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On a Pseudodifferentional Calculation in Group Algebras

Andronik A. Arutyunov[1,2*]

[1] *V.A. Trapeznikov Institute of Control Sciences of Russian Academy of Sciences, Moscow, Russia*
[2] *Moscow Institute of Physics and Technology, Moscow, Russia*

Abstract: The paper is devoted to the construction of pseudodifferentional operators over group algebras. An algebra of such operators is generated by central derivations. The equipping with an order function of this algebra is also shown. Some examples of such operators are given and some further ideas for the development of the new pseudodifferentional calculus are presented.

Keywords: pseudodifferential operators, group algebras, derivations, pseudodifferential calsulus

1. INTRODUCTION

In A.N. Parshin's paper [12] the following construction was proposed. Let d be a derivation of some ring R. Then the left module of formal expressions $L = \sum\limits_{i \in \mathbb{Z}} a_i d^i$ can be understood as an analog of the pseudo-differential operators algebra. Further development of this logic leads us to the Shur-Sato theory. For the most complete and up-to-date study, see [14].

Note also that similar ideas to describe pseudodifferential operators as noncommutative power series in which derivations are substituted as variables have been previously proposed in [9]. In particular, the formalism of the so-called μ-calculus and the corresponding μ-algebras was studied.

The study of differentials over noncompact spaces is motivated by the author's earlier work in which the studies of classical and nonlocal pseudodifferential operators over a Euclidean space [3, 6]. The study of operators on Euclidean spaces has also been carried in [11]. Issues related to the calculus of Pseudodifferentional operators on abelian groups were considered in [8, 10].

In this paper, we will propose a construction of the algebra of pseudo-differential operators in group algebra, using previously obtained results to describe derivations from papers [2, 4, 5]. Using the aforementioned idea from [12] we will take as derivations the central derivations that were studied earlier in [2].

The description of all sorts of derivations as variables is hampered by considerable computational difficulties and the problematic nature of the study of outer derivations (see results of [4] and [5]).

In the first section we give a short overview of the concept of central derivations and prove their necessary conditions. Especially the formula for the power (see lemma 2.2). In the second section we build firstly the construction for 1-dimensional algebra of pseudodifferentional operators generated by single derivation. Then by induction we are constructing the algebra of "large" algebra generated by the cortege of central derivations.

*Corresponding author: andronick.arutyunov@gmail.com

In the theorem we prove that this is a well-ordered algebra. We give some examples, the most intresting is the analog of the Laplace operator (see examples 3.1 and 3.4).

2. CENTRAL DERIVATIONS

Recall that a derivation $d : \mathbb{C}[G] \to \mathbb{C}[G]$ is a such linear operator that

$$d(uv) = d(u)v + ud(v), \quad \forall u, v \in \mathbb{C}[G].$$

Central derivations were presented in [2]. It is easy to see that they form a Lie algebra, i.e. the commutator of central derivations is a central derivation. Central derivations give an example of derivations which are not inner – they cannot be represented as a commutator $x \to [a, x]$ for some $a \in \mathbb{C}[G]$. In other words, the intersection of the central derivations space and inner derivations space is trivial.

Moreover, such derivations are, in a sense, fundamentally outer, i.e., they cannot in any sense be approximated by inner derivations. For more details on this, see the paper [4] which shows that central derivations are quasi-outer.

Recall the definition of central derivations. We are fixing $z \in Z(G)$ central element, and a homomorphism $\tau : G \to \mathbb{C}$ i.e. $\tau(uv) = \tau(u) + \tau(v), u, v \in G$.

And a formula for the action of central derivations on the basis element $g \in G \subset \mathbb{C}[G]$ is

$$d_{\tau,z} : g \mapsto \tau(g)gz. \tag{2.1}$$

Sure the following identity holds for central elements z, z_1

$$d_{\tau,z} \cdot z_1 = d_{\tau,z\cdot z_1}. \tag{2.2}$$

From the definition we can get the formula for the derivation commutator.

