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Identifiability of a Family of Dynamical Systems: Application to Crops Identification

Abdes Bernoussi[1*], Edyta Woźniak[2], Abdelaâziz Belfekih[1]

[1] *Faculty of Sciences and Techniques Tangier, University Abdelmalek Essaadi, Tangier, Morocco*
[2] *Centrum Badań Kosmicznych PAN, Warsaw, Poland*

Abstract: In this paper we consider the problem of identifying a system among a family of given systems. Thus, from measurements collected on an unidentified system but that is part of a family of known model systems, we seek to determine this unidentified system. This differs from identifying the parameters of a given system through experimental observations [15]. The determination (identification) in a given family not always being possible, we refer to the identifiable family as any family for which this identification is possible. We thus introduce the concept of identifiability of a family of systems through a given measurement function. For localized linear systems we give algebraic characterizations that use the notion of system observability. We then propose algorithms which, in case of identifiability of the family and by a process of elimination, identify the system to which the collected measurements correspond. We have given some examples to illustrate these algorithms. We have also added an exemplified extension to discrete localized systems.

Keywords: dynamical systems, identifiability, observability, remote sensing

1. INTRODUCTION

The modeling of a given phenomenon consists in describing it mathematically by the equations that describe its dynamics. Moreover, in the model we have to take into account various input-output parameters that allow interaction with the system. For the output, the information are obtained via a measurement function. This function actually models the sensors installed on this system and provides the collected measurements.

In general, the purpose of collecting measurements through these means of observation is to estimate, when possible, the state of the system which evolves over time, which subsequently allows corrections or adjustments of actions that one might have to exert on the system. So knowing the model of the dynamics of the system and the measurement function, the observability concept consists in the possibility to reconstruct the state of the system and even predict its future evolution and this in order to control it to achieve predetermined goals. The observability concept was introduced and characterized for localized systems by Kalman [11, 12]. Due to the interest of this concept, many works were devoted to it in many aspects for numerous applications [4, 19]. It was later generalized to distributed systems [2, 8, 18]. Different degrees of observability were then introduced depending on the considered problem: ”total” observability ([10]), partial observability ([6]) and regional observability ([1, 7]). The global state of the system can be determined by measurements in the case of total observability; the ”visible” part of this state can be determined in the case

*Corresponding author: abernoussi@uae.ac.ma

of partial observability and the trace of the state of the distributed system over an observable region can be determined in the case of regional observability.

The problem that we are considering in this paper differs both in the data of the problem and in the goal to be achieved. We assume that we have a family of models and a measurement function, then the problem is the following: can we determine, from the collected measurements, the model of our system knowing that it is in the given family? If we manage to determine it, we will be able to do everything that could be done on a system of which we have both the model and the measurements.

As it is not always possible to identify the system (or, more precisely, the model) that can generate the measurements among any family of systems (see an example below), we introduce the mathematical concept of identifiability of a family of systems: this is the ability to differentiate the systems of this family from one another by a given measurement function, which then makes it possible to identify exactly which of them corresponds to the phenomenon under study. We define this concept for dynamic systems governed by equations of the "Cauchy problem" type in a fairly general way.

We will then distinguish two levels of identifiability: the first level is the identifiability relative to a given function of measurement g, which we will refer to as the g identifiability (or the C identifiability for linear systems); while the second level is absolute identifiability which corresponds to the existence of a measurement function g which ensures g identifiability. It is immediate to see that the first level, which relates to a measurement function, can only be achieved if the second level is achieved, which is an intrinsic property of the family of systems independently of the measurement function. We noted that the observability was generally characterized by a rank condition of a given matrix. The computation of such rank depends on the particularity of the matrices and for large matrices many numerical methods are used [9, 14].

The study of the identifiability of a family of localized linear systems, both continuous (in time) and discrete, reveals an abstract system and uses its "observability".

The concept of identifiability, introduced in this paper, and observability one differ in their philosophies. While observability is the possibility for a measurement function that is associated with a system to reconstruct the state of that system from collected measurements, identifiability is the possibility to differentiate a group of systems from one another using a set of measurements, and to identify the system that generated those measurements. This study is motivated by several real issues. Let's quote two:

- In agriculture, can we identify the crop growing on a given plot of land from measurements collected with satellite images? This is the problem of identifying a crop by comparing its signature with the signatures of a number of crops.
- Among a certain number of physical systems, only one of these systems emits (or has emitted) a signal; can we know which of these systems is sending this signal?

This paper is organized as follow: In a first part we define the identifiability of a family of systems and we give a characterization for the linear localized systems by a rank condition, similar to the one given for the observability or the controllability of the linear systems. We then propose algorithms that test the identifiability of a family of systems, and other algorithms which determine, for an identifiable family, the evolving system that has generated the collected measurements. These algorithms are based on an elimination process. Numerical simulations and an application are provided to illustrate our approach by applying the developed algorithms. We then propose algorithms that test the identifiability of a family of systems, and other algorithms that determine, for an identifiable family, the evolving system that has generated the collected measurements. These algorithms are based on a process of elimination. Numerical simulations and an application are provided to illustrate our approach by applying the developed algorithms.

2. IDENTIFIABILITY OF A FAMILY OF SYSTEMS: INTRODUCTION OF THE CONCEPT

In this section we introduce the concept of identifiability of a family of two, and more then many systems and this for systems that are not necessarily linear. Next, we give some characteristics of the general case before dedicating ourselves to finite dimensional linear systems in the following sections.

Consider two systems (S_1) and (S_2) governed by the following state equations $(t > 0)$:

$$(S_1)\begin{cases} \dot{z}\left(t\right) = f_1\left(z\left(t\right)\right) \\ \\ z\left(0\right) = z_0^1 \in Z \end{cases} \qquad (S_2)\begin{cases} \dot{z}\left(t\right) = f_2\left(z\left(t\right)\right) \\ \\ z\left(0\right) = z_0^2 \in Z \end{cases}$$

where $f_i : Z \to Z$ and Z an Hilbert space (states space). We assume that for each $z_0^1 \in Z$, the first equation admits a unique solution $z^1\left(t\right), t \geqslant 0$, and that for each $z_0^2 \in Z$, the second equation admits a unique solution $z^2\left(t\right), t \geqslant 0$.

Consider on the other hand an output function

$$y\left(t\right) = g\left(z\left(t\right)\right) \quad , \quad t \in [0, T] \tag{2.1}$$

where $g : Z \to Y$ and Y an Hilbert space (observations space). $z\left(t\right)$ in (2.1) is one of the states of these systems (S_1) or (S_2), but we do not know which one. Actually, we have one of two statements:

$$y\left(t\right) = g\left(z^1\left(t\right)\right) \quad , \quad t \in [0, T] \qquad \text{or} \qquad y\left(t\right) = g\left(z^2\left(t\right)\right) \quad , \quad t \in [0, T]$$

To be able to identify the system that has generated the collected measurements, it is necessary that through the output $y(t)$, the systems (S_1) and (S_2) give different signals. Moreover, it is necessary that this difference appears for any state of (S_1) and any state of (S_2) because when they have the same state then their outputs coincide and in this case the origin of the measurements cannot be detected.

These conditions allow us to define the *identifiable families* concept to which we give the following definition:

Definition 2.1:
(i) *we say that the family of the two systems $\mathscr{F} = \{(S_1), (S_2)\}$ is identifiable on $[0, T]$ by the observation (2.1), or $g-$identifiable on $[0, T]$, if the obtained measurements of (S_1) are different from the obtained measurements of (S_2) and this for all possible states of (S_1) and all possible states of (S_2), provided that at least one of the states is non-zero:*

$$\begin{array}{c} \forall z^1\left(.\right) \text{ state of } (S_1) \text{ and } \forall z^2\left(.\right) \text{ state of } (S_2) \text{ checking} \\ z^1\left(.\right) \neq 0 \text{ or } z^2\left(.\right) \neq 0 \text{ on } [0, T], \text{ we have } g\left(z^1\left(.\right)\right) \neq g\left(z^2\left(.\right)\right) \text{ on } [0, T] \end{array} \tag{2.2}$$

(ii) *We say that the family of the two systems $\mathscr{F} = \{(S_1), (S_2)\}$ is identifiable on $[0, T]$, if there exist a space Y and an application $g : Z \to Y$ such as the family $\mathscr{F}$ i.e. $g-$identifiable on $[0, T]$.*

Remark 2.1:
According to (2.2), the family $\mathscr{F} = \{(S_1), (S_2)\}$ is not $g-$identifiable if there exist a state $z^1\left(t\right)$ of the system (S_1) and a state $z^2\left(t\right)$ of the system (S_2), non-zero, such that $g\left(z^1(t)\right) = g\left(z^2(t)\right), \forall t \in [0, T]$. In this case we can have the same measurement for one or more states of (S_1) and one or more states of (S_2).

We have an important particular case which consists in taking $Y = Z$ and $g = Id$; the observation in this case is $y\left(t\right) = z\left(t\right), t \in [0, T]$, which corresponds to direct and complete access to the state of the system. In this case we speak of $Id-$identifiable family whose definition is as follows

$$\begin{array}{c}\forall z^{1}(.) \text{ state of } (S_1) \text{ and } \forall z^{2}(.) \text{ state of } (S_2) \text{ satisfying} \\ z^{1}(.) \neq 0 \text{ or } z^{2}(.) \neq 0 \text{ on } [0,T] \text{ we have } z^{1}(.) \neq z^{2}(.) \text{ on } [0,T]\end{array} \tag{2.3}$$

We then have the following proposition:

Proposition 2.1:
The family $\mathscr{F} = \{(S_1), (S_2)\}$ is identifiable on $[0,T]$ if, and only if it is $Id-$identifiable on $[0,T]$.

Proof

- It is immediate to see that if $\mathscr{F}$ is $Id-$ identifiable on $[0,T]$, then it is identifiable on $[0,T]$ since it is $g-$identifiable with $Y = Z$ and $g = Id$.
- Conversely if $\mathscr{F}$ is identifiable on $[0,T]$, then there exists Y and $g : Z \to Y$ such that $\mathscr{F}$ be $g-$identifiable on $[0,T]$. For a state $z^{1}(.)$ of (S_1) and a state $z^{2}(.)$ of (S_2), with $z^{1}(.) \neq 0$ or $z^{2}(.) \neq 0$ on $[0,T]$, we have $g(z^{1}(.)) \neq g(z^{2}(.))$ on $[0,T]$. We deduce that $z^{1}(.) \neq z^{2}(.)$ on $[0,T]$ because $z^{1}(.) = z^{2}(.)$ on $[0,T]$ would give $g(z^{1}(.)) = g(z^{2}(.))$ on $[0,T]$. This proves that $\mathscr{F}$ is $Id-$identifiable on $[0,T]$.

□

Remark 2.2:
It is therefore sufficient, to know if $\mathscr{F} = \{(S_1), (S_2)\}$ is identifiable on $[0,T]$ to check (2.3) *(if it is $Id-$identifiable on $[0,T]$). However, since in practice the measurement function is given, it is necessary to test if the family is $C-$identifiable.*

This concept can be generalised to several systems. Consider a family $\mathscr{F} = \{(S_1), \ldots, (S_N)\}$ of N systems and suppose that each system (S_i) is governed by the state equation

$$(S_i) \quad \begin{cases} \dot{z}(t) = f_i(z(t)) \quad , \qquad t > 0 \\ z(0) = z_0^i \in Z \end{cases}$$

with $f_i : Z \to Z$. We also assume that for each $z_0^i \in Z$, this equation admits a unique solution denoted by $z^i(.)$. The output equation is always given by (2.1), we consider the following definitions which generalize the definition 2.1.

