Adv Syst Sci Appl 2023; **01**:61–68
Published online at https://ijassa.ipu.ru.

The Problem of Controllability with the Phase Space Change

Irina Maximova

Peoples' Friendship University of Russia (RUDN University), Moscow, Russia

Abstract: In this paper, we consider differential systems of the following structure: at two consecutive time intervals the motion of the object is described by two different systems of differential equations. We study the controllability of the object described by such system from the initial set in one space to the given set in another space through so-called "transition hypersurface". The transition of an object from one space to another one is given by a certain reflection. Sufficient conditions of the controllability of such differential systems in the problem with phase space change are obtained. Approaches to the study of both nonlinear and linear systems are considered.

Keywords: controllability, reachability set, multiple-valued mapping, support function

1. INTRODUCTION

Problems with the change of the phase space are an important class of so-called hybrid (composite) systems, they are characterized by the condition that at different time intervals the motion of the object is described by different differential equations and some links for trajectory mating. The initial source of such problems was the multistage processes of space flight (see [16]). Problems with the phase space change arise in various applied problems, including aircraft engineering, robotics, economics, etc. For example, the problem of launching a missile from an underwater object: in this case the dimension of the space does not change, but the environment and conditions of the motion do.

The change of the phase spaces can occur in the mathematical modeling of complex dynamic systems, for example, large production complexes, multistage technological processes. The process of controlling a system of chemical reactors, models of the dynamics of metapopulations, and economic systems with a variable structure are also characterized by a sequence of consecutive stages. The dimensionality of dynamic systems used when modelling these processes depends on their state and can change over time, i.e. the decomposition of the complex system is taking place.

The possibility of using systems with the variable dimensionality in modeling the dynamics of biological communities was pointed out in the work [15]. During the construction of realization theory, R. Kalman proposed to generalize the notion of dynamic system that the dimensionality of its space of states could change over time (see [9]). Problems of optimal control of composite systems were studied at different times by, for example, V.G. Boltyansky, L.T. Ashchepkov, and V.N. Rozova.

Necessary optimality conditions were derived in the general problem of control with intermediate constraints on the trajectory (see [2]). In the paper [7], the necessary optimality conditions for the problem with a phase space change were obtained.

The paper [14] considers the problem of optimal control of several objects with sequential mode of operation. The initial state of each subsequent object depends on the final state of

*Corresponding author: maximova-is@rudn.ru

the previous one. Each object is described by a system of ordinary differential equations on the interval of its action. Necessary and sufficient optimality conditions were obtained for the problem of optimal control with a criterion of sufficiently general form.

Necessary optimality conditions for the problem with a change of phase space were obtained by Maximova in [11]. The results were generalized to the case of several spaces.

In many works devoted to problems of this kind, the optimization issue is mainly studied. Meanwhile, typical theorems of the existence of optimal control assume the existence of at least one admissible control that generates a trajectory that satisfies the given boundary conditions, for example, a control that translates a trajectory from one given position to another. The latter problem is the essence of the controllability problem. Thus, the problem of controllability is important and relevant in solving optimal control problems.

In the paper [3], Bargsegyan V.R. considered a mathematical model of control of linear composite systems described at different time intervals by different differential equations and some finite relations for continuity of motion of composite systems. The analytical view of motion of composite systems is constructed, the properties of motion and the geometrical structure of the reachability region were investigated. Necessary and sufficient conditions for the complete controllability were formulated. The method of solving the problem of control of composite systems and the method of solving the problem of optimal control were proposed.

The monograph [4] is devoted to the control problems of composite linear dynamical systems and systems with multipoint intermediate conditions. Particular attention is paid to necessary and sufficient conditions for the complete controllability and observability of composite linear systems, which in the stationary case are comparable to the Kalman conditions by their completeness. Qualitative properties of controllability and observability of composite systems are revealed. Constructive methods for solving control problems of composite systems, systems with undivided multipoint intermediate conditions, with constraints on the values of different parts of the phase vector coordinates at intermediate points in time are proposed.

The issue of controllability for some classes of nonlinear systems with a phase space change was studied by the author in [12]. The paper consists of two sections devoted to the cases of nonlinear and linear systems, respectively. An example illustrating this approach is considered.

