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Oscillating and Proper Solutions of Singular Quasi-Linear Differential Equations

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Abstract: We study the behaviour of solutions of quasi-linear differential equations of the second order at their singular points, where the coefficient of the second-order derivative vanishes. We consider solutions entering a singular point either with definite tangential direction (proper solutions) or without definite tangential direction (oscillating solutions). Equations of this type appear in many problems arising in analysis, geometry, dynamical systems theory, and physics. First, we prove that a generic equation of the considered type has no oscillating solutions. Then we concentrate on proper solutions, which can enter a singular point in admissible tangential directions only. Great attention is paid to second-order differential equations, whose right-hand sides are cubic polynomials by the first-order derivative. We obtain local representations for solutions of such equations in a form similar to Newton–Puiseux series – series with fractional exponents (and, in a special case, with logarithmic terms).

Keywords: differential equations, singular points, normal forms, oscillating solutions

1. INTRODUCTION

1.1. A brief look at quasi-linear equations

The study of singular quasi-linear differential equations goes back to the middle of the XIX century. One of the first examples is the equation

$$x^m y' = f(x, y) \tag{1.1}$$

named after Briot and Bouquet (French mathematicians of the 19th century), who carried out a detailed study of such equations for analytic functions f; the main achievement of this study was representation of solutions in the form of power series with fractional exponents. In that time, the theory developed by Briot and Bouquet seemed so significant that one of the problems at the famous prize competition (1885) sponsored by Oscar II[†], was to develop a comparable theory for non-linear equations of the general form $F(x, y, y') = 0$. In the same year, Poincaré published his third memoir "*On Curves Defined by Differential Equations*", where he gave a start to the geometric approach in the theory of differential equations.[‡]

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[†] King of Sweden (1872–1907) and Norway (1872–1905). In his youth he studied at Uppsala University, where he distinguished himself in mathematics.
[‡] It is worth observing that Poincaré was awarded Oscar II prize for solution of another problem.

Later on, the interest to singular quasi-linear differential equations was motivated either by pure theoretical aspects or by various applications. For example, equations

$$\Delta(x,y)y'' = M(x,y,y') \tag{1.2}$$

appear in many problems arising in analysis, geometry, dynamical systems theory, and physics. Here one can name the Bessel equation and the Gaussian hypergeometric equation, to name a few. We also notice equations appearing in two recent works: a generalized Ginzburg-Landau model for liquid crystals [8] and Newton's aerodynamic problem [16].

Definition 1.1:
A point $q_0 = (x_0, y_0)$ *is called a regular point of equation* (1.2) *if* $\Delta(x_0, y_0) \neq 0$, *and it is called a singular point of equation* (1.2) *if* $\Delta(x_0, y_0) = 0$.

Among equations (1.2), a great attention is paid to the case that the right-hand side is a cubic polynomial in y', that is,

$$M(x,y,y') = \sum_{i=0}^{3} \mu_i(x,y)(y')^i. \tag{1.3}$$

An attention to equations (1.2), (1.3) is motivated by their role in physics and geometry, for instance, the description of various geometric structures (geodesic flows in affine or projective connection, etc.). Equations of this class were studies by Sophus Lie, A. Tresse, J. Liouville, E. Cartan, etc.

Among others, an important case is the Levi–Civita connection, which is generated by the metric tensor

$$ds^2 = a(x,y)\, dx^2 + 2b(x,y)\, dxdy + c(x,y)\, dy^2. \tag{1.4}$$

Geodesics in the Levi–Civita connection are solutions of equation (1.2), (1.3), where

$$\Delta = ac - b^2$$

and the coefficients of the cubic polynomial M are polynomials on the coefficients a, b, c and their first-order derivatives:

$$\begin{aligned} \mu_0 &= a(a_y - 2b_x) + a_x b, \\ \mu_1 &= b(3a_y - 2b_x) + a_x c - 2ac_x, \\ \mu_2 &= b(2b_y - 3c_x) + 2a_y c - ac_y, \\ \mu_3 &= c(2b_y - c_x) - bc_y. \end{aligned} \tag{1.5}$$

Here Δ vanishes at points where the quadratic form (1.4) degenerates. In Riemannian geometry, such is not the case, since the inequality $\Delta > 0$ holds true everywhere. However, metrics (quadratic forms) with varying signature generically appear on surfaces embedded into pseudo-Euclidean spaces: the metric induced on a surface into pseudo-Euclidean space degenerates at those points where the surface tangents the light cone of the ambient space. Singularities of the geodesic equation appearing at degenerate points of the metric (1.4) are studied in the series of papers [22]– [19].