Lemma 2.1:
The commutator of central derivations is the central derivation such that

$$[d_{\tau_1,z_1}, d_{\tau_2,z_2}] = d_{\tau_2(g)\tau_1(z_2)-\tau_1(g)\tau_2(z_1),z_1z_2}. \tag{2.3}$$

Furthermore the derivations corresponding to the same homomorphism τ commute.

Proof
Directly from the formula (2.1) we have the following

$$\begin{aligned}[d_{\tau_1,z_1}, d_{\tau_2,z_2}](g) = (\tau_1(gz_1)(\tau_2(g) - \tau_2(gz_1)\tau_1(g))gz_1z_2 = \\ = (\tau_2(g)\tau_1(z_2) - \tau_2(z_1)\tau_1(g))gz_1z_2.\end{aligned}$$

Central elements z_1, z_2 are fixed so $\tau_1(z_2)$ and $\tau_2(z_1)$ do not depend from the element g. So the mapping $(\tau_2(g)\tau_1(z_2) - \tau_2(z_1)\tau_1(g)) : G \to \mathbb{C}$ is a linear combination of homomorphisms so it as also a homomorphism.

The permutability of differentiations corresponding to the same homomorphism follows from the proved formula. □

We can also calculate the power of the central derivation.

Lemma 2.2:
The formula for the central derivation of the power $n \in \mathbb{Z}$ is following

$$d^n_{\tau,z}(g) = \prod_{j=0}^{n-1}(\tau(g) + j\tau(z))^{\mathrm{sign}(n)}gz^n.$$

Note that in the case $n = -1$ we have

$$d^{-1}_{\tau,z}(g) = \frac{1}{\tau(g) - \tau(z)} g z^{-1}. \tag{2.4}$$

Proof
First we consider the case $n > 0$.
Let us fix central derivations for $i = 1 \dots n$:

$$d_i : g \mapsto \tau_i(g) g z_i. \tag{2.5}$$

So the composition is

$$d_1 d_2(g) = d_1(\tau_2(g) g z_2) = \tau_2(g) d_1(g z_2) = \tau_2(g)\tau_1(g z_2) g z_1 z_2.$$

That means that for n central derivatons we get the following

$$d_1 \dots d_n(g) = \tau_n(g)\tau_{n-1}(g z_{n-1}) \dots \tau_1(g z_{n-1} \cdot \dots \cdot z_1) g z_1 \dots z_n.$$

We are calculating the power so $\tau_i \equiv \tau_j, z_i \equiv z_j$. Hence:

$$d^n(g) = \tau(g)\tau(gz) \dots \tau(g z^{n-1}) g z^n. \tag{2.6}$$

We can rewrite the last formula in the following way considering that τ is a homomorphism to $(\mathbb{C}, +)$ we have that $\tau(z^n) = n\tau(z)$:

$$d^n(g) = \tau(g)(\tau(g) + \tau(z)) \dots (\tau(g) + (n-1)\tau(z)) g z^n. \tag{2.7}$$

In other words

$$d^n(g) = \prod_{j=0}^{n-1} (\tau(g) + j\tau(z)) g z^n. \tag{2.8}$$

That gives us the statement.
The proof for the case $n < 0$ is simillar. □

Example 2.1:
If $\tau(z) = 0$ then we get

$$d^n(g) = \tau^n(g) g z^n. \tag{2.9}$$

We will use the following coefficient-function $\theta^n_\tau(g) : G \to \mathbb{C}$,

$$\theta^n_{\tau,z}(g) := \prod_{j=0}^{n-1} (\tau(g) + j\tau(z)) \tag{2.10}$$

This function satisfies the following asymptotic condition

Proposition 2.1:
For $n \neq 0$

$$\frac{|\theta^n_{\tau,z}(g)|}{|g|^n} \longrightarrow \mathbf{const}\,, |g| \to \infty. \tag{2.11}$$

Proof
Follows from Lemma 2.2. □

3. PSEUDODIFFERENTIONAL OPERATORS OVER $\mathbb{C}[G]$

3.1. Case of one derivation

Let us consider a Laurent polynomial $f(x)$ with coefficients in $\mathbb{C}$

$$f(x) = \sum_{k=-\infty}^{N} f_k x^k, \quad f_k \in \mathbb{C}. \tag{3.12}$$

For such denoted f with $f_N \neq 0$ we will say that $\deg(f) = N$.