2. NONLINEAR SYSTEMS

Let x and y be two phase variables:

$$x = (x_1, \dots, x_n) \in X = \mathbb{R}^n, \quad y = (y_1, \dots, y_m) \in Y = \mathbb{R}^m.$$

The motion of the object is described by the following non-linear systems of differential equations:

$$\dot{x} = f(t, x(t), u(t)), \quad u(t) \in U, \quad t \in [0, \tau], \quad x \in X, \tag{2.1}$$

$$\dot{y} = g(t, y(t), v(t)), \quad v(t) \in V, \quad t \in [\tau, T], \quad y \in Y. \tag{2.2}$$

The class of admissible controls consists of all functions

$$u(\cdot) \in U = \{u(t) \in \mathbb{R}^n \mid u(\cdot) \in L_\infty[0, \tau];\ u(t) \in U_1 \subset \mathbb{R}^n,\ t \in [0, \tau]\},$$

$$v(\cdot) \in V = \{v(t) \in \mathbb{R}^m \mid v(\cdot) \in L_\infty[\tau, T];\ v(t) \in V_1 \subset \mathbb{R}^m,\ t \in [\tau, T]\},$$

where $U_1 \in \Omega(\mathbb{R}^n), V_1 \in \Omega(\mathbb{R}^m)$. Here $\Omega(\mathbb{R}^n)$ and $\Omega(\mathbb{R}^m)$ are the sets of all nonempty convex compact subsets of the spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

The functions $f(t, x(t), u(t))$, $g(t, y(t), v(t))$ are such that for systems (2.1) and (2.2) the existence and uniqueness theorem of the Cauchy problem is satisfied. Solutions of systems (2.1) and (2.2) at $t \in [0, \tau]$ and $t \in [\tau, T]$ are absolutely continuous functions that satisfy systems (2.1) and (2.2) almost everywhere on $[0, \tau]$ and $[\tau, T]$, respectively.

There is an initial set $M_0 \in \Omega(\mathbb{R}^n)$ and a non- intersecting convex "transition hypersurface" in the space X Γ. Let's suppose that τ is the smallest time moment at which the object reaches the hypersurface Γ. When an object moving according to the law (2.1) reaches the hypersurface Γ, it moves to the space Y given by the mapping $q : X \to Y$ (it is assumed that this mapping translates the convex set into the convex one), and further motion is performed in the space Y according to the law (2.2). Finally, the terminal set $M_1 \in \Omega(\mathbb{R}^m)$ (not overlapping with the set $q(\Gamma)$) is given in Y. A similar scheme of motion of an object was considered in [7]. The problem is to find the conditions under which the object described by the systems (2.1) and (2.2) will be controllable from M_0 to M_1.

Definition 2.1:
An object described by systems (2.1) and (2.2) is called controllable from M_0 to M_1, if there exist allowable controls $u(\cdot)$ and $v(\cdot)$ such that the corresponding solutions of the systems satisfy the boundary conditions $x(0) \in M_0$, $x(\tau) \in \Gamma$ and $y(\tau) = q(x(\tau))$, $y(T) \in M_1$.

Remark 2.1:
For system (2.1), consider the set $f(t, x(t), U)$ consisting of all vectors $f(t, x(t), u(t))$, where $u(t) \in U$. If $x(t)$ is a trajectory of system (2.1) corresponding to an admissible control $u(t)$, then for almost all $t \in [0, \tau]$ the inclusion

$$\dot{x}(t) \in f(t, x(t), U) \tag{2.3}$$

holds true. This leads us to the differential inclusion of

$$\dot{x} \in f(t, x, U). \tag{2.4}$$

Solutions of the differential inclusion (2.4) are absolutely continuous functions $x(t)$ defined on the interval $[0, \tau]$ that satisfy the inclusion (2.3) for almost all $t \in [0, \tau]$.

So, under rather general assumptions, system (2.1) is equivalent to a differential inclusion (2.4), i.e. for any solution $x(\cdot)$ of the inclusion (2.4) there exists a valid control $u(\cdot)$ such that the function $x(\cdot)$ is the path of system (2.1) with the control $u(\cdot)$. This question is discussed in detail in [8].