1.2. Phenomenon of oscillating solutions

A. F. Filippov showed [5] that a system of ordinary differential equations

$$F(x,y,p) = 0, \;\; p = dy/dx, \tag{1.6}$$

where $x \in \mathbb{R}^1$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $p = (p_1, \ldots, p_n)$, and $F : \mathbb{R}^{2n+1} \to \mathbb{R}^n$ is a smooth (by smooth we mean C^∞, if the otherwise is not stated) vector function, may have solutions without definite tangential direction at a certain point. This motivates the following terminology.

A solution $y(x)$ of system (1.6) is called *oscillating* at a point (x_0, y_0), if $y(x)$ is a vector function differentiable on the interval $(x_0, x_0 + \delta)$ or $(x_0 - \delta, x_0)$, $\delta > 0$, such that $y(x) \to y_0$ but $y'(x)$ has neither finite not infinite limit as $x \to x_0$. A solution $y(x)$ is called *proper* at a point (x_0, y_0), if $y'(x)$ has limit (finite or infinite) as $x \to x_0$.

Let J^1 be the 1-jet space of smooth functions $y(x)$, with the coordinates (x, y, p). Filippov showed that if for a point $T_0 \in J^1$ there exists $T_0' \in J^1$ such that $x_0 = x_0'$, $y_0 = y_0'$, $p_0 \neq p_0'$, and the matrix F_p degenerates at T_0', then besides a unique proper solution passing through (x_0, y_0) with the tangential direction p_0, system (1.6) may have oscillating solutions, which pass through (x_0, y_0) and even have definite tangential direction p_0 at (x_0, y_0).

The following example is taken from [5]. Consider solutions of the system

$$\begin{cases} p_1(1 - p_1^2 - p_2^2) + 8xy_1 + 4y_2 = 0, \\ p_2(1 - p_1^2 - p_2^2) + 8xy_2 - 4y_1 = 0, \end{cases} \tag{1.7}$$

that pass through the origin with the tangential direction $p_i = 0$, that is, satisfy the conditions $y_i(0) = y_i'(0) = 0,\ i = 1, 2$. The matrix F_p is non-degenerate at the point $T_0 = 0$, and system (1.7) has a unique proper solution $y_i(x) \equiv 0$, which obviously satisfies the required conditions. However, system (1.7) has an infinite number of oscillating solutions given by the formula

$$y_1(x) = x^2 \cos(x^{-1} + c),\ \ y_2(x) = x^2 \sin(x^{-1} + c),\ \ c = \text{const},$$

for $x \neq 0$, and $y_i(0) = 0$. The both derivatives $y_i'(x)$ are zero at $x = 0$, but the limits $y_i'(x)$ as $x \to 0$ do not exist.

On the other hand, Filippov showed that systems with oscillating solutions are a sort of exception. Given a point $q_0 = (x_0, y_0) \in \mathbb{R}^{n+1}$, we define the set

$$Q(q_0) = \{T = (q_0, p) \in J^1 : F(T) = 0,\ \det(F_p(T)) = 0\}.$$

Theorem 1.1 ([5]):
Assume that at least one of the following conditions holds true:

1. $p_0 \notin \operatorname{co} Q(q_0)$, where co *denotes the convex hull,*

2. the set $Q(q_0)$ is at most countable (i.e., countable, or finite, or empty).

Then system (1.6) *has no solution oscillating at q_0 with tangential direction p_0 at q_0. Generically, the condition 2 holds true at all points, and system* (1.6) *has no oscillating solutions.*

Although Theorem 1.1 has a very general character, it gives no information for some special classes of systems. For instance, it gives only a trivial result for quasi-linear implicit systems

$$A(x, y)\, p = b(x, y),\ \ p = dy/dx, \tag{1.8}$$

where A is a $n \times n$ matrix depending on (x, y), b is a vector function. It is worth observing that system (1.8) are of great interest due to various applications, especially in electrical circuit theory. See, e.g., the papers [27] – [21] and the references therein).

Theorem 1.1 guarantees the absence of oscillating solution at all points where $\det A \neq 0$, which obviously follows from the fact that then system (1.8) is locally equivalent to the explicit system $p = A^{-1}(x, y)b(x, y)$. However, it says nothing about points where the matrix A degenerates, since $Q(q_0) = \{q_0\} \times \mathbb{R}^n$ if $\det A(q_0) = 0$, and the both conditions in Theorem 1.1 fail.