Proposition 3.1:
For a central derivation d *the operator* $f(d) : \mathbb{C}[G] \to \mathbb{C}[G]$ *is well defined for all* g *such that* $\theta^n_{\tau,z}(g) \neq 0$.

Proof
Almost all coefficients f_k in (3.12) are equal to zero. Hence, if $\theta^n_{\tau,z}(g) \neq 0$ then each term in (3.12) is well defined by lemma 2.2. □

Definition 3.1:
For fixed $N \in \mathbb{Z}$ *and fixed homomorphism* $\tau : G \to \mathbb{C}$ *the algebra generated by operators* $f(d_{\tau,*})$ *from* (3.12) *and multiplying on central elements will be denoted as* $\Psi^N_\tau \mathbb{C}[G]$,

$$\Psi^N_\tau \mathbb{C}[G] = \langle f(d_{\tau,z}, \times z \rangle, \quad \deg f \leq N, z \in Z(G).$$

Therefore, $\Psi^\infty_\tau \mathbb{C}[G] := \cup_{N\in\mathbb{Z}} \Psi^N_\tau \mathbb{C}[G]$.

Certainly holds the embedding $\Psi^N_\tau \mathbb{C}[G] \subset \Psi^{N+k}_\tau \mathbb{C}[G]$, for $k \in \mathbb{N}$.

Considering the identity (2.2) we get that each operator $A \in \Psi^N$ can be written for the basis element g as

$$A(g) = \sum_{i\leq N} a_i d^i_{\tau,e}(g) z_i, \quad z_i \in Z(G).$$

So we can define the order for operators in following classic form.

Definition 3.2:
For the operator $A \in \Psi^\infty_\tau \mathbb{C}[G]$

$$\mathrm{ord}(A) := \min\{N \in \mathbb{Z} | A \in \Psi^N_\tau \mathbb{C}[G]\}. \tag{3.13}$$

It is well known from the classical PDO-calculation theory that for a good symbol-calculus we need the order function which transforms the space of pseudo-differential operators into a graded algebra.

Proposition 3.2:
$\Psi^\infty_\tau \mathbb{C}[G]$ *is a commutative algebra such that*

$$\mathrm{ord}(A \circ B) \leq \mathrm{ord}(A) + \mathrm{ord}(B); \tag{3.14}$$

Proof
The commutativity of algebra follows from Lemma 2.1. The formula for orders follows from Lemma 2.2. □

Example 3.1:
Let us give some examples of operators from $\Psi^\infty_\tau \mathbb{C}[G]$ *with their orders*

operator	*N=*
$\Delta_{\tau,e}(g) := \tau^2(g)g$	*2*
$\Delta_{\tau,z}(g) := \tau^2(g)gz, z \in Z(G)$	*2*
$id_z(g) := gz, z \in Z(G)$	*0*
$\Delta_{\tau,z}^{-1}(g) := \frac{1}{\tau^2(g)}gz^{-1}$	*-2*

It is easy to see that for such g that $\tau(g) \neq 0$ holds $\Delta_{\tau,z} \circ \Delta_{\tau,z}^{-1} = id_e$. It is also important to note that multiplying operator id_z saves the order

$$\forall z \in Z(G): \quad id_z(\Psi_\tau^N \mathbb{C}[G]) = \Psi_\tau^N \mathbb{C}[G]. \tag{3.15}$$

We can also show an analogue of the composition formula.