Definition 2.2:
(i) *We say that the family of systems $\mathscr{F} = \{(S_1), \ldots, (S_N)\}$ is $g-$identifiable on $[0,T]$ if every subfamily $\{(S_i), (S_j)\}$, $i \neq j$, is $g-$identifiable on $[0,T]$.*
(ii) *We say that the family of systems $\mathscr{F} = \{(S_1), \ldots, (S_N)\}$ is identifiable on $[0,T]$ if every subfamily $\{(S_i), (S_j)\}$ is identifiable on $[0,T]$.*

As in the case of two systems, we will say that the family $\{(S_1), \ldots, (S_N)\}$ is $Id-$identifiable on $[0,T]$ if each subfamily $\{(S_i), (S_j)\}$ is $Id-$identifiable on $[0,T]$.

Proposition 2.2:
The family of systems $\mathscr{F} = \{(S_1), \ldots, (S_N)\}$ is identifiable on $[0,T]$ if and only if it is $Id-$identifiable on $[0,T]$.

Proof
The family $\mathscr{F}$ is identifiable on $[0,T]$ if and only if each subfamily $\{(S_i), (S_j)\}$, $i \neq j$, is identifiable on $[0,T]$, or if and only if, thanks to the proposition 2.1, each $\{(S_i), (S_j)\}$, $i \neq j$, is $Id-$identifiable on $[0,T]$, which is equivalent to what the family $\mathscr{F}$ i.e. $Id-$identifiable on $[0,T]$. □

3. IDENTIFIABILITY OF FAMILIES OF FINITE DIMENSIONAL LINEAR SYSTEMS

Consider a family of two systems whose dynamics are governed by the two following linear state equations:

$$(S_1)\begin{cases} \dot{z}(t) = A_1 z(t) \\ z(0) = z_0^1 \in \mathbb{R}^n \end{cases} \quad , \quad (S_2)\begin{cases} \dot{z}(t) = A_2 z(t) \\ z(0) = z_0^2 \in \mathbb{R}^n \end{cases}$$

with A_1 and A_2 are two square matrices of order n. The state of (S_1), noted by $z^1(.)$, is given by $z^1(t) = \exp(tA_1)\, z_0^1$ and the state of (S_2), noted by $z^2(.)$, is given by $z^2(t) = \exp(tA_2)\, z_0^2$. The system is augmented by the following linear measurement function

$$(E) \qquad y(t) = Cz(t) \quad , \quad 0 \leqslant t \leqslant T$$

with C a matrix with q rows and n columns. $z(t)$ is either $z^1(t)$ or $z^2(t)$; So we have $y(t) = Cz^1(t), t \in [0,T]$, or $y(t) = Cz^2(t), t \in [0,T]$.

Remark 3.1:
The matrices A_1 and A_2 are necessarily of the same order n since we use the same output (and therefore the same matrix C) to measure each of the two systems.

Note that the output (E) is of the form (2.1) with the choice $g(u) = Cu$, $u \in \mathbb{R}^n$; hence, and in all that follows, the families $g-$identifiable will be called $C-$*identifiable* and families $Id-$identifiable will be called I_n-*identifiable.*

Let us go back to the definition 2.1. The characterisation (2.2) takes the form

$$\begin{gathered}\forall z^1(.) \text{ state of } (S_1) \text{ and } \forall z^2(.) \text{ state of } (S_2) \text{ satisfying} \\ z^1(.) \neq 0 \text{ or } z^2(.) \neq 0 \text{ on } [0,T] \text{ we have } Cz^1(.) \neq Cz^2(.) \text{ on } [0,T]\end{gathered}$$

By contraposition, and denoting $z_0^k = z^k(0)$ the initial state of the system (S_k), we obtain the characterisation of the $C-$*identifiability* of a family of two systems as given in the following proposition:

Proposition 3.1:
(i) *The family $\mathscr{F} = \{(S_1), (S_2)\}$ is $C-$identifiable on $[0,T]$ if and only if the only states of the two systems which give the same measurements are the null states (generated by zero initial conditions):*

$$\left.\begin{array}{c} z^1(.) \textit{ state of } (S_1),\ z^2(.) \textit{ state of } (S_2) \\ \textit{and } Cz^1(.) = Cz^2(.) \textit{ on } [0,T] \end{array}\right\} \Rightarrow \quad z_0^1 = z_0^2 = 0 \tag{3.4}$$

(ii) *The family $\mathscr{F} = \{(S_1), (S_2)\}$ is identifiable on $[0,T]$ if, and only if, there exists an integer $q \geqslant 1$ and a matrix $C \in \mathscr{M}_{q,n}(\mathbb{R})$ such that $\mathscr{F}$ is $C-$identifiable on $[0,T]$.*
(iii) *$\mathscr{F} = \{(S_1), (S_2)\}$ is identifiable on $[0,T]$ if, and only if, it is I_n-identifiable:*

$$\left.\begin{array}{c} z^1(.) \textit{ state of } (S_1)\ z^2(.) \textit{ state of } (S_2) \\ \textit{and } z^1(.) = z^2(.) \textit{ on } [0,T] \end{array}\right\} \Rightarrow \quad z_0^1 = z_0^2 = 0$$

This proposition which characterises $C-$identifiable (resp. identifiable) families of two linear systems can be easily generalised to families of several linear systems.

Example 3.1:
(i) Consider the two systems $\{(S_1), (S_2)\}$ given by the following state equations:

$$(S_1)\begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \quad , \quad (S_2)\begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}$$

with the measurement function $y(t) = z_1(t) + z_2(t)$, $0 \leqslant t \leqslant T$. *This corresponds to* $C = \begin{pmatrix} 1 & 1 \end{pmatrix}$. *For* $z_0^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $z_0^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ *the respective states are* $z^1(t) = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$ *and* $z^2(t) = \begin{pmatrix} 0 \\ e^t \end{pmatrix}$ *which give* $Cz^1(t) - Cz^2(t) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} e^t \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ e^t \end{pmatrix} = 0$, $\forall t \geqslant 0$*; the family* $\{(S_1), (S_2)\}$ *is therefore not* $C-$*identifiable.*

(ii) Consider now the two systems $\{(S_1), (S_2')\}$ *given by the following state equations:*

$$(S_1) \begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \text{ and } (S_2') \begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}$$

with the same measurement function $y(t) = z_1(t) + z_2(t)$, $0 \leqslant t \leqslant T$. *For* $z_0^1 = \begin{pmatrix} a \\ b \end{pmatrix}$ *and* $z_0^2 = \begin{pmatrix} c \\ d \end{pmatrix}$ *the respective states are* $z^1(t) = \begin{pmatrix} ae^t \\ be^{2t} \end{pmatrix}$ *and* $z^2(t) = \begin{pmatrix} ce^{3t} \\ de^{4t} \end{pmatrix}$ *which give* $Cz^1(t) - Cz^2(t) = ae^t + be^{2t} - ce^{3t} - de^{4t}$. *The family of functions* $\{e^t, e^{2t}, e^{3t}, e^{4t}\}$ *is free, equality* $ae^t + be^{2t} - ce^{3t} - de^{4t} = 0$, $t \in]0, T[$, *gives* $a = b = c = d = 0$ *or* $z_0^1 = z_0^2 = 0$, *which proves that the family* $\{(S_1), (S_2)\}$ *is* $C-$*identifiable.*

Before characterising the C-identifiable families, we will recall in the following section the definition of the observable system as well as its algebraic characterisation. Then we will, in this case, give the expression of the state from the collected measurements. This will be useful for us to characterize algebraically the $C-$identifiable families.

3.1. Observable System and State Reconstruction

The observability concept of a localised system reflects the ability of an output to reconstruct the state of the system in a "faithful"manner. Let us now the definition of observable system.

Consider a linear system whose dynamics is described by the following state equation

$$(S) \quad \begin{cases} \dot{z}(t) = Az(t) & , \quad t > 0 \\ z(0) = z_0 \in Z = \mathbb{R}^n \end{cases}$$

where A is a square matrix of order n representing the dynamics of the system, and consider the output function given by the following equation:

$$y(t) = Cz(t) \quad , \quad t \in [0, T] \tag{3.5}$$

where C is a given matrix with q rows and n columns.

Definition 3.1:
The system (S) *is said to be observable for the measurement function (3.5) on* $[0, T]$ *if two different states of the system give two different measurements:*

$$z(t) \neq \widetilde{z}(t) \text{ on } [0, T] \quad \Rightarrow \quad Cz(t) \neq C\widetilde{z}(t) \text{ on } [0, T]$$

We then say that (A, C) *is observable on* $[0, T]$.

Thanks to the linear character of the above equations, we can see that this definition is equivalent to the following

$$(Cz(t) = 0, \quad t \in [0, T]) \quad \Rightarrow \quad z_0 = 0 \tag{3.6}$$

where $z_0 = z(0)$ is the initial state of the system. Let us introduce the matrix $\mathscr{O}$ with nq rows and n columns

$$\mathscr{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

called observability matrix of the system (S), and the symmetric square matrix M of order n:

$$M = \int_0^T \exp\left(tA^{\mathrm{T}}\right) C^{\mathrm{T}} C \exp(tA)\,\mathrm{d}t \tag{3.7}$$

Then we have the following result ([1]):

Proposition 3.2:
(i) *The system (S) is observable on $[0,T]$ for the measurement function (3.5) if and only if* $\mathrm{rank}(\mathscr{O}) = n$, *or again if and only if the matrix M is invertible.*
(ii) *If the system (S) is observable on $[0,T]$ for the observation (3.5) and if we have the measurements $y^{mes}(t)$, $t \in [0,T]$, then the state of the system can be determined by*

$$z(t) = \exp(tA)\, z_0\ , \quad t \geqslant 0, \tag{3.8}$$

where

$$z_0 = M^{-1} \int_0^T \exp\left(sA^{\mathrm{T}}\right) C^{\mathrm{T}} y^{mes}(s)\,\mathrm{d}s \tag{3.9}$$

The observability of a localised system is often characterised by the rank condition: $\mathrm{rank}(\mathscr{O}) = n$.