Taking into account previous remarks, we shall consider differential inclusion (2.4) instead of nonlinear system (2.1). Let us denote $f(t, x, U)$ by $F(t, x)$, then the motion of the controlled object is described by the differential inclusion

$$\dot{x} \in F(t, x), \;\; t \in [0, \tau]. \tag{2.5}$$

Similarly, the motion of the controlled object in the space $Y = \mathbb{R}^m$ is described by the differential inclusion

$$\dot{y} \in G(t, y), \;\; t \in [\tau, T]. \tag{2.6}$$

The motion of an object from the space X to the space Y is described above.

Definition 2.2 ([6])**:**
A multi-valued mapping $F(t, x)$ is called concave by x on a set $M \subset X$ if for any points $x_1, x_2 \in M$ and any $\lambda \in [0, 1]$ the condition

$$\lambda F(t, x_1) + (1 - \lambda)F(t, x_2) \subset F(t, \lambda x_1 + (1 - \lambda)x_2)$$

holds true.

We note that the above condition implies the set $F(t, x)$ is convex for every $x \in M$ (see [6]).

The reachability set $K(t)$ for each $t \in [0, \tau]$ consists of all points $x(t) \in \mathbb{R}^n$, where $x(t)$ is the solution of the inclusion (2.5) with initial condition $x(0) \in M_0$.

Consider the motion of an object in the space X from the initial set M_0 to the transition hypersurface Γ. Let us suppose that the mapping $F(t, x)$ is concave by x on the reachability set $K(\tau)$ for all $t \in [0,\tau]$. It is known (see [6]) that in this case the family of all solutions on the interval $[0,\tau]$ with initial condition $x(0) \in M_0$ is a convex set in the space $C[0,\tau]$. The convexity of the solution family implies that the reachability set $K(\tau)$ is convex (the reverse is not true).

So, under the above assumptions, the reachability set $K(\tau)$ is convex. The intersection of $K(\tau)$ with the transition hypersurface Γ, we obtain the set

$$K_1(\tau) = K(\tau) \cap \Gamma.$$

Suppose that there exists $\tau : K_1(\tau) \neq \emptyset$. Then $K_1(\tau)$ is convex as intersection of two convex sets. Let us transform the set $K_1(\tau)$ as follows: $K_2(\tau) = q(K_1(\tau))$, where $q : X \to Y$.

The resulting set $K_2(\tau)$ is convex due to the properties of mapping q. The set $K_2(\tau)$ is the initial set when the object moves in the space Y into the set M_1.

In the space Y we obtain the following controllability problem: whether the system (2.2) is controllable from the set $K_2(\tau)$ to the set M_1 at the time interval $[\tau, T]$. Let us denote the system reachability set (2.2) from $K_2(\tau)$ at time T by $K_3(T)$. Let us suppose that the mapping $G(t, y)$ is concave on y on the reachability set $K_3(T)$ and $K_3(T)$ is compact. Then, for the controllability of the system (2.2) it is sufficient that $K_3(T) \cap M_1 \neq \emptyset$ or, in the terms of support functions, the inequality

$$c(K_3(T), \psi) + c(M_1, -\psi) \geq 0$$

holds true for any $\psi \in \mathbb{R}^m$ (see [5]).

Thus, the controllability conditions from the set $M_0 \subset X$ to the set $M_1 \subset Y$ for systems (2.1) and (2.2) can be expressed as the following statement:

Theorem 2.1:
Under the above assumptions, for the controllability of an object described by systems (2.1) and (2.2) on the time interval $[0, T]$ it is sufficient that

$$c(K_3(T), \psi) + c(M_1, -\psi) \geq 0,$$

for any $\psi \in \mathbb{R}^m$.

Remark 2.2:
For autonomous system (2.2), one can consider the motion of an object in Y space in backward time and get the reachability set $K_4(T)$ from the set M_1 on the of the transition hypersurface Γ. Then the controllability condition for the systems (2.1) and (2.2) on the time interval $[0, T]$ is a non-empty intersection of the sets $K_4(T)$ and $K_2(\tau)$.