Proposition 3.3:
For $z \in Z(G)$ – central element of infinite order, and Laurant polynomials f, h the following composition formula holds for the central derivation $d := d_{\tau,z}$:

$$((h \circ f)(d))(g) = f(h(d))(g) = \sum_{i\in\mathbf{Z}} \sum_{k+l=i} f_k h_l \theta_\tau^k(g)\theta_\tau^l(g) g z^i. \tag{3.16}$$

Proof
Immediately follows from Lemma 2.2 □

For a Laurant polynomial from (3.12) using Lemma 2.2 we get that

Example 3.2:
For a central derivation $d_{\tau,e}$ for a neutral group element e

$$f(d_{\tau,e})(g) = \sum_{k=-\infty}^{N} f_k(\tau(g))^k g = f(\tau(g))g. \tag{3.17}$$

The situation is arranged similarly for the finite-ordered central elements.

Example 3.3:
Let $z \in Z(G)$ such that $z^k = e$. We want to calculate $f(d_{\tau,z})$. First we have to rewrite $f = f^1 + \cdots + f^k$ where

$$f^i(x) = \sum_{j\in\mathbf{Z}} f_{jk+i} x^{jk+i}. \tag{3.18}$$

After simple calcultaions we get the following

$$f(d_{\tau,z})(g) = \sum_{j=1}^{k} f^j(\tau(g)) g z^j. \tag{3.19}$$

3.2. Generalized $\Psi^\infty(\mathbb{C}[G])$ algebra

If we are working with the pair of central derivations their commutator is also a central derivaion by the formula (2.3). Because this commutator is not trivial, the algebra which is generated by all Laurant polynimials with the values in central derivations, is not commutative and is arranged more cunningly.

We will define a generalized algebra $\Psi^\infty_{\tau_1,\ldots,\tau_k}(\mathbb{C}[G])$ by induction.

Fix an algebra $\Psi^{\infty}_{\tau_1,\dots,\tau_{k-1}}(\mathbb{C}[G])$ with given homomorphisms $\tau_1,\dots,\tau_{k-1}: G\to\mathbb{C}$ and equipped with the order function $\mathrm{ord}:\Psi^{\infty}_{\tau_1,\dots,\tau_{k-1}}(\mathbb{C}[G])\to\mathbb{Z}$ such that the following holds

$$\mathrm{ord}(A\circ B)\le \mathrm{ord}(A)+\mathrm{ord}(B);$$
$$\mathrm{ord}([A,B])\le \mathrm{ord}(A)+\mathrm{ord}(B)-1.$$

Then for a homomorphism $\tau_k: G\to\mathbb{C}$ we will define an algebra $\Psi^{\infty}_{\tau_1,\dots,\tau_k}(\mathbb{C}[G])$ as

$$\Psi^{\infty}_{\tau_1,\dots,\tau_k}(\mathbb{C}[G])=\langle\Psi^{\infty}_{\tau_1,\dots,\tau_{k-1}}(\mathbb{C}[G]),d_{\tau,e}\rangle.$$

The order function is defined as following

$$\mathrm{ord}(A\circ d^n_{\tau,e}):=\mathrm{ord}(A)+n, n\in\mathbb{Z}, \tag{3.20}$$

and naturally continues on linear combinations .

So we are ready to formulate and prove the theorem

Theorem 3.1:
Algebra $\Psi^{\infty}_{\tau_1,\dots,\tau_k}(\mathbb{C}[G])$ is well defined and the order function satisfies conditions

$$\mathrm{ord}(A\circ B)\le \mathrm{ord}(A)+\mathrm{ord}(B);$$
$$\mathrm{ord}([A,B])\le \mathrm{ord}(A)+\mathrm{ord}(B)-1.$$

Proof

We need to check for $\varepsilon_{1,2}=\pm1$ that the following inequality holds

$$\mathrm{ord}([d^{\varepsilon_1}_{\tau_1,z_1}\circ d^{\varepsilon_2}_{\tau_2,z_2}])<\varepsilon_1+\varepsilon_2. \tag{3.21}$$

If $\varepsilon_1=\varepsilon_2=1$ it immediately follows from 2.1.