3.2. Characterisation of Identifiable Families

Let us consider the family of two systems $\mathscr{F} = \{(S_1), (S_2)\}$ as well as the observation (E) and let us look for a condition equivalent to condition (3.4) of the $C-$identifiability of $\mathscr{F}$, where $z^1(t)$ is the state of (S_1) and $z^2(t)$ is the state of (S_2). We have

$$\begin{aligned} Cz^1(t) - Cz^2(t) &= C\exp(tA_1)\, z_0^1 - C\exp(tA_2)\, z_0^2 \\ &= \begin{bmatrix} C\exp(tA_1) & -C\exp(tA_2) \end{bmatrix} \begin{bmatrix} z_0^1 \\ z_0^2 \end{bmatrix} \\ &= \begin{bmatrix} C & -C \end{bmatrix} \exp\left(t \begin{bmatrix} A_1 & \mathrm{O} \\ \mathrm{O} & A_2 \end{bmatrix}\right) \begin{bmatrix} z_0^1 \\ z_0^2 \end{bmatrix} = \mathscr{C}\exp(t\mathscr{A})\,\xi_0 \end{aligned}$$

where we have noted that

$$\mathscr{A} = \begin{bmatrix} A_1 & \mathrm{O} \\ \mathrm{O} & A_2 \end{bmatrix} \quad , \quad \mathscr{C} = \begin{bmatrix} C & -C \end{bmatrix} \quad , \quad \xi_0 = \begin{bmatrix} z_0^1 \\ z_0^2 \end{bmatrix}$$

The condition given by (3.4) then becomes

$$\left(\mathscr{C}\exp(t\mathscr{A})\,\xi_0 = 0\ , \quad t \in [0,T]\right) \quad \Rightarrow \quad \xi_0 = 0$$

which is exactly the characterisation of the observability of the couple $(\mathscr{A}, \mathscr{C})$. The matrices $\mathscr{A}$ and $\mathscr{C}$ can be associated to an abstract linear system of order $2n$ and an observation of order q; then we have the following:

Proposition 3.3:
The family $\mathscr{F} = \{(S_1), (S_2)\}$ is $C-$identifiable on $[0, T]$ if, and only if, the following $2n$ order system:

$$(S_0) \quad \begin{cases} \dot{\xi}(t) = \mathscr{A}\xi(t) & , \quad t > 0 \\ \xi(0) = \xi_0 \in \mathbb{R}^n \times \mathbb{R}^n \end{cases} \tag{3.10}$$

is observable on $[0, T]$ for the output

$$\eta(t) = \mathscr{C}\xi(t) \quad , \quad t \in [0, T] \tag{3.11}$$

We deduce a characterisation of the $C-$identifiability of $\{(S_1), (S_2)\}$ via the observability of (S_0). This observability is verifiable by calculating the rank of the observability matrix of the latter system. Indeed the system (S_0) is observable if and only if its observability matrix

$$\mathscr{O} = \begin{bmatrix} \mathscr{C} \\ \mathscr{C}\mathscr{A} \\ \mathscr{C}\mathscr{A}^2 \\ \vdots \\ \mathscr{C}\mathscr{A}^{2n-1} \end{bmatrix}$$

has rank $2n$. And since for any integer k we have $\mathscr{C}\mathscr{A}^k = \begin{bmatrix} C(A_1)^k & -C(A_2)^k \end{bmatrix}$, then $\mathscr{O}$ is expressed as a function of A_1, A_2 and C. Noting it $\mathscr{D}_C$, this matrix becomes

$$\mathscr{D}_C = \begin{bmatrix} C & -C \\ CA_1 & -CA_2 \\ C(A_1)^2 & -C(A_2)^2 \\ \vdots & \vdots \\ C(A_1)^{2n-1} & -C(A_2)^{2n-1} \end{bmatrix} \tag{3.12}$$

Thus, we obtain an algebraic characterisation of the $C-$identifiability of a family of two systems and thus of its identifiability. This gives us an algebraic characterisation of the $C-$identifiability of a family of two systems and hence of its identifiability:

Theorem 3.1:
(i) The family $\mathscr{F} = \{(S_1), (S_2)\}$ is $C-$identifiable on $[0, T]$ if and only if the matrix $\mathscr{D}_C$, called the $C-$identifiability matrix of $\mathscr{F}$, has rank $2n$:

$$\operatorname{rank}(\mathscr{D}_C) = 2n$$

(ii) The family $\mathscr{F} = \{(S_1), (S_2)\}$ is identifiable if and only if the matrix $\mathscr{D}_{\mathrm{I}_n}$, called matrix of identifiability matrix of the family $\mathscr{F}$, is of rank $2n$:

$$\operatorname{rank}(\mathscr{D}_{\mathrm{I}_n}) = 2n$$

Remark 3.2:
1. The (ii) in theorem 1 shows that the $C-$identifiability and identifiability do not depend on the observation time interval; it suffices that it is of non-zero length.
2. If $A_1 = A_2$, the problem of identifying does not arise because we have the same system. Note that in this case, and as one might expect, the family $\mathscr{F}$ is not $C-$identifiable since the matrix of $C-$identifiability is in this case of rank $\leqslant n$ (by adding in $\mathscr{D}_C$ each column k to column $n + k$ we get n null columns that can be eliminated).

The theorem 3.1 proves that:

- If $\operatorname{rank}(\mathscr{D}_{\mathrm{I}_n}) < 2n$ then the family $\{(S_1),(S_2)\}$ is non identifiable and consequently non $C-$identifiable for none matrix C (i.e. none given output).
- If $\operatorname{rank}(\mathscr{D}_{\mathrm{I}_n}) = 2n$ then the family $\{(S_1),(S_2)\}$ is identifiable and therefore there are C matrices for which this family is $C-$identifiable i.e. $\operatorname{rank}(\mathscr{D}_C) = 2n$.

Example 3.2:

(i) Let us take the example 3.1 where $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ *and* $A_2 = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$. *The identifiability matrix is then*

$$\mathscr{D}_{\mathrm{I}_n} = \begin{bmatrix} \mathrm{I}_2 & -\mathrm{I}_2 \\ A_1 & -A_2 \\ (A_1)^2 & -(A_2)^2 \\ (A_1)^3 & -(A_2)^3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -3 & 0 \\ 0 & 2 & 0 & -4 \\ 1 & 0 & -9 & 0 \\ 0 & 4 & 0 & -16 \\ 1 & 0 & -27 & 0 \\ 0 & 8 & 0 & -64 \end{pmatrix}$$

$\mathscr{D}_{\mathrm{I}_n}$ *is of rank* 4 *so the family* $\{(S_1),(S_2)\}$ *is identifiable.*
(ii) Now take $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$*; the* $C-$*identifiability matrix is given by*

$$\mathscr{D}_C = \begin{bmatrix} C & -C \\ CA_1 & -CA_2 \\ C(A_1)^2 & -C(A_2)^2 \\ C(A_1)^3 & -C(A_2)^3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & -3 & 0 \\ 1 & 0 & -9 & 0 \\ 1 & 0 & -27 & 0 \end{pmatrix}$$

which is of rank 2. *The family* $\{(S_1),(S_2)\}$ *is therefore not* $C-$*identifiable (although it is identifiable).*

Remark 3.3:
Consider the matrix $\widetilde{\mathscr{C}}$ *defined by blocks of type* (q,n), *having* $2n$ *row blocks,* $2n$ *column blocks, the diagonal blocks are equal to* C *and non-diagonal blocks are zero (*$\widetilde{\mathscr{C}}$ *is therefore of type* $(2nq, 2n^2)$*). So we have* $\mathscr{D}_C = \widetilde{\mathscr{C}}\mathscr{D}_{\mathrm{I}_n}$ *which gives us*

$$\operatorname{rank}(\mathscr{D}_C) \leqslant \operatorname{rank}(\mathscr{D}_{\mathrm{I}_n}) \leqslant 2n \quad , \quad \forall C$$

Inequality allowing to find that any C-identifiable family is identifiable.

In the case where the matrices A_1 and A_2 commute, we have a characterisation of identifiability with a reduced dimension matrix. The proof uses the following lemma:

Lemma 3.1:
Let A *be a square matrix of order* n *with real coefficients. So*

$$(\exp(tA)\, z_0 = \mathrm{Ct} \ \textit{on} \ ]0,T[) \quad \Rightarrow \quad z_0 = 0 \tag{3.13}$$

if and only if the following matrix has rank n

$$\begin{bmatrix} A \\ A^2 \\ \vdots \\ A^n \end{bmatrix} \tag{3.14}$$

where Ct *denotes a constant vector.*

Proof
Since $\frac{\mathrm{d}}{\mathrm{d}t}\exp(tA)\,z_0 = A\exp(tA)\,z_0$, $\forall t \geqslant 0$, then we have equivalence

$$(\exp(tA)\,z_0 = \mathrm{Ct}\ ,\ \forall t \in [0,T]) \quad \Leftrightarrow \quad (A\exp(tA)\,z_0 = 0\ ,\ \forall t \in [0,T])$$

and the implication (3.13) becomes

$$(A\exp(tA)\,z_0 = 0\ ,\ \forall t \in [0,T]) \quad \Rightarrow \quad z_0 = 0$$

This implication is none other than the characterisation (3.6) of the observability for the following observed system

$$\begin{cases} \eta'(t) = A\eta(t)\ , \quad t \geqslant 0 \\ \eta_0 \in \mathbb{R}^n \end{cases} ; \qquad \begin{array}{c} y(t) = A\eta(t) \\ t \in [0,T] \end{array}$$

whose observability matrix is exactly the matrix (3.14). The proposition 3.2 then makes it possible to obtain the equivalence. □

Using this lemma, we have the following proposition:

Proposition 3.4:
Suppose that the matrices A_1 and A_2 commute. Then family $\{(S_1),(S_2)\}$ is identifiable if and only if the following matrix has rank n

$$\mathscr{M} = \begin{bmatrix} A_2 - A_1 \\ (A_2 - A_1)^2 \\ \vdots \\ (A_2 - A_1)^n \end{bmatrix}$$

Proof
The characterisation of identifiability given by (3.4) becomes, when $C = \mathrm{I}_n$,

$$\big(\exp(tA_1)\,z_0^1 = \exp(tA_2)\,z_0^2\ ,\ \forall t \in [0,T]\big) \quad \Rightarrow \quad z_0^1 = z_0^2 = 0$$

Since A_1 and A_2 commute the equality $\exp(tA_1)\,z_0^1 = \exp(tA_2)\,z_0^2$ is equivalent to $\exp\big(t(A_2 - A_1)\big)\,z_0^2 = z_0^1$, $\forall t \in [0,T]$. Taking $A = A_2 - A_1$ in the lemma 3.1, the family $\{(S_1),(S_2)\}$ is identifiable by the observation (E) on $[0,T]$ if and only the matrix $\mathscr{M}$ has rank n. □

Example 3.3:
Consider the systems (S_1) and (S_2) whose dynamics are defined by the matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad A_2 = \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}$$

In this case the identifiability matrix $\mathscr{D}_{\mathrm{I}_2}$ admits 8 lines. Note that the matrices A_1 and A_2 commute, and let us apply the previous proposition; the matrix $\mathscr{M}$ admits 4 rows and it is equal to

$$\mathscr{M} = \begin{bmatrix} A_2 - A_1 \\ (A_2 - A_1)^2 \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \\ 3 & -1 \\ -2 & 6 \end{pmatrix}$$

$\mathscr{M}$ is of rank 2, so the family $\{(S_1),(S_2)\}$ is identifiable.

On the other hand, the observability of both systems (S_1) and (S_2) cannot guarantee the $C-$identifiability of the family $\{(S_1),(S_2)\}$; however this observability is necessary as shown the following proposition:

Proposition 3.5:
If the family $\{(S_1),(S_2)\}$ is $C-$identifiable then each system of this family is observable by the observation defined by C. The converse is not true.