3. LINEAR SYSTEMS

In this part of the paper we consider the case where the motion of the controlled object is described by a linear system of differential equations. Let us suppose that systems (2.1) and (2.2) are linear, then we obtain the following problem.

3.1. Problem statement

Let x and y be two phase variables:

$$x = (x_1, \ldots, x_n) \in X = \mathbb{R}^n, \quad y = (y_1, \ldots, y_m) \in Y = \mathbb{R}^m.$$

The motion of the object is described by the following linear systems of differential equations:

$$\dot{x} = Ax + u, \quad u(t) \in U, \quad t \in [0, \tau], \quad x \in X. \tag{3.7}$$

$$\dot{y} = By + v, \quad v(t) \in V, \quad t \in [\tau, T], \quad y \in Y. \tag{3.8}$$

The class of admissible controls is the sets of functions

$$\{u(\cdot) \in L_\infty([0,\tau],\mathbb{R}^n) \mid u(t) \in U,\, t \in [0,\tau]\},$$

$$\{v(\cdot) \in L_\infty([\tau,T],\mathbb{R}^m) \mid v(t) \in V,\, t \in [\tau,T]\}.$$

In the space X there is an initial set $M_0 \in \Omega(\mathbb{R}^n)$ and a non-overlapping convex the transition hypersurface Γ. Let τ be the smallest time moment at which the object reaches the hypersurface Γ. The motion of the object from one space to another one occurs in the same way as in the first part of the paper. The problem is to find the conditions under which the object described by systems (3.7) and (3.8) is controllable from M_0 to M_1.

3.2. Main result

The reachability set $K(\tau)$ for system (3.7) is the set of endpoints of trajectories of system (3.7) with initial set M_0, corresponding to all possible admissible controls $u(\cdot) \in U$ and considered at time τ. Due to linearity of system (3.7), the reachability set can be presented in the explicit form:

$$K(\tau) = e^{\tau A} M_0 + \int_0^\tau e^{(\tau-s)A} U ds, \tag{3.9}$$

Here $e^{\tau A}M_0$ is the image of the set M_0 under the linear transformation $e^{\tau A}$, and the integrand is a multivalued mapping obtained for all $s \in [0,\tau]$ as the image of the set U under the linear transformation $e^{(\tau-s)A}$. To find the reachability set with the initial convex set M_0, let us first calculate its support function and then reconstruct the set $K(\tau)$ by its support function. Thus, the support function of the reachability set has the form

$$c(K(\tau),\psi) = c(M_0, e^{\tau A^*}\psi) + \int_0^\tau c(U, e^{sA^*}\psi) ds. \tag{3.10}$$

Since the initial set is convex, the reachability set is also convex (see [5]). Then the set $K(\tau)$ reconstructed from the support function is intersected at the time τ with the the transition hypersurface Γ. By assumption, this yields the convex set

$$K_1(\tau) = K(\tau) \cap \Gamma.$$

Assume that there exists τ such that $K_1(\tau) \neq \emptyset$. Let us transform the set $K_1(\tau)$ as follows: $K_2(\tau) = q(K_1(\tau))$, where $q : X \to Y$.

Then the set $K_2(\tau)$ is convex due to properties of q. The set $K_2(\tau)$ is the initial set for the system (3.8) when the object moves in space Y to the set M_1.

So, we have formulated the following controllability problem in the space Y: whether system (3.8) is controllable from the set $K_2(\tau)$ to the set M_1 on the time interval $[\tau,T]$. Let us define the controllability function $\varphi : \mathbb{R}^m \to \mathbb{R}^1$ by the relation

$$\varphi(\psi) = c(K_2(\tau), e^{(T-\tau)B^*}\psi) + c(M_1,-\psi) + \int_0^{T-\tau} c(V, e^{sB^*}\psi) ds, \tag{3.11}$$

see [5]). According to the controllability theorem [5], an object is controllable on the time segment $[\tau,T]$ from the set $K_2(\tau)$ to the set M_1 if and only if for the controllability function is non-negative, i.e.

