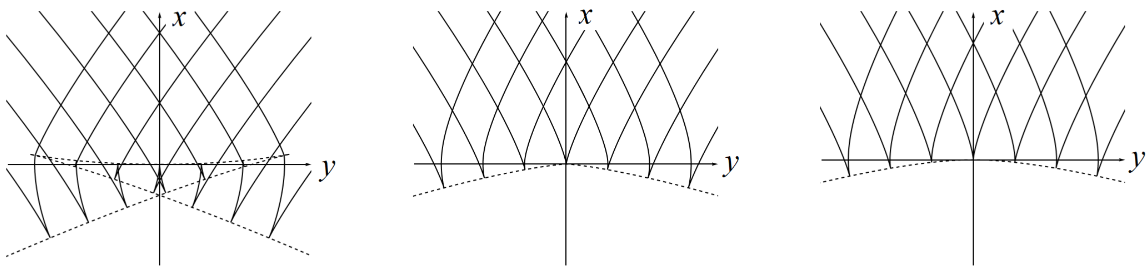

Fig. 5.11. Bifurcation of a generic family (5.23): the phase portrait on the (x,y)-plane, the dotted line is the discriminant curve. From the left to right: $\mu < 0$, $\mu = 0$, $\mu > 0$.

5.2. *Bifurcations of lips and beaks*

Consider a generic one-parameter perturbation of equation (4.18) with singularity of the type lips or beaks:

$$F(x, y, p) = p^3 + a(x, y)p^2 + b(x, y)p + c(x, y), \;\; p = dy/dx, \tag{5.25}$$

where μ is a small parameter. In the space with cartesian coordinates a, b, c consider the family of surfaces S_μ defined by the coefficients $a(x, y, \mu)$, $b(x, y, \mu)$, $c(x, y, \mu)$ of equation (5.25). In the same space consider the discriminant surface W of the polynomial $M(p) = p^3 + ap^2 + bp + c$, which is the cuspidal edge. The bifurcation of the phase portrait of equation (5.25) is determined by the intersection of the family S_μ with W.

Assume that

$$\frac{\partial b}{\partial \mu}(0, 0, 0) \neq 0,$$

Then the function $b(0, 0, \mu)$ changes its sign when the parameters μ passes through zero. Without loss of generality we shall further assume that the sign of the function $b(0, 0, \mu)$ coincides with the sign of μ.

The mutual position of S_μ and W for a generic family (5.25) with singularity of the type *lips* is presented in Fig. 5.12. The corresponding bifurcation of the phase portrait is presented in Fig. 5.13. For all $\mu > 0$ there exists a narrow bounded domain on the (x, y)-play (which looks like lips), where integral curves constitute a 3-web. As $\mu \to +0$ this domain tends to a point, and for $\mu \geq 0$ integral curves have no triple intersections.

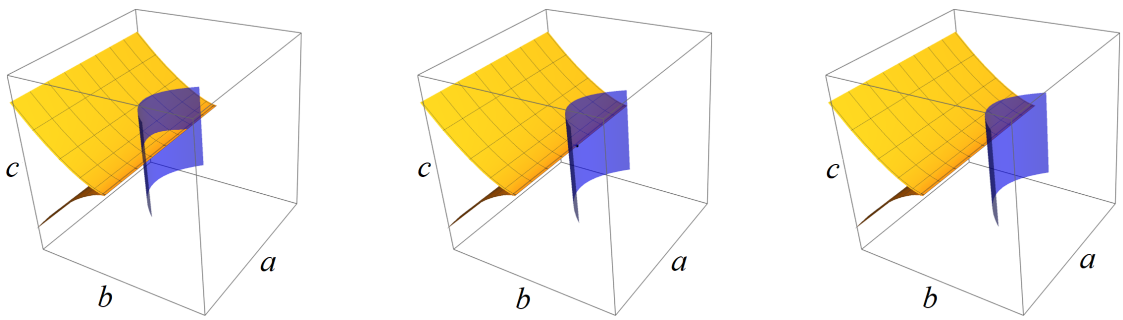

Fig. 5.12. The surfaces S_μ and W for a generic family (5.25) with lips depicted in blue and orange respectively. From the left to right: $\mu > 0$, $\mu = 0$, $\mu < 0$.