Proof
• Let $z^1(.)$ be a state of (S_1) such that $Cz^1(t)=0,\ t\in[0,T]$, and let $z^2(.)$ be a state of (S_2) such that $Cz^2(t)=0,\ t\in[0,T]$; then $Cz^1(t)-Cz^2(t)=0,\ t\in[0,T]$. The family $\{(S_1),(S_2)\}$ being $C-$identifiable, we get $z_0^1=0$ and $z_0^2=0$. This proves that both (S_1) are observable and (S_2) is observable.
• The converse is not true. Simply consider both systems in $\mathbb{R}^2$ and the observation defined by

$$A_1=\begin{pmatrix}1&0\\0&2\end{pmatrix},\ A_2=\begin{pmatrix}2&0\\1&1\end{pmatrix}\quad,\quad C=\begin{pmatrix}1&-1\end{pmatrix}$$

We verify that $\{(S_1),(S_2)\}$ is not $C-$identifiable while every system is observable. □

As a consequence of proposition 3.5 we have:

Corollary 3.1:
If one of the two systems is not observable for the observation given by C then the family $\{(S_1),(S_2)\}$ is not $C-$identifiable.

Remark 3.4:
If the family $\{(S_1),(S_2)\}$ is identifiable then we can find at least one matrix C which makes it possible to differentiate these systems (and sometimes several matrices). By introducing a comparison criterion between these C matrices, we can take the one that verifies a certain optimality. We take this into account in our outlook

Using the above we can give a first algorithm which detects if a family of two systems is $C-$identifiable or not:

Algorithm 1 $C-$identifiability test of a family of 2 systems.

1. Form the matrix $\mathscr{D}_C$;
2. If $\operatorname{rank}(\mathscr{D}_C)=2n$: Display "*Family $C-$identifiable.*"; STOP
3. Else, Display "*Family non $C-$identifiable.*"

3.3. Algorithm for Identifying the Searched System in a Family

Once the possibility of differentiating between two specific systems is acquired, the problem arises of the effective determination of the system which has generated the collected measurements $\{y^{mes}(t),t\in[0,T]\}$. The algorithm we present allows to identify this system. It relies on the $C-$identifiability of the family $\{(S_1),(S_2)\}$ and its unfolding is done in two steps. First, we estimate the states of the two systems from measurements collected over an arbitrary interval $[0,T_1]$; As a second step, and with measurements collected over a second interval $[T_1,T_2]$ that we will determine, comparisons are made which eliminate the system which cannot generate these measurements.
Principle. Suppose the family $\{(S_1),(S_2)\}$ is $C-$identifiable over an interval of time $[0,T]$.

a) We assume that measurements are taken $y^{mes}(t)$ on a time interval $[0, T_1]$ $(T_1 < T)$ and supposing that these measurements are non-zero. If the system (S_1) generated these measurements then its state, which is necessarily non-zero, is given by

$$z^1(t) = \exp(tA_1)\, z_0^1\,, \quad t \geqslant 0$$

with

$$z_0^1 = (M_1)^{-1} \int_0^{T_1} \exp\left(sA_1^{\mathrm{T}}\right) C^{\mathrm{T}} y^{mes}(s)\, \mathrm{d}s$$

where we have noted $(k = 1, 2)$

$$M_k = \int_0^{T_1} \exp\left(sA_k^{\mathrm{T}}\right) C^{\mathrm{T}} C \exp(sA_k)\, \mathrm{d}s \tag{3.15}$$

If on the other hand it is the system (S_2) which generated these measurements then its state, which is necessarily non-zero, is given by

$$z^2(t) = \exp(tA_2)\, z_0^2\,, \quad t \geqslant 0$$

with

$$z_0^2 = (M_2)^{-1} \int_0^{T_1} \exp\left(sA_2^{\mathrm{T}}\right) C^{\mathrm{T}} y^{mes}(s)\, \mathrm{d}s$$

b) If we have the information that the system sought is one of the two systems then we have only two cases to study. In the general case, we have four cases (the first three of which lead to outputs):

Case 1. $Cz^1(T_1) = y^{mes}(T_1)$ and $Cz^2(T_1) \neq y^{mes}(T_1)$, then the measurements come from (S_1) and not from (S_2); (S_1) is therefore the sought system;

Case 2. $Cz^1(T_1) \neq y^{mes}(T_1)$ et $Cz^2(T_1) = y^{mes}(T_1)$, then the measurements come from (S_2) and not from (S_1); (S_2) is therefore the sought system;

Case 3. $Cz^1(T_1) \neq y^{mes}(T_1)$ and $Cz^2(T_1) \neq y^{mes}(T_1)$, then the measurements come neither from (S_1) nor from (S_2);

Case 4. $Cz^1(T_1) = y^{mes}(T_1)\, Cz^2(T_1) = y^{mes}(T_1)$, so we cannot differentiate these systems on $[0, T_1]$; we then apply the following:

c) The family $\{(S_1), (S_2)\}$ is $C-$identifiable on $[T_1, T]$ and one of the states is non-zero, we can find an instant T_2 such that

$$T_2 \in \,]T_1, T[\quad \text{and} \quad Cz^1(T_2) \neq Cz^2(T_2) \tag{3.16}$$

Indeed in the opposite case we will have $Cz^1(t) = Cz^2(t)\,,\ \forall t \in \,]T_1, T[\,,$ and $C-$identifiability from $\{(S_1), (S_2)\}$ gives $z_0^1 = z_0^2 = 0$ in contradiction with non-zero measurements.

d) We resume step ***b)*** with T_2 instead of T_1; the output is then made by one of the first three cases only, the last case cannot be realised thanks to (3.16).

In this algorithm, tolerance plays a key role. We set a tolerance $\varepsilon > 0$ and for two vectors $u, v \in \mathbb{R}^n$, we will try to fulfill the condition $u = v$ by the test $\|u - v\| < \varepsilon$ and to fulfill the condition $u \neq v$ by the test $\|u - v\| \geqslant \varepsilon$, where $\|u\|$ denotes the norm of the vector u. This gives the following algorithm:

Algorithm 2 Identification of the system sought in a family $C-$identifiable from 2 systems.

1. Take measurements $y^{mes}(t)$ on $[0, T_1]$;
2. Compute initial states z_0^1 and z_0^2, and the states $z^1(t)$ and $z^2(t)$;
3. If $\left\|Cz^1(T_1) - y^{mes}(T_1)\right\| \geqslant \varepsilon$: Reject the system (S_1);
4. If $\left\|Cz^2(T_1) - y^{mes}(T_1)\right\| \geqslant \varepsilon$: Reject the system (S_2);
5. If only one system is not rejected: It is the sought system; STOP
6. If both systems are rejected: Go to 13
7. Determine $T_2 > T_1$ such that $\left\|Cz^1(T_2) - Cz^2(T_2)\right\| \geqslant \varepsilon$;
8. Take the measurements $y^{mes}(t)$ on $[T_1, T_2]$;
9. If $\left\|Cz^1(T_2) - y^{mes}(T_2)\right\| \geqslant \varepsilon$: Reject the system (S_1);
10. If $\left\|Cz^2(T_2) - y^{mes}(T_2)\right\| \geqslant \varepsilon$: Reject the system (S_2);
11. If only one system is not rejected: It is the sought system; STOP
12. If no system is rejected: Display "*Tolerance too large.*"; STOP
13. Display "*No system matches the measurements or tolerance too low.*"

Determination of the moment T_2

The existence of T_2 is ensured by the $C-$identifiability of the family but this moment is unknown. To determine this we present two methods, both at random:

Method 1 (Monte-Carlo method) we choose $T_2 \in]T_1, T[$ randomly (according to the normal law) and, as long as we have $Cz^1(T_2) = Cz^2(T_2)$, we do a another choice of T_2.

Method 2

- We choose a time step $\delta > 0$; we start with $T_2 = T_1 + \delta$ and while we have $Cz^1(T_2) = Cz^2(T_2)$ we increase T_2 by δ: $T_2 \leftarrow T_2 + \delta$. If we do not fulfill this condition, we have

$$Cz^1(k\delta) = Cz^2(k\delta) \quad , \quad k = 0, 1, 2, \ldots, \frac{T}{\delta}$$

  We can then avoid this situation by slightly changing δ.
- We can also take δ large enough and $T_2 = T_1 + \delta$, and as long as we have $Cz^1(T_2) = Cz^2(T_2)$, we change δ by $\delta/2$ and resume $T_2 = T_1 + \delta$; the condition $Cz^1(T_2) = Cz^2(T_2)$ will be then checked at a certain step, except in very special cases where the changing δ fixes the problem.

3.4. Generalisation to Families of Several Systems

In applications, it is rare to have to identify one system among two systems. The most usual way is to seek to identify it among several given systems (in a database). Therefore, we assume that a given family $\mathscr{F} = \{(S_1), \ldots, (S_N)\}$ of N linear systems is governed by

$$(S_i) \quad \begin{cases} \dot{z}(t) = A_i z(t) \quad , \quad t > 0 \\ z(0) = z_0^i \in \mathbb{R}^n \end{cases} \qquad i = 1, \ldots, N$$

where the matrices A_i which model their dynamics are all of dimension n. We also assume that we have an observation

$$(E) \qquad y(t) = Cz(t) \quad , \quad t \in [0, T] \tag{3.17}$$

where $z(t)$ is the state of the system and C a matrix of type (q, n). We can generalise the theorem 3.1 which characterises $C-$identifiability and identifiability to families of several systems.

Let us introduce the matrices

$$\mathscr{D}_C^{ij} = \begin{bmatrix} C & -C \\ CA_i & -CA_j \\ C(A_i)^2 & -C(A_j)^2 \\ \vdots & \vdots \\ C(A_i)^{2n-1} & -C(A_j)^{2n-1} \end{bmatrix} \qquad \begin{matrix} i, j = 1, \ldots, N \\ i \neq j \end{matrix}$$

Then we have the following characterisations:

Theorem 3.2:
(i) *The family of systems* $\mathscr{F} = \{(S_1), \ldots, (S_N)\}$ *is* $C-$*identifiable if and only if*

$$\text{rank}\left(\mathscr{D}_C^{ij}\right) = 2n \quad , \quad i, j = 1, \ldots, N \quad , \quad i \neq j$$

(ii) *The family of systems* $\mathscr{F} = \{(S_1), \ldots, (S_N)\}$ *is identifiable if and only if*

$$\text{rank}\left(\mathscr{D}_{\mathrm{I}_n}^{ij}\right) = 2n \qquad , \qquad i, j = 1, \ldots, N \quad , \quad i \neq j$$

From this theorem we deduce the following algorithm which allows to test the $C-$identifiability of family $\mathscr{F}$:

Algorithm 3 $C-$identifiability test for a family of systems.

---

1. For $i = 1, \ldots, N$ and for $j = 1, \ldots, N, j \neq i$
2. Determine the matrix $\mathscr{D}_C^{ij}$;
3. If $\text{rank}\left(\mathscr{D}_C^{ij}\right) < 2n$: Display "*Family* $\{(S_i), (S_j)\}$ *non* $C-$*identifiable.*"; STOP
4. next j ; next i
5. Display "*Family* $C-$*identifiable.*"

For a family $\{(S_1), \ldots, (S_N)\}$ $C-$identifiable we generalise the previous algorithm to the case of N systems with a slight adaptation imposed by the case of several systems using the algorithm below.