$$\varphi(\psi) \geq 0 \quad \forall \psi \in S,$$

where S is the unit sphere in $\mathbb{R}^n$. In turn, this is equivalent to the condition

$$\varphi_0 = \min_{\psi \in S} \varphi(\psi) \geq 0.$$

Applying this theorem, we obtain the following result:

Theorem 3.1:
Under the above assumptions, for the controllability of an object described by systems (3.7) and (3.8) on the interval $[0, T]$, it is sufficient that the controllability function

$$\varphi(\psi) = c(K_2(\tau), e^{(T-\tau)B^*}\psi) + c(M_1, -\psi) + \int_0^{T-\tau} c(V, e^{sB^*}\psi)\, ds$$

is non-negative for any $\psi \in S$.

3.3. Example

Let $X = \mathbb{R}^3$ and $Y = \mathbb{R}^2$ and the motion of an object is described by the following systems of equations:

$$\begin{cases} \dot{x}_1 = x_2 + u_1, \\ \dot{x}_2 = -x_1 + u_2, \qquad |u| \leq 1, \;\; u = (u_1, u_2, u_3) \in \mathbb{R}^3, \\ \dot{x}_3 = u_3, \qquad t \in [0, \tau], \end{cases} \tag{3.12}$$

$$\begin{cases} \dot{y}_1 = y_1 + v_1, \qquad |v| \leq 1, \;\; v = (v_1, v_2) \in \mathbb{R}^2, \\ \dot{y}_2 = v_2, \qquad t \in [\tau, T]. \end{cases} \tag{3.13}$$

In the space $\mathbb{R}^3$, consider the initial set $M_0 = \{(0, -1, 0)\}$ and the transition hypersurface

$$\Gamma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_2 = 0, x_3 \geq 0\}.$$

By τ denote the smallest time moment at which the object reaches the hypersurface Γ. The mapping that makes the transition from $\mathbb{R}^3$ to $\mathbb{R}^2$ has the form

$$q(x_1, x_2, x_3) = (x_1 + \sin\tau, x_3) = (y_1, y_2).$$

In the space $\mathbb{R}^2$, consider the set $M_1 = S_1(0, -3)$. The problem is to find conditions for an object described by systems (3.12) and (3.13) be controllable from M_0 to M_1 on the time interval $[0, T]$.

Denote by $K(\tau)$ the reachability set of system (3.12) from the set M_0 at time τ. First, consider the motion of the object in the space $\mathbb{R}^3$. The support function of the reachability set $K(\tau)$ from (3.10) reads

$$c(K(\tau), \psi) = -\psi_1 \sin\tau - \psi_2 \cos\tau + \tau\|\psi\|,$$

where $\|\psi\|$ is the standard Euclidean norm of the vector ψ. Restoring the set $K(\tau)$ by the its support function $c(K(\tau), \psi)$, one can easily see that the reachability set $K(\tau)$ is the circle of radius τ with the center at the point $(-\sin\tau, -\cos\tau, 0)$. The intersection of the reachability set with the transition hypersurface Γ occurs at $\tau > 1$ and it is a segment with the endpoints

$$(-\sin\tau - \sqrt{\tau^2 - \cos^2\tau}, 0, 0); \quad (-\sin\tau + \sqrt{\tau^2 - \cos^2\tau}, 0, 0).$$

We pass to the space $\mathbb{R}^2$ under the action of the mapping q. We obtain the set $K_1(\tau)$, which is the initial for system (3.13) when the object moves in the space $\mathbb{R}^2$. The set $K_1(\tau)$

is the segment with the endpoints

$$(-\sqrt{\tau^2 - \cos^2\tau}, 0), \quad (\sqrt{\tau^2 - \cos^2\tau}, 0).$$

Thus, we have the following controllability problem in $\mathbb{R}^2$: find such conditions on systems (3.12) and (3.13) that the object described by these systems is controllable from the set $K_1(\tau)$ to the set M_1. The controllability function in this case has the form

$$\varphi(\psi) = \sqrt{\tau^2 - \cos^2\tau} \cdot |\psi_1| + \|\psi\| + 3\psi_2 + \int\limits_0^{T-\tau} \sqrt{\psi_1^2 + (s\psi_1 + \psi_2)^2}\, ds, \tag{3.14}$$

where $\|\psi\| = 1$, $\psi = (\psi_1, \psi_2, \psi_3)$. The controllability function (3.14) reaches its minimum at $\psi_2 = -1$, then due to the fact that $\|\psi\| = 1$, $\psi_1 = \psi_3 = 0$. This yields

$$\min_{\psi \in S} \varphi(\psi) = T - (2 + \tau).$$

Therefore, when $T > 2 + \tau$, the object described by systems (3.12) and (3.13) is controllable on the time interval $[0, T]$ from the set M_0 to the set M_1.