Now let us consider $\varepsilon_1=1,\varepsilon_2=-1$. Then we have the following calculation for the meaning of derivations $d_{\tau_1,z_1}, d_{\tau_2,z_2}$ on the basis element g:

$$(d_{\tau_1,z_1}\circ d^{-1}_{\tau_2,z_2})(g)=d_{\tau_1,z_1}\left(\frac{gz_2^{-1}}{\tau_2(g)-\tau_(z_2)}\right)=\frac{\tau_1(g)-\tau_1(z_2)}{\tau_2(g)-\tau_(z_2)}gz_1z_2^{-1}=$$
$$=\frac{z_2^{-1}}{\tau_2(g)-\tau_(z_2)}\left(d_{\tau_1,z_1}(g)-\tau_1(z_2)gz_1\right)=(d^{-1}_{\tau_2,z_2}\circ d_{\tau_1,z_1})(g)-\tau_1(z_2)d^{-1}_{\tau_2,z_2}(g)z_1.$$

Hence we have the following operator view of the commutator:

$$\left[d_{\tau_1,z_1},d^{-1}_{\tau_2,z_2}\right]=-\tau_1(z_2)d^{-1}_{\tau_2,z_2}\times z_1. \tag{3.22}$$

Considering that $\tau_1(z_2)$ is a constant we get that $\mathrm{ord}(\left[d_{\tau_1,z_1},d^{-1}_{\tau_2,z_2}\right])=-1$.

Similarly for $\varepsilon_1=\varepsilon_2=-1$ we get that

$$(d^{-1}_{\tau_1,z_1}\circ d^{-1}_{\tau_2,z_2})(g)=d^{-1}_{\tau_1,z_1}\left(\frac{gz_2^{-1}}{\tau_2(g)-\tau_2(z_2)}\right)=$$
$$=\frac{gz_1^{-1}z_2^{-1}}{(\tau_2(g)-\tau_2(z_2))(\tau_1(gz_2^{-1})-\tau_1(z_1))}=\frac{1}{\tau_2(g)-\tau_2(z_2)}d^{-1}_{\tau_1,z_1z_2}(g)=$$
$$=z_2\left(d^{-1}_{\tau_2,z_2}\circ d^{-1}_{\tau_1,z_1z_2}\right)(g).$$

Hence for the commutator we have the following formula:

$$[d^{-1}_{\tau_1,z_1}, d^{-1}_{\tau_2,z_2}] = \left(d^{-1}_{\tau_2,z_2} \circ d^{-1}_{\tau_1,z_1z_2}\right)(z_2 - e). \tag{3.23}$$

The operator on the right hand has the order -2, which is the required result.

Summarizing Lemma 2.1 and formulas (3.22),(3.23) we see that order function defined by (3.20) satisfies conditions of our theorem. □

We can define the spaces $\Psi^N_{\tau_1,\ldots,\tau_k}(\mathbb{C}[G])$ with the help of out order function in following way:

$$\Psi^N_{\tau_1,\ldots,\tau_k}(\mathbb{C}[G]) := \{A \in \Psi^\infty_{\tau_1,\ldots,\tau_k}(\mathbb{C}[G]) | ord(A) \leq N\}.$$

That gives us the structure of filtred algebra in $\Psi^\infty_{\tau_1,\ldots,\tau_k}(\mathbb{C}[G])$ by the theorem.

Also it is easy to see that if, for instance, τ_k are equal (or can be presented as linear combination as in Lemma 2.1) then new algebra is precisely the algebra from the previous step. But it is superfluous variable for the general construction.

We can improve Example 3.1 for the generalized construction of pseudodifferentional operators for the cortege of homomorphisms $\{\tau_1, \ldots, \tau_k\}$

Example 3.4:
Let us give some examples of operators from $\Psi^\infty_{\tau_1,\ldots,\tau_k}\mathbb{C}[G]$ with their orders:

operator	*N=*
$\Delta_{\tau_1,\ldots,\tau_k,z}(g) := (\tau_1^2(g) + \cdots + \tau_k^2(g))gz, z \in Z(G)$	*2*
$\Delta^{-1}_{\tau_1,\ldots,\tau_k,z}(g) := \frac{1}{\tau_1^2(g)+\cdots+\tau_k^2(g)} g z^{-1}$	*-2*

We can define the main symbol of the operator $A \in \Psi^N_{\tau_1,\ldots,\tau_k}\mathbb{C}[G]$ as an element in factor algebra $\Psi^N_{\tau_1,\ldots,\tau_k}\mathbb{C}[G]/\Psi^{N-1}_{\tau_1,\ldots,\tau_k}\mathbb{C}[G]$. Immediately from Theorem 3.1 we get that

Corollary 3.1:
Main symbols form a commutative algebra.