To identify the system sought we take in the first step the measurements on $[0, T_1]$ and we calculate the state of each system; then, using the measurements collected on a second interval $[T_1, T_2]$ that is determined by one of the methods we make comparisons that eliminate systems that cannot generate these measurements. This last operation is repeated for more than one system, as long as the observation generated by its state coincides in T_2 with the measurement obtained. In more details:

a) We take measurements on $[0, T_1]$ (arbitrary), and we determine the state $z^i(t)$ of each system (S_i) by

$$z_0^i(t) = \exp(tA_1)\, z_0^i \quad , \quad t \geqslant 0, \quad i = 1, \ldots, N$$

with

$$z_0^i = (M_i)^{-1} \int_0^{T_1} \exp\left(sA_i^{\mathrm{T}}\right) C^{\mathrm{T}} y^{mes}(s)\,\mathrm{d}s$$

b) At each step τ we reject each system whose output associated with its state at the time T_τ does not coincide with the measurement at this instant. If a single system is not rejected, it is the system sought.

c) If all the systems are rejected it is that, or none of the systems does not correspond to the measurements, or the tolerance is too low.

d) Otherwise, we take two non-rejected systems, we determine an instant $T_{\tau+1} > T_\tau$ where their observations at this instant are distinct, we take measurements on $[T_\tau, T_{\tau+1}]$ and we start again at step ***b)*** with this time $T_{\tau+1}$ instead of T_τ.
Step by step the number of non-rejected systems is reduced, until one is reached and then and only then the sought system is found.
If at some stage the number of non-rejected systems is $\geqslant 2$ and is not reduced, it is because the tolerance is too large: we change tolerance and resume at step ***b)***.

Remark 3.5:
In some cases the only tests in T_1 are enough to determine the sought system (see the example in section 3.5).

This gives the algorithm 4.

Algorithm 4 Identification of the sought system in a family $C-$identifiable.

---

1. Take the measurements $y^{mes}(t)$ on $[0, T_1]$;
2. Compute the initial states z_0^i and the states $z^i(t)$, $1 \leqslant i \leqslant N$;
3. Reject each system (S_k) that checks $\left\|Cz^k(T_1) - y^{mes}(T_1)\right\| \geqslant \varepsilon$;
4. If all systems are rejected: Go to 14
5. If only one system is not rejected: this is the sought system; STOP
6. Else, choose 2 non-rejected systems (S_i) and (S_j), $i \neq j$;
7. Determine $T_2 > T_1$ such that $\left\|Cz^i(T_2) - Cz^j(T_2)\right\| \geqslant \varepsilon$;
8. Take measurements $y^{mes}(t)$ on $[T_1, T_2]$;
9. Reject every system (S_k) that matches $\left\|Cz^k(T_2) - y^{mes}(T_2)\right\| \geqslant \varepsilon$;
10. If all systems are rejected: Go to 14
11. If only one system is not rejected: this is the sought system; STOP
12. If the number of unrejected systems has not decreased: Display "*Tolerance too large.*"; STOP
13. Do $T_1 \leftarrow T_2$ and resume at 6
14. Display "*None of the systems in the family matches the measurements or tolerance too low.*"

Remark 3.6:
The algorithm 3is applied once to find out if the systems registered in the database form a $C-$identifiable family. It is in the affirmative that we apply algorithm 4 and we reapply it each time we have an identification to make both with this family of systems and for any measurements collected.

3.5. Numerical Example

Consider a family of three systems whose dynamics are modeled by the matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \ A_2 = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}, \ A_3 = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \ C = \begin{pmatrix} 1 & 1 \end{pmatrix} \tag{3.18}$$

where C intervenes in the measurement function. To verify the $C-$identifiability of the family $\{(S_1), (S_2), (S_3)\}$ we calculate the matrices $\mathscr{D}_C^{ij}$

$$\begin{gathered} \mathscr{D}_C^{12} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 2 & -3 & -4 \\ 1 & 4 & -9 & -16 \\ 1 & 8 & -27 & -64 \end{pmatrix}, \quad \mathscr{D}_C^{13} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 4 & -7 & 1 \\ 1 & 8 & -13 & 9 \end{pmatrix} \\ \mathscr{D}_C^{23} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 3 & 4 & -3 & -1 \\ 9 & 16 & -7 & 1 \\ 27 & 64 & -13 & 9 \end{pmatrix} \end{gathered} \tag{3.19}$$

Since they are all of rank 4 then the family $\{(S_1), (S_2), (S_3)\}$ is $C-$identifiable. We consider the tolerance $\varepsilon = 0.01$.

- **First example of measurements**

1) We take measurement on $[0, T_1]$ with $T_1 = 1$. We obtain $y^{mes}(t) = 2e^{4t} - e^{3t}$, $t \in [0, 1]$.

2) We compute the matrices M_i by (3.15), we obtain

$$\begin{gathered} M_1 = \begin{pmatrix} \frac{1}{2}e^2 - \frac{1}{2} & \frac{1}{3}e^3 - \frac{1}{3} \\ \frac{1}{3}e^3 - \frac{1}{3} & \frac{1}{4}e^4 - \frac{1}{4} \end{pmatrix}, \quad M_2 = \begin{pmatrix} \frac{1}{6}e^6 - \frac{1}{6} & \frac{1}{7}e^7 - \frac{1}{7} \\ \frac{1}{7}e^7 - \frac{1}{7} & \frac{1}{8}e^8 - \frac{1}{8} \end{pmatrix} \\ M_3 \simeq \begin{pmatrix} 25.70 & 0.22 \\ 0.22 & 1.09 \end{pmatrix} \end{gathered} \tag{3.20}$$

which make it possible to calculate the initial states

$$z_0^1 \simeq \begin{pmatrix} -23.70 \\ 19.06 \end{pmatrix}, \ z_0^2 = \begin{pmatrix} -1.0 \\ 2.0 \end{pmatrix}, \ z_0^3 \simeq \begin{pmatrix} 5.76 \\ -9.08 \end{pmatrix}$$

and from this the states of the three systems

$$z^1(t) \simeq \begin{pmatrix} -23.7e^t \\ 19.06e^{2t} \end{pmatrix}, \ z^2(t) = \begin{pmatrix} -e^{3t} \\ 2e^{4t} \end{pmatrix}, \ z^3(t) \simeq \begin{pmatrix} 5.76\cos t + 9.08\sin t \\ 5.76\sin t - 9.08\cos t \end{pmatrix}$$

3) *(i)* For (S_1) : since

$$\left|Cz^1(T_1) - \left(2e^{4T_1} - e^{3T_1}\right)\right| \simeq 12.69 \geqslant \varepsilon$$

we reject (S_1).

(ii) For (S_2): since $\left|Cz^2(T_1) - \left(2e^{4T_1} - e^{3T_1}\right)\right| = 0.0 < \varepsilon$, we keep (S_2).

(iii) For (S_3) : since $\left|Cz^3(T_1) - \left(2e^{4T_1} - e^{3T_1}\right)\right| \simeq 298.95 \geqslant \varepsilon$, we reject (S_3).

4) Only (S_2) is not rejected: therefore (S_2) is the sought system.

- **Second example of measurements**

1) We take the measurement on $[0, T_1]$ with $T_1 = 1$. We get $y^{mes}(t) = 4e^{2t}\cos(t)$, $t \in [0, 1]$.

2) The matrices M_i are already calculated by (3.20); we calculate the initial states:

$$z_0^1 \simeq \begin{pmatrix} 12.52 \\ -5.88 \end{pmatrix} , \; z_0^2 \simeq \begin{pmatrix} 4.27 \\ -1.31 \end{pmatrix} , \; z_0^3 = \begin{pmatrix} 2.0 \\ 2.0 \end{pmatrix}$$

and the states

$$z^1(t) \simeq \begin{pmatrix} 12.52e^{t} \\ -5.88e^{2t} \end{pmatrix} , \; z^2(t) \simeq \begin{pmatrix} 4.27e^{3t} \\ -1.31e^{4t} \end{pmatrix} , \; z^3(t) = e^{2t} \begin{pmatrix} 2\cos t - 2\sin t \\ 2\cos t + 2\sin t \end{pmatrix}$$

3) *(i)* For (S_1) : since $\left|Cz^1(T_1) - 4e^{2T_1}\cos(T_1)\right| \simeq 25.38 \geqslant \varepsilon$, we reject (S_1).

(ii) For (S_2) : since $\left|Cz^2(T_1) - 4e^{2T_1}\cos(T_1)\right| \simeq 32.47 \geqslant \varepsilon$, we reject (S_2).

(iii) For (S_3) : since $\left|Cz^3(T_1) - 4e^{2T_1}\cos(T_1)\right| = 0.0 < \varepsilon$, we keep (S_3).

4) (S_3) is the only system not rejected; so (S_3) is the sought system.

Finally we obtain the following assignments:

plot	1	2
System	(S_2)	(S_3)

4. CASE OF DISCRETE FINITE DIMENSIONAL LINEAR SYSTEMS

We note that the majority of environmental phenomena has a very slow evolution; this allows us to assume for one of these systems that are modelling these phenomena that, after an instant t_k, its state only shows a significant change at an instant t_{k+1}. The measurements can then be considered as taken at instants t_k. All these considerations lead us to consider what becomes of everything discussed earlier when the systems have discrete models.

4.1. Identifiability of a Family of Finite Dimensional Discrete Linear Systems

Suppose the two dynamical systems are governed by the following two recurring equations

$$(S_1) \quad \begin{cases} z_{k+1} = A_1 z_k & k = 0, 1, \ldots \\ z_0 \in Z \end{cases} \qquad (S_2) \quad \begin{cases} z_{k+1} = A_2 z_k & k = 0, 1, \ldots \\ z_0 \in Z \end{cases}$$

The state of (S_1) is denoted z_k^1 and the state of (S_2) is denoted z_k^2. Suppose also that the measurements are discrete

$$(E) \qquad y_k = Cz_k \quad , \quad k = 0, 1, \ldots, m-1$$

where z_k is either z_k^1 or z_k^2. If y_k^{mes}, $k = 0, 1, \ldots, m-1$, are collected measurements; so we have either $y_k^{mes} = Cz_k^1$, $k = 0, 1, \ldots, m-1$ or else $y_k^{my} = Cz_k^2$, $k = 0, 1, \ldots, m-1$. We take up the same problem of the identification posed in the previous sections.

Definition 4.1:

(i) *We say that the family $\{(S_1), (S_2)\}$ is $C-$identifiable for the first m measurements if*

$$\left[\begin{array}{c} z_k^1 \textit{ state of } (S_1), \, z_k^2 \textit{ state of } (S_2) \textit{ and} \\ \exists k \leqslant m-1 \textit{ such as } z_k^1 \neq 0 \textit{ or } z_k^2 \neq 0 \end{array} \right] \Rightarrow \left(\exists j \leqslant m-1 \textit{ such as } Cz_j^1 \neq Cz_j^2\right)$$

(ii) *We say that the family $\{(S_1), (S_2)\}$ is identifiable for the first m measurements if there exist an integer $q \geqslant 1$ and a matrix C of type (q, n) such that this family is $C-$identifiable for the first m measurements.*

By taking the contrapositive, and making the initial states z_0^1 and z_0^2 appear, the definition of $C-$identifiability becomes

$$\left[\begin{array}{c} z_k^1 \text{ state of } (S_1)\,,\, z_k^2 \text{ state of } (S_2) \\ \text{and } Cz_k^1 = Cz_k^2, \quad k = 0,\ldots,m-1 \end{array}\right] \quad \Rightarrow \quad z_0^1 = z_0^2 = 0 \tag{4.21}$$

We will adopt this implication to characterize the families $C-$identifiable formed by two discrete systems.