1.3. The aim, scope and structure of the paper

The main emphasis of this paper is on oscillating solutions of quasi-linear second-order differential equations (1.2). It should be remarked that in all papers mentioned above (as well as in other works known to us) only proper geodesics were considered, while the possibility of oscillating geodesics is not studied yet. In the present paper, we shall fill this gap.

The paper is organized as follows. In Section 2, we establish a necessary condition for quasi-linear equation (1.2) to have an oscillating solution – Theorem 2.1, which plays the same role as that played by Theorem 1.1 for non-linear systems (1.6). This condition shows that a generic equation (1.2) with M analytic in p has no oscillating solutions, but proper solutions only. In particular, equation (1.2) with a cubic polynomial (1.3) that describes geodesics in the Levi–Civita connection generated by the metric tensor (1.4) has no oscillating solutions if $\Delta, d\Delta$ do not vanish simultaneously.

In Section 3, we briefly describe proper solutions of equation (1.2) with the right-hand side (1.3) entering its generic singular points. Proper solutions cannot enter singular points in arbitrary tangential directions, but only at so-called *admissible* directions p that correspond to real roots of the cubic polynomial M in p. To every admissible direction p, where p is a primer root of M, corresponds either a unique smooth solution or an infinite family of solutions with the common tangential direction and power singularity. Theorem 3.3 gives the local representation of such solutions in a form similar to Newton–Puiseux series. This can be considered as a far development of the results obtained by Poincaré and his predecessors (Briot, Bouquet, Fuchs, etc.) for differential equations (1.1). However, in studying second-order equations we encounter a special difficulty that does not appear in the case of the first order equations: vector fields with non-isolated singular points. A systematic study of such vector fields begins in the end of XX century, the first known works are [25, 28]. A brief survey of results on the local classification of such fields can be found in [24] (Appendix).

In Section 4, we present a natural generalization of the results obtained before for quasi-linear differential equations of higher orders. Finally, we should remark the closely related papers [9] – [29] and especially [26].

2. OSCILLATING SOLUTIONS

In this section, we shall deal with differential equation (1.2). Following the tradition to denote derivatives by single letters going back to Gaspard Monge, we write this equation in the following form:

$$\Delta(x,y)\frac{dp}{dx} = M(x,y,p), \quad p = dy/dx. \tag{2.9}$$

The functions $\Delta(x,y)$, $M(x,y,p)$ are supposed to be smooth.

Denote the locus of singular points of equation (2.9) by Γ. Generically, the set

$$\Gamma = \{(x,y) : \Delta(x,y) = 0\}$$

is a curve on the (x,y)-plane, which is called *singular curve* or *degenerate curve* of the equation. We shall assume this in what follows.

Consider the initial value problem for equation (2.9) with the initial condition $y(x_0) = y_0$. If the point $q_0 = (x_0, y_0)$ is regular, for every p_0 this problem has a unique solution satisfying the additional condition $p(x_0) = p_0$, which is defined and smooth on a real interval including the point x_0, and it has no solutions without $\lim p(x)$ as $x \to x_0$. The situation becomes different if q_0 is singular (see the examples below). This makes sense to refine the notion of solutions for the case of singular points. Throughout this paper, we use the following notation: if I is an open real interval, then $\overline{I}$ denotes the closed interval (segment) with the same endpoints.

Definition 2.1:
Let $q_0 = (x_0, y_0) \in \Gamma$. A solution $y(x)$ of equation (2.9) *enters* the point q_0 if the function $y(x)$ satisfies the following conditions:

1. It is continuous on a non-empty segment $\overline{I}_\varepsilon$ with endpoints x_0 and $x_0 + \varepsilon$, where ε can be either positive or negative, $y(x_0) = y_0$.
2. The function $y(x)$ is differentiable and it satisfies (2.9) on the open interval I_ε.
3. The graph of $y(x)$ has no common points with Γ except for q_0.

If in addition to the above conditions, the derivative $p(x)$ has a (finite or infinite) limit as $x \to x_0$, the solution $y(x)$ is called *proper*. Otherwise the solution $y(x)$ is called *oscillating* at the point q_0. See Fig. 2.1.

Definition 2.2:
A solution $y(x)$ is called *passing through* the point q_0 if the function $y(x)$ is differentiable and it satisfies (2.9) at all points of an open interval I that contains x_0, $y(x_0) = y_0$, and the graph of $y(x)$, $x \in I$, intersects Γ at the point q_0 only.

The significance of the given definitions will become clear in the examples presented below. Obviously, every solution passing through q_0 is the union of two solutions entering q_0. It is worth observing that solutions from Definitions 2.1, 2.2 are also called *one-sided* or *two-sided* solutions, respectively (for example, in [26]). In this paper, we prefer the terminology given above, which shows the geometric flavour of these notions.