Conclusion

In conclusion, we would like to note the following. For such a well-defined algebra of pseudo-differential operators, it becomes correct to investigate the index of elliptic operators: operators with invertible main symbol. There is also an opportunity to construct a symbolic calculus. In particular, the question of calculating the parametrix is of interest.

The areas of interest seem to be the following

- Is there a clear formula for the composition?
- Is there an equipment of the group algebra with the Sobolev structure such that an pseudodifferentional operator is bounded as an operator between the corresponding spaces (with respect to the order)?
- Is there an analogue of the index formula for such operators?

Another area of activity may also be the transfer of the results obtained to other classes of operators. In particular the case of $(\sigma, \tau)-$twisted operators. The case of derivations was investigated in [1]. For a more complete overview see [7].

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REFERENCES

1. Alekseev, A., Arutyunov, A. & Silvestrov, S. (2020, August 2) *On* (σ, τ)-*derivations of group algebra as category characters*. [Online] Available: https://arxiv.org/abs/2008.00390
2. Arutyunov, A. (2020) Derivation Algebra in Noncommutative Group Algebras, *Differential equations and dynamical systems. Collected papers. Proc. Steklov Inst. Math. Steklova*, **308**, 28–41.
3. Arutyunov, A. (2015) Reduction of Nonlocal Pseudodifferential Operators on a Noncompact Manifold to Classical Pseudodifferential Operators on a Double-Dimensional Compact Manifold, *Math. Notes*, **97**(4), 502–509.
4. Arutyunov, A. & Alekseev, A. (2019) Cohomology of n-categories and derivations in group algebras, *Topology and its Applications*, **275**, 107002.
5. Arutyunov, A. & Mishchenko, A. (2019) A smooth version of Johnson's problem on derivations of group algebras, *Sb. Math.*, **210**(6), 756–782.
6. Arutyunov A., & Mishchenko, A. (2013) Reduction of the Calculus of Pseudodifferential Operators on a Noncompact Manifold to the Calculus on a Compact Manifold of Doubled Dimension, *Math. Notes*, **94**(4), 455–469.
7. Hartwig, J., Larsson, D. & Silvestrov, S. (2006) Deformations of Lie algebras using σ derivations, *J. Algebra*, **295**(2), 314–361.
8. Kumar V. & Wong M. (2019) C^* -algebras, H^*-algebras and trace ideals of pseudo-differential operators on locally compact, Hausdorff and abelian groups, *Journal of Pseudo-Differential Operators and Applications*, **10**, 269–283.
9. Maslov, V. (1973) *Operatorniye metodi* [Operator methods]. Moscow, Russia: Nauka, [in Russian].
10. Molahajloo, S. & Wong, K. (2015) Pseudo-differential operators on finite Abelian groups, *Journal of Pseudo-Differential Operators and Applications*, **6**, 1–9.
11. Nicola, F. & Rodino, L. (2010) *Global Pseudo-Differential Calculus on Euclidian Spaces*. Basel, Switzerland: Springer Basel AG.
12. Parshin, A. (1999) On a Ring of Formal Pseudodifferential Operators, *Algebra. Topology. Differential equations and their applications*, **224**, 266–280.
13. Shubin, M. (2005) *Psevdodiffenecialniye operatori i spektralnaya teoriya* [Pseudodifferential operators and spectral theory]. Moscow, Russia: Dobrosvet, [in Russian].
14. Zheglov, A. (2022, May 13) *Schur-Sato theory for quasi-elliptic rings* [Online] Available https://arxiv.org/abs/2205.06790.