In the general case, the sought system is among N systems of a given family. We assume that these systems have the equations

$$(S_i) \quad \begin{cases} z_{k+1} = A_i z_k \quad , \quad k = 0,1,2,\ldots \\ z_0 = z_0^i \in \mathbb{R}^n \qquad i = 1,\ldots,N \end{cases}$$

where the matrices A_i, which model their dynamics, are all of dimension n. We can generalize the definition of identifiability to families of several systems as follows:

Definition 4.2:
(i) *The family of systems $\{(S_1),\ldots,(S_N)\}$ is said to be $C-$identifiable for the first m measurements if each subfamily $\{(S_i),(S_j)\}$, $i \neq j$, is $C-$identifiable for the observation via the first m measurements.*
(ii) *The family of systems $\{(S_1),\ldots,(S_N)\}$ is said to be identifiable for the first m measurements if each subfamily of systems $\{(S_i),(S_j)\}$, $i \neq j$, is identifiable for the first m measurements.*

4.2. Observable Discrete System and State Reconstruction

As for continuous systems in time, we use the observability of an abstract discrete system for the characterisation of $C-$identifiability.
Consider a dynamical system governed by the following recurring equation of state:

$$(S) \quad \begin{cases} z_{k+1} = A z_k \quad , \quad k = 0,1,2,\ldots \\ z_0 \in Z = \mathbb{R}^n \end{cases}$$

where A is a given square matrix of order n with real coefficients. The state of the system has the expression $z_k = A^k z_0$, $k = 0,1,2,\ldots$. This equation of state is augmented by a measurement function (output equation):

$$y_k = C z_k \quad , \quad k = 0,1\ldots,m-1 \tag{4.22}$$

where C is a given matrix with q rows and n columns.

Definition 4.3:
The system (S) is said to be observable for observation (4.22) on the discrete interval $0,1,\ldots,m-1$ if for two states any distinct z_k and $\widetilde{z}_k$ of this system correspond to two distinct measurements:

$$(\exists k \leqslant m-1 \text{ such as } z_k \neq \widetilde{z}_k) \quad \Rightarrow \quad (\exists j \leqslant m-1 \quad \text{such as} \quad Cz_j \neq C\widetilde{z}_j)$$

We then say that the couple (A,C) is observable for the first m measurements (or on $0,1,\ldots,m-1$).

One can easily see that this definition is equivalent to the following characterisation:

$$(Cz_k = 0\,, \quad k = 0,1,\ldots,m-1) \;\Rightarrow\; z_0 = 0$$

Consider the linear map $\mathscr{K} : z_0 \in \mathbb{R}^n \to \{y_k\}_{0\leqslant k\leqslant m-1} \in (\mathbb{R}^q)^m$. Since $y_k = CA^k z_0$, $0 \leqslant k \leqslant m-1$, the matrix expression $\mathscr{O}$ of the application $\mathscr{K}$ is given by

$$\mathscr{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{m-1} \end{bmatrix}$$

The system (S) is then observable with m measurements if, and only if, $\mathscr{K}$ is injective, which is equivalent to $\operatorname{rank}(\mathscr{O}) = n$. The matrix $M = \mathscr{O}^{\mathrm{T}}\mathscr{O}$ is symmetric; it is expressed as functions of A and C by

$$M = \sum_{k=1}^{m} \left(A^{k-1}\right)^{\mathrm{T}} C^{\mathrm{T}} C A^{k-1}$$

Then we have:

Proposition 4.1:
(i) *The system (S) is observable for the observation (4.22) with the first m measurements if and only if $\operatorname{rank}(\mathscr{O}) = n$, or even if and only if the matrix M is invertible.*
(ii) *If the system (S) is observable for the observation (4.22) with the first m measurements and if we have the measurements y_k^{mes}, $k = 0, 1, \ldots, m-1$, then the state of the system is given by*

$$z_k = A^k z_0 \quad ; \quad k = 0, 1, 2, \ldots$$

where

$$z_0 = (M)^{-1} \sum_{j=0}^{m-1} \left(A^j\right)^{\mathrm{T}} C^{\mathrm{T}} y_j^{mes}$$

Corollary 4.1:
The system (S) is not observable for the observation (4.22) on the discrete interval $0, 1, \ldots, m-1$ if

$$mq < n$$

This shows that for the system to be observable, a necessary but not sufficient condition must exist and that the number of measurements should be greater than or equal to the value n/q.

4.3. Characterisation of the $C-$identifiability of a Family

Similar to the 3.2 section, we have, in the case of a family with two systems,

$$Cz_k^1 - Cz_k^2 = CA_1^k z_0^1 - CA_2^k z_0^2 = \begin{bmatrix} C\,(A_1)^k & -C\,(A_2)^k \end{bmatrix} \begin{bmatrix} z_0^1 \\ z_0^2 \end{bmatrix}$$
$$= \begin{bmatrix} C & -C \end{bmatrix} \begin{bmatrix} A_1 & \mathrm{O} \\ \mathrm{O} & A_2 \end{bmatrix}^k \begin{bmatrix} z_0^1 \\ z_0^2 \end{bmatrix} = \mathscr{C}\mathscr{A}^k \xi_0$$

where

$$\mathscr{A} = \begin{bmatrix} A_1 & \mathrm{O} \\ \mathrm{O} & A_2 \end{bmatrix} \quad , \quad \mathscr{C} = \begin{bmatrix} C & -C \end{bmatrix} \quad , \quad \xi_0 = \begin{bmatrix} z_0^1 \\ z_0^2 \end{bmatrix}$$

The characterisation (4.21) of the $C-$identifiability of the family $\{(S_1), (S_2)\}$ then becomes

$$\left(\mathscr{C}\mathscr{A}^k\xi_0 = 0,\ k = 0, 1, \ldots, m-1\right) \quad \Rightarrow \quad \xi_0 = 0$$

which characterises the observability of the couple $(\mathscr{A}, \mathscr{C})$ over the time interval $0, 1, \ldots, m-1$. This couple models the dynamics of a certain abstract discrete system and a certain observation, which allows us to have the following characterisation:

Proposition 4.2:
The family $\{(S_1), (S_2)\}$ is $C-$identifiable for the first m measurements if and only if the following system of order $2n$

$$(S_0) \quad \begin{cases} \xi_{k+1} = \mathscr{A}\xi_k & k = 0, 1, \ldots \\ \xi_0 \in Z \times Z = \mathbb{R}^n \times \mathbb{R}^n \end{cases}$$

is observable during the time interval $0, 1, \ldots, m-1$ by the observation

$$\eta_k = \mathscr{C}\xi_k \quad , \quad k = 0, 1, \ldots, m-1$$

We know (proposition 4.1) that the observability of the system (S_0) over $0, 1, \ldots, m-1$ is equivalent to its matrix observability $\mathscr{O}$ of rank $2n$.

$$\mathscr{O} = \begin{bmatrix} \mathscr{C} \\ \mathscr{C}\mathscr{A} \\ \vdots \\ \mathscr{C}\mathscr{A}^{m-1} \end{bmatrix}$$

With the remark that $\mathscr{C}\mathscr{A}^k = \begin{bmatrix} C(A_1)^k & -C(A_2)^k \end{bmatrix}$ the matrix $\mathscr{O}$ can be expressed in terms of A_1, A_2 and C. This matrix becomes

$$\mathscr{D}_C = \begin{bmatrix} C & -C \\ CA_1 & -CA_2 \\ C(A_1)^2 & -C(A_2)^2 \\ \vdots & \vdots \\ C(A_1)^{m-1} & -C(A_2)^{m-1} \end{bmatrix}$$

We then get the following theorem:

Theorem 4.1:
(i) *The family $\{(S_1), (S_2)\}$ is $C-$identifiable for the first m measurements if, and only if, the so-called $C-$identifiability matrix has rank $2n$:*

$$\operatorname{rank}(\mathscr{D}_C) = 2n$$

(ii) *The family $\{(S_1), (S_2)\}$ is identifiable for the first m measurements if and only if, the so-called identifiability matrix has rank $2n$:*

$$\operatorname{rank}(\mathscr{D}_{\mathrm{I}_n}) = 2n$$

We deduce from this theorem that for a given number m of measurements, if $\operatorname{rank}(\mathscr{D}_{\mathrm{I}_n}) < 2n$ then the family $\{(S_1), (S_2)\}$ is not identifiable and therefore it is not $C-$identifiable for any matrix C. If on the other hand $\operatorname{rank}(\mathscr{D}_{\mathrm{I}_n}) = 2n$ then this family is identifiable and therefore there are matrices C for which this family $\{(S_1), (S_2)\}$ is $C-$identifiable.

Corollary 4.2:
The family $\{(S_1),(S_2)\}$ cannot be $C-$identifiable with the first m measurements if

$$m < \frac{2n}{q} \tag{4.23}$$

Proof
$\mathscr{D}_C$ admits mq rows so $\operatorname{rank}(\mathscr{D}_C) \leqslant mq$. If $mq < 2n$ then $\operatorname{rank}(\mathscr{D}_C) < 2n$ and in this case the family $\{(S_1),(S_2)\}$ is not $C-$identifiable. □

Remark 4.1:
1. In the particular case of a scalar measurement ($q = 1$) the condition (4.23) becomes $m < 2n$; a minimum of $2n$ measurements is therefore necessary. If besides the system states are scalars ($n = 1$) a minimum of 2 measurements is required.

2. There exist non-identifiable families for a given number m of measurements. An example of such families is given by the family of the two systems modeled by the matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \textit{and} \quad A_2 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

with $m = 4$ since its identifiability matrix $\mathscr{D}_{\mathrm{I}_n}$ is of rank < 4. However, and contrary to the case of continuous models in time, it is possible that an unidentifiable family of discrete systems for a number of measurements m becomes identifiable for a greater number $m' \geqslant m$.

3. As for continuous linear systems in time (remark 3.3) we can show that

$$\operatorname{rank}(\mathscr{D}_C) \leqslant \operatorname{rank}(\mathscr{D}_{\mathrm{I}_n}) \leqslant 2n \quad , \quad \forall C$$

inequality allowing to find that any family $C-$identifiable is identifiable.