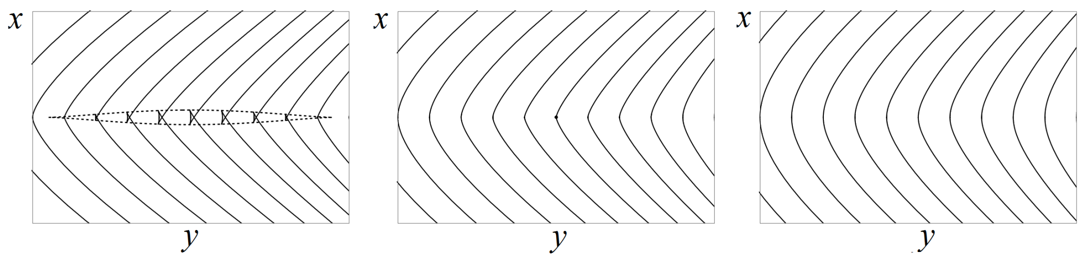

Fig. 5.13. Bifurcation of a generic family (5.25) with lips. From the left to right: $\mu > 0$, $\mu = 0$, $\mu < 0$. The closed bold curve in the left image is the discriminant set, which degenerates into a point when $\mu = 0$.

The mutual position of S_μ and W for a generic family (5.25) with singularity of the type *beaks* is presented in Fig. 5.14. The corresponding bifurcation of the phase portrait is presented in Fig. 5.15.

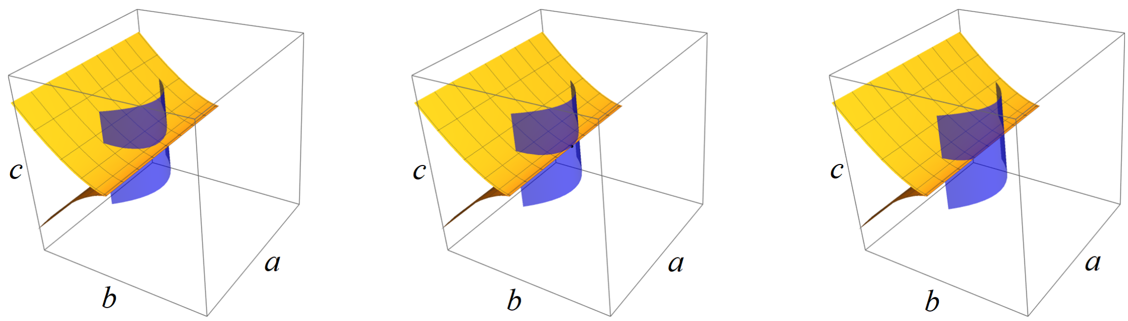

Fig. 5.14. The surfaces S_μ and W for a generic family (5.25) with beaks depicted in blue and orange respectively. From the left to right: $\mu > 0$, $\mu = 0$, $\mu < 0$.

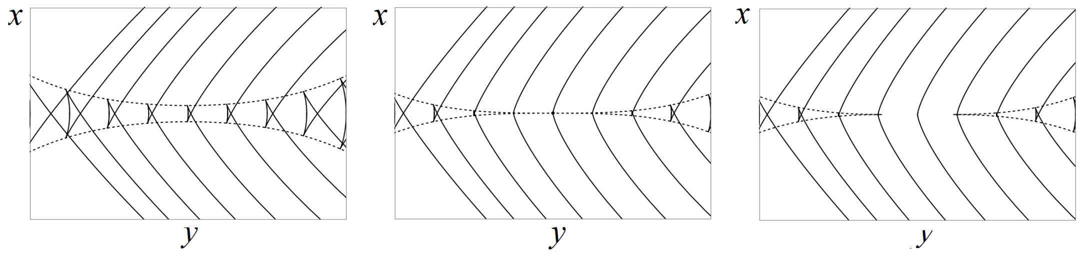

Fig. 5.15. Bifurcation of a generic family (5.25) with beaks. From the left to right: $\mu > 0$, $\mu = 0$, $\mu < 0$. The dotted curve is the discriminant set.

6. CONCLUSION

In this paper, we studied local phase portraits of first-order implicit differential equations, that is, equations in the form $F(x, y, y') = 0$, in a neighborhood of their proper singular points of codimension 3. This study, together with previous works, completes qualitative investigation of singularities of such equations up to codimension— 3. Moreover, we investigated generic bifurcations in one-parameter families of such equations.

It is worth observing that an interest to such equations is motivated by their applications in various mathematical and technical problems; see, for example, [6] and the lists of references in [11, 14].

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