4. CONCLUSION

A special class of controlled differential systems called *hybrid* or *composite* is considered. Such systems are characterized by the condition that at different time intervals the motion of the object is described by different differential equations and some links for trajectory mating.

Sufficient conditions of the controllability of such systems from the initial set in one space to the terminal set of the other space are obtained. Sufficient controllability conditions for the nonlinear case are obtained using the apparatus of convex analysis, the theory of multivalued mappings, and the control theory. This class of composite systems has not been considered before.

An actual application of the proposed approaches and results lies in the field of mathematical biology (see [15]) and mathematical economics with control input, for instance, dynamic market models with continuous, see, e.g., [1, 13].

REFERENCES

1. Arutyunov, A. V. & Pavlova, N. G. (2022). Equilibrium in market models described by differential equations, *Differ. Equ.*, **58**(9), 1267–1276.
2. Ashepkov, L. T. (1981). Optimal system control with intermediate conditions, *Applied mathem. mech.*, **45**(2), 215–222. [in Russian]
3. Bargsegayn, V. R. (2012). Constructive approach to the study of control problems of linear composite systems, *Control problems*, **4**, 11–17.
4. Bargsegayn, V. R. (2016). *Controls of composite dynamical systems with multipoint intermediate conditions*, Moscow, Russia: Nauka [in Russian].
5. Blagodatskikh, V. I. (2001). *Introduction to optimal control*, Moscow, Russia: Visshaya shkola, [in Russian].
6. Blagodatskih, V. I. & Filippov A. F. (1986). Differential inclusions and optimal control, *Proc. Steklov Inst. Math.*, **169**, 199–259.
7. Boltyanskii, V. G. (1983). The optimization problem with phase space change, *Differ. Equ.*, **19**(3), 518–521.
8. Borisovich, Yu. G., Gel'man, B. D., Myshkis, A. D. & Obukhovskii, V. V. (1999). *Introduction to the Theory of Multivalued Mappings and Differential Inclusions*, Dordrecht, Netherlands: Kluwer Academic Publishers.

9. Kalman, R., Falb, P., & Arbib, M. (1969). *Topics in mathematical system theory*, New York, NY: McGraw-Hill.
10. Kramar, V., Alchakov, V., & Osadchenko, A. (2021). The Optimal Control of the Vessel's Automatic Dynamic Positioning System Under Deviation, *Adv. Syst. Sci. Appl.*, **21**(1), 1–10.
11. Maksimova, I. S. (2012). Necessary conditions of optimality in the problem with changing phase spaces, *RUDN Reports. Series: Mathematics. Computer science. Physics*, **4**, 5–14. [in Russian]
12. Maksimova, I. S. & Rozova, V. N. (2017). Local controllability in the problem with phase space change, *RUDN Reports. Series: Natural and Technical Sciences* **25**(4), 331–338. [in Russian]
13. Pavlova, N. G. (2019). Study of the continuous-time open dynamic Leontief model as a linear dynamical control system, *Differ. Equ.*, **55**(1), 113–119.
14. Rozova, V. N. (2006). Optimal control of step systems, *RUDN Reports. Series: Natural and Technical Sciences*, **3**, 15–23. [in Russian]
15. Svirezhev, Yu. M. (1987). *Nonlinear Waves, Dissipative Structures, and Catastrophes in Ecology*, Moscow, Russia: Nauka. [in Russian]
16. Velichenko, V. V. (1966). On optimal control problems for equations with discontinuous right sides, *Automation and telemechanics*, **7**, 20–30. [in Russian]