Example 4.1:
Consider the two systems

$$(S_1)\ \ z_{k+1} = A_1 z_k \quad , \quad (S_2)\ \ z_{k+1} = A_2 z_k$$

where

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad , \quad A_2 = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \quad \textit{and} \quad m = 4$$

• *The identifiability matrix of the family $\{(S_1),(S_2)\}$ is given by*

$$\mathscr{D}_{\mathrm{I}_n} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -3 & 0 \\ 0 & 2 & 0 & -4 \\ 1 & 0 & -9 & 0 \\ 0 & 4 & 0 & -16 \\ 1 & 0 & -27 & 0 \\ 0 & 8 & 0 & -64 \end{pmatrix}$$

and its rank is 4. The family $\{(S_1),(S_2)\}$ is therefore identifiable for 4 measurements.
• *For observation*

$$(E) \quad y_k = C z_k\ , \quad k = 0,1,2,3 \quad \textit{with} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

The $C-$identifiability matrix of the family $\{(S_1),(S_2)\}$ is given by

$$\mathscr{D}_C = \begin{bmatrix} C & -C \\ CA_1 & -CA_2 \\ C(A_1)^2 & -C(A_2)^2 \\ C(A_1)^3 & -C(A_2)^3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & -3 & 0 \\ 1 & 0 & -9 & 0 \\ 1 & 0 & -27 & 0 \end{pmatrix}$$

which is of rank 2. The family $\{(S_1),(S_2)\}$ is therefore not $C-$identifiable.
• Since the family $\{(S_1),(S_2)\}$ is identifiable we can find a matrix $C' \neq C$ such that this family is $C'-$identifiable. It is the case of the following observation:

$$(E') \quad y_k = C' z_k\ ,\ k = 0,1,2,3 \qquad \textit{with } C' = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

The $C'-$identifiability matrix of this family is given by

$$\mathscr{D}_{C'} = \begin{bmatrix} C' & -C' \\ C'A_1 & -C'A_2 \\ C'(A_1)^2 & -C'(A_2)^2 \\ C'(A_1)^3 & -C'(A_2)^3 \end{bmatrix} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 2 & -3 & -4 \\ 1 & 4 & -9 & -16 \\ 1 & 8 & -27 & -64 \end{pmatrix}$$

which is of rank 4. The family $\{(S_1),(S_2)\}$ is therefore $C'-$identifiable.

We have similar results for the case of continuous systems, which we give in the following proposition:

Proposition 4.3:
(i) When A_1 and A_2 commute the family $\{(S_1),(S_2)\}$ is identifiable by the first m measurements if, and only if, the matrix with mn rows

$$\mathscr{M} = \begin{bmatrix} A_2 - A_1 \\ (A_2 - A_1)^2 \\ \vdots \\ (A_2 - A_1)^m \end{bmatrix}$$

is of rank n.
(ii) For the first m measurements, if the family $\{(S_1),(S_2)\}$ is $C-$identifiable then each system of this family is observable. The opposite is false.

We therefore deduce from this proposition, as well as from Corollary 4.2, that if one of the systems is not observable by the first m measurements or if $m < n = q$ then the family formed by these two systems cannot be $C-$identifiable for these measurements.

For families with several systems the algebraic characterisation of the $C-$identifiability and identifiability of a family of systems $\{(S_1),\ldots,(S_N)\}$ is done through the following matrices

$$\mathscr{D}_C^{ij} = \begin{bmatrix} C & -C \\ CA_i & -CA_j \\ C(A_i)^2 & -C(A_j)^2 \\ \vdots & \vdots \\ C(A_i)^{m-1} & -C(A_j)^{m-1} \end{bmatrix} \qquad \begin{matrix} i,j = 1,\ldots,N \\ i \neq j \end{matrix}$$

Theorem 4.2:
***(i)** The family of systems $\{(S_1),\ldots,(S_N)\}$ is $C-$identifiable for first m measurements if and*

only if

$$\operatorname{rank}\left(\mathscr{D}_C^{ij}\right) = 2n \quad , \quad i,j = 1,\ldots,N \quad , \quad i \neq j$$

(ii) *The family of systems* $\{(S_1),\ldots,(S_N)\}$ *is identifiable for the first* m *measurements if and only if*

$$\operatorname{rank}\left(\mathscr{D}_{\mathrm{I}_n}^{ij}\right) = 2n \quad , \quad i,j = 1,\ldots,N \quad , \quad i \neq j$$

4.4. Algorithm for the Identification of the Sought System

For a given family $\mathscr{F} = \{(S_1),\ldots,(S_N)\}$ of N discrete systems, we give below the first algorithm that determines, when it exists, the first integer for which $\mathscr{F}$ is $C-$identifiable. A second algorithm identifies in the $\mathscr{F}$ family the system that generated the measurements, assuming that this family is $C-$identifiable.

Consider an integer $m_0 \geqslant 2$ (to be set by the user). We search for the first integer $m \leqslant m_0$, if it exists, that ensures the $C-$identifiability of the family $\mathscr{F}$. $E[x]$ is always the integer part of x we have the following algorithm:

Algorithm 5 $C-$identifiability test of a family and determination of m.

1. If $\frac{2n}{q}$ is an integer, take $m \leftarrow \frac{2n}{q}$ Else take $m \leftarrow E\left[\frac{2n}{q}\right] + 1$
2. For $i = 1,\ldots,N$ and for $j = 1,\ldots,N$, $j \neq i$
3. Determine $\mathscr{D}_C^{ij}$; If rank $\left(\mathscr{D}_C^{ij}\right) < 2n$ go to 6
4. next j ; next i
5. Display "*Family* $C-$*identifiable for* $m =$"m; STOP
6. $m \leftarrow m+1$; Si $m \leqslant m_0$ resume in 2
7. Else, Display "*Family non* $C-$*identifiable for* $m \leqslant$"m_0

Suppose in the following that the family $\mathscr{F}$ is $C-$identifiable for an integer m, determined by the algorithm 5, and that we have collected m measurements $y_0^{mes},\ldots,y_{m-1}^{mes}$ assumed to be non-zero. For each system (S_i) of the family, we can estimate the state z_k^i by

$$z_k^i = A_i^k z_0^i \tag{4.24}$$

where

$$z_0^i = (M_i)^{-1} \sum_{k=0}^{m-1} \left(A_i^k\right)^{\mathrm{T}} C^{\mathrm{T}} y_k^{mes} \quad \text{and} \quad M_i = \sum_{k=0}^{m-1} \left(A_i^k\right)^{\mathrm{T}} C^{\mathrm{T}} C A_i^k \tag{4.25}$$

Let us then denote by (S_{i_0}) the system (still unknown) which has given these measurements.

So, other than system (S_{i_0}), the state of each system (S_i), $i \neq i_0$, check $Cz_k^i \neq y_k^{mes}$ for at least one integer $k \leqslant m-1$; otherwise we would have

$$\left(Cz_k^{i_0} = y_k^{mes}\ ,\ k = 0,1,\ldots,m-1\right) \text{ and } \left(Cz_k^{i} = y_k^{mes}\ ,\ k = 0,1,\ldots,m-1\right)$$

which cannot be achieved because of the $C-$identifiability of the subfamily $\{(S_{i_0}),(S_i)\}$.

So just test $(Cz_k^i = y_k^{mes}\ ,\ k = 0,1,\ldots,m-1)$ for each system (S_i) and to eliminate it if it does not fulfill this test. At the end of this step we should, theoretically, obtain a single not eliminated system. Practically three cases can arise:

Case 1. Several systems check for equalities $Cz_k^i = y_k^{mes}$, i.e. $\|Cz_k^i - y_k^{mes}\| < \varepsilon$, $k = 0, \ldots, m-1$. In this case the tolerance ε is too large to be able to identify the sought system;

Case 2. No system fulfills these equalities: $\|Cz_k^i - y_k^{mes}\| \geqslant \varepsilon$, $k = 0, 1, \ldots, m-1$, $1 \leqslant i \leqslant N$. In this case, either the tolerance is too low, or no system in the family has generated these measurements;

Case 3. Only one system checks for ties; this is then the sought system.We deduce from all the above the following algorithm:

Algorithm 6 Identification of the sought system in a family $C-$identifiable by m measurements.

1. Take the measurements y_k^{mes}; $k = 0, 1, \ldots, m-1$;
2. Compute the states z_k^i , $i = 1, \ldots, N$;
3. For each system (S_i)
4. For $k = 0, 1, \ldots, m-1$
5. If $\left\|Cz_k^i - y_k^{mes}\right\| \geqslant \varepsilon$: Reject (S_i) and go to 7
6. next k
7. next system (S_i)
8. If all systems are rejected: Display "*None of the systems generated these measurements, or tolerance was too low.*"; STOP
9. If the number of non-rejected systems is $\geqslant 2$: Display "*Tolerance too large.*"; STOP
10. The only non-rejected system is the sought system.

Remark 4.2:
The algorithms of the discrete case are slightly different from those of the continuous case (in time). This results from the very nature of these systems.

4.5. Numerical Example

Consider a family of three systems whose dynamics are modeled by the matrices (3.18).

We determine the first integer m for which the family of these 3 systems is $C-$identifiable. For $m < 4$ the matrices $\mathscr{D}_C^{ij}$ have less of 4 rows and therefore their ranks are < 4. For $m = 4$ the $\mathscr{D}_C^{ij}$ are given by (3.19) which all have rank $4 = 2n$; hence the $C-$identifiability of this family for measurements in $k = 0, 1, 2, 3$. In what follows we take $\varepsilon = 0.01$ as tolerance.

- **First example of measurements**

The observation at moments $k = 0, 1, 2, 3$ gave the measurements $\{y_0^{mes}, y_1^{mes}, y_2^{mes}, y_3^{mes}\} = \{1, 5, 23, 101\}$.

1) Compute of the states: We first calculate the matrices M_i by (4.25); we obtain

$$M_1 = \begin{pmatrix} 4 & 15 \\ 15 & 85 \end{pmatrix} , \ M_2 = \begin{pmatrix} 820 & 1885 \\ 1885 & 4369 \end{pmatrix} , \ M_3 = \begin{pmatrix} 228 & -120 \\ -120 & 84 \end{pmatrix} \tag{4.26}$$

so the initial states, computed by (4.24), are

$$z_0^1 = \begin{pmatrix} -\frac{523}{23} \\ \frac{1694}{115} \end{pmatrix} , \ z_0^2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} , \ z_0^3 = \begin{pmatrix} \frac{65}{22} \\ -\frac{449}{66} \end{pmatrix}$$

2) For (S_1) : we test
(i) $|Cz_0^1 - y_0^{mes}| = \frac{1036}{115} \geqslant \varepsilon$: we reject (S_1) .
3) For (S_2) : we test
(i) $|Cz_0^2 - y_0^{mes}| = 0.0 < \varepsilon$;
(ii) $|Cz_1^2 - y_1^{mes}| = |CA_2 z_0^2 - 5| = 0.0 < \varepsilon$;
(iii) $|Cz_2^2 - y_2^{mes}| = \left|C\left(A_2\right)^2 z_0^2 - 23\right| = 0.0 < \varepsilon$;
(iv) $|Cz_3^2 - y_3^{mes}| = C\left(A_2\right)^3 z_0^2 = 0.0 < \varepsilon$.
On conserve (S_2).
4) For (S_3) : we test
(i) $|Cz_0^3 - y_0^{mes}| = \frac{160}{33} \geqslant \varepsilon$: We reject (S_3) .
(S_2) is the only system not rejected; so (S_2) is the sought system.

- **Second example of measurements**

The observation at moments $k = 0, 1, 2, 3$ gave the measurements $\{y_0^{mes}, y_1^{mes}, y_2^{mes}, y_3^{mes}\} = \{1, 5, 15, 35\}$.
1) The matrices M_i are always given by (4.26); we calculate the initial states by (4.24):

$$z_0^1 = \begin{pmatrix} -\frac{101}{23} \\ \frac{564}{115} \end{pmatrix} \; , \; z_0^2 = \begin{pmatrix} \frac{74\,039}{29\,355} \\ -\frac{3028}{5871} \end{pmatrix} \; , \; z_0^3 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

2) For (S_1) we test :
(i) $|Cz_0^1 - y_0^{mes}| = \frac{56}{115} \simeq 0.48 \geqslant \varepsilon$: we reject (S_1) .
3) For (S_2) we test:
(i) $|Cz_0^2 - y_0^{mes}| = 0.0 < \varepsilon$;
(ii) $|Cz_1^2 - y_1^{mes}| = |CA_2 z_0^2 - 5| = \frac{778}{1545} \simeq 0.5 > \varepsilon$: we reject (S_2) .
4) For (S_3) , we keep it because:
(i) $|Cz_0^3 - y_0^{mes}| = 0.0 < \varepsilon$;
(ii) $|Cz_1^3 - y_1^{mes}| = |CA_3 z_0^3 - 5| = 0.0 < \varepsilon$;
(iii) $|Cz_2^3 - y_2^{mes}| = \left|C\left(A_3\right)^2 z_0^3 - 15\right| = 0.0 < \varepsilon$;
(iv) $|Cz_3^3 - y_3^{mes}| = \left|C\left(A_3\right)^3 z_0^3 - 35\right| = 0.0 < \varepsilon$.
(S_3) is the only system not rejected; so (S_3) is the sought system.
Finally we obtain the following assignments:

plot	1	2
System	(S_2)	(S_3)

5. APPLICATION TO THE CROP IDENTIFICATION USING RADAR

In this section we present the use of Radar for crop identification as an application of the general approach given in the previous sections. We recall the Radar principle.

The Radar, considered as a sensor, is a satellite-based Synthetic Aperture Radar (SAR). It transmits microwave pulses and receives the signals backscattered from the objects on the Earth, which are recorded as an image data and data products. Radar signal backscatter depends mainly on the physical properties of the observed objects: their roughness, geometry and dielectric constant.

In case of vegetation it is mostly linked to the plant external structure, biomass and water content ([13]). The possibility of observation of crop characteristics is also dependent on sensor properties, e.g.:

- the wavelength used for the observation and the associated depth of penetration, for instant: when using C-band (5,6 cm) we detect mainly plant's leaves, but when using L-band (23 cm) mostly only branches are visible.
- the polarization of the transmitted and received signal. Using different polarization we can observe better different kind of objects. Cross-polarization is better to use for observation of non-structured crops (cereals, maize), co-polarization is more suitable for detection of structured crops (vineyards, hop).

To use the Radar for crop identification, it is necessary to develop a classification algorithm/approach. First, it should be noted that even if crops evolve continuously in time we study them using discrete measurements as shown in figure 5.1.

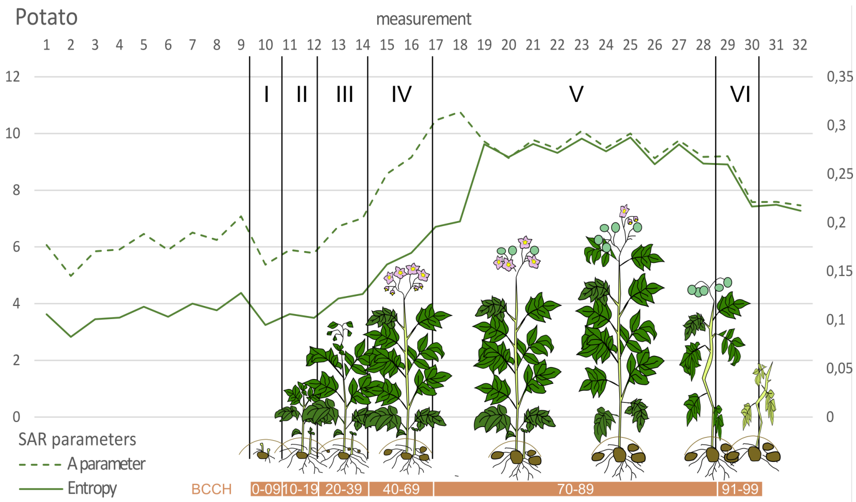

Fig. 5.1. Evolution of a crop (Potato) (BCCH - the phenological development stages of plants)

For identification of crops using Radar we proceed in two steps as in the mathematical approach:

1. First step: dataset and crops dynamics. To assume that the dynamics of the crops, and consequently the discrete states, are known, a reference dataset is needed. This requires "a first" measurement to construct the database.

In the case of crops, ground truth data are needed to know the different types of crops, their dynamics and states in specific fields, as well as the measurements of radar backscatter must be acquired for these fields using satellite sensors (e.g.: Sentinel-1). It must be noticed that the SAR measurement can be realized using different polarization at the same time so several parameters can be derived to extract more information about crops, e.g.: polarimetric coherence matrices and H/α ([5]), which show the degree of the depolarization of the signal in the contact with the object, its scatter type or entropy. Such multidimensional observation increases the accuracy of the systems identification ([16]). Also the speckle noise must be removed. Using these measurements we estimate the standard dynamic of each crop (reference models of crops) which we want to identify (figure 1). So the dataset consists to have the equivalent of figure 1 for each crop as in figure 2.

2. Second step: identification. To identify a system on a given plot, where the crop type is unknown, we consider two steps:

2.1 We must ensure that our sensor can distinguish crops. This is done through the analysis of our reference database. In figures 5.3 we show accordingly a case where one crop can be identified whereas the other cannot. It must be noticed that different parameters can or cannot identify the systems.

2.2 If the sensor can identify the crops, a "second measurement" is made. We collect measurements of all parameters for all the fields in the area where we want to identify

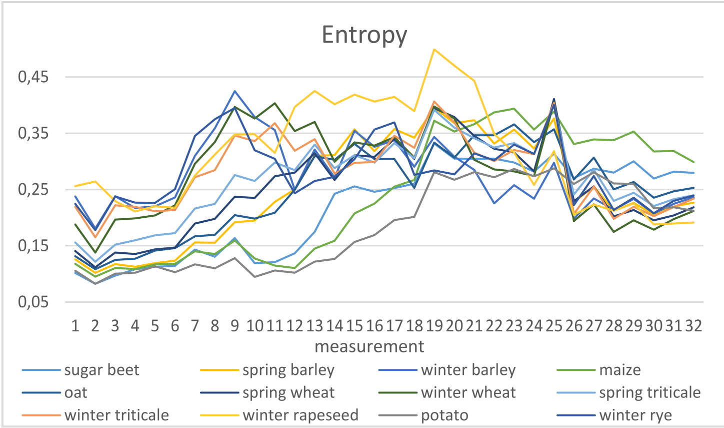

Fig. 5.2. Reference models of different crops (dataset)

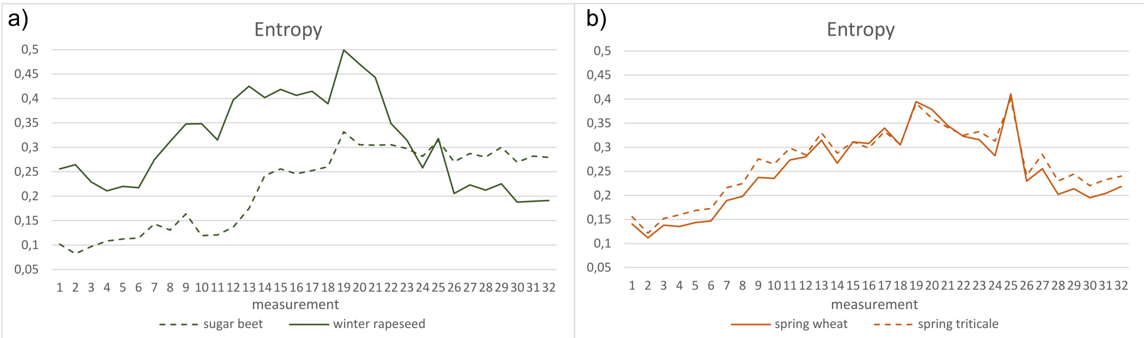

Fig. 5.3. *Examples of identifiable crops (a) and non-identifiable crops (b).*

unknown crop types (systems) and use the equivalent of the algorithm given in the mathematical approach. In the case of crop classification using radar we will look for all fields which can give the similar measurements given by the standard crops (reference models of crops). This step is carried out using one specific classifier (e.g. Random Forest (RF) ([3]) which compares directly measurements to the reference models of the systems (crops) and identifies crops.

In the mathematical approach, we have pointed out that measurements must be made over a time interval $I =]0, T]$ in the continuous case and in the discrete cases, the measurements must be greater than a given threshold iterations. It is the case for crop identification using Radar. As an example, in figure 5.4, we show that in order to identify the systems we have to consider a relatively greater time.

In fact, we could include this (the time of the measurements) in our classification algorithm and choose this time according to this data. We can mention that different times of measurement have better or worse potential for the identification.

The case study carried out in Poland using Sentinel-1 SAR images ([17]) showed that, e.g. sugar beet and winter rapeseed systems can be very well discriminated from other 23 crops (systems) – F1 score which evaluate an accuracy of the classification of a specific class is 0.98 and 0.99, respectively. So Sentinel-1 can be considered as an identifying sensor for these systems. On the contrary, the cases of spring triticale and spring wheat systems, which were very poorly discriminated 0.53 and 0.6, respectively.

In our mathematical approach, the first measurement consists in searching for standard states of systems (reference models of crops) given through the measurement of radar parameters. The second measurement consists in the identification among systems of both: mathematical and radar algorithms.

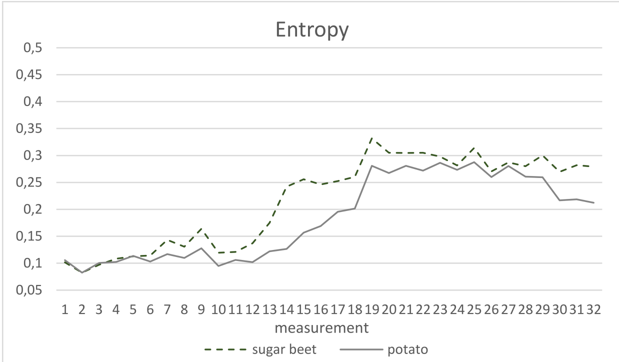

Fig. 5.4. Example of two systems identifiable or non-identifiable depending on the time of measurement.

6. CONCLUSION

In this paper, we considered the problem of identifying a family of dynamical systems. For the case of finite dimensional linear systems, we have given characterizations and proposed algorithms to determine the sought system, among N systems, from the collected measurements.

We have developed numerical examples and an application to illustrate our approach. It would be interesting to extend this work, and in particular the algorithms, to nonlinear systems and to distributed parameter systems. It would also be very useful to automate the identification of the system sought for a possible application to the automation of the identification of cultures using Radar. These issues are currently under investigation.

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