Example 2.1:
The parabolas $y = \alpha x^2$, $\alpha = \text{const}$, are integral curves of the second-order equation

$$2y\,\frac{dp}{dx} = p^2, \quad p = dy/dx,$$

passing through the origin. However, the graphs of functions defined by the formula

$$f(x) = \begin{cases} \alpha_1 x^2, & x \le 0, \\ \alpha_2 x^2, & x > 0, \end{cases}$$

are also integral curves, and all such functions are solution of the given equation passing through the origin. To avoid such ambiguity, it is sufficient to use the notion of a solution entering a point defined above. Then solutions entering the origin are the branches of the parabolas $y = \alpha x^2$, $\alpha = \text{const}$, and only them.

In the previous example, all solutions are proper. Now we give two examples of equations (2.9) that possess oscillating solutions.

Example 2.2:
The equation

$$x^4\,\frac{dp}{dx} = 2x^3 p - (2x^2 + 1)y, \quad p = dy/dx,$$

has the family of solutions defined by the formula

$$y = x^2(\alpha \cos x^{-1} + \beta \sin x^{-1}) \;\text{ for }\; x \ne 0, \quad \alpha, \beta = \text{const}, \tag{2.10}$$

and zero for $x = 0$. These functions are differentiable on the whole real axis, but their derivatives are not continuous at the origin except for $\alpha = \beta = 0$. According to the given definitions, formula (2.10) gives a family of solutions passing through the origin with the tangential direction $p = 0$. For $\alpha = \beta = 0$, we have the regular solution $y = 0$, while for all others α, β the solutions (2.10) are oscillating at the origin.

Example 2.3:
The equation

$$x^2\,\frac{dp}{dx} = xp - 2y, \quad p = dy/dx,$$

has the family of solutions

$$y = x(\alpha \cos \ln |x| + \beta \sin \ln |x|), \quad \alpha, \beta = \mathrm{const}, \tag{2.11}$$

entering the origin. Except for $\alpha = \beta = 0$, all these solutions are oscillating at the origin and (unlike the previous example) they have no definite tangential directions at the origin.

Moreover, using formula (2.11), one can construct an equation of the form (2.9) with oscillating solution, whose right-hand side is a polynomial of arbitrary degree in p. For instance, one can see that the function (2.11) with $\alpha = \beta = 1$ is a solution of the first-order equations $f_i(x, y, p) = 0$, where

$$f_2 = (xp)^2 - 2xyp + 2(y^2 - x^2),$$
$$f_3 = (xp)^3 + y(xp)^2 - 2xp(x^2 + 2y^2) + 6y(y^2 - x^2).$$

Therefore, $y = x(\cos \ln |x| + \sin \ln |x|)$ is an oscillating solution of the second-order equations

$$x^2 \frac{dp}{dx} = xp - 2y + f_i(x, y, p), \quad p = dy/dx.$$

We presented several examples demonstrating the existence of oscillating solutions. The following theorem shows that oscillating solutions do not appear generically.

Theorem 2.1:
Let $q_0 \in \Gamma$ and $M(q_0, p)$ be an analytic function not identically zero. Then equation (2.9) *has no oscillating solutions entering q_0.*

Proof
Assume that the above conditions hold true, but equation (2.9) has an oscillating solution $y(x)$ that enters q_0 without definite tangential direction. By the definition, $y(x)$ is defined on a segment $I_\varepsilon = [x_0, x_0 + \varepsilon]$ and $\Delta(x, y(x)) \neq 0$ at all its inner points: $x_0 < x < x_0 + \varepsilon$. Without loss of generality, assume that $\Delta(x, y(x)) > 0$ for all x such that $x_0 < x < x_0 + \varepsilon$ (otherwise we multiply the both sides of (2.9) by -1). From the absence of the limit $y'(x)$ as $x \to x_0$ it follows that there exist two sequences

$$x'_n \to x_0 + 0 \ \text{ and } \ x''_n \to x_0 + 0$$

such that

$$p(x'_n) \to p' \ \text{ and } \ p(x''_n) \to p'', \quad p' \neq p''.$$

For definiteness, assume that $p' < p''$. Since the function $y'(x)$ is continuous on the interval $x_0 < x < x_0 + \varepsilon$, for every $p_* \in (p', p'')$ there exist two sequences

$$\xi_n \to x_0 + 0 \ \text{ and } \ \xi'_n \to x_0 + 0$$

such that

$$\lim_{n\to\infty} p(\xi_n) = \lim_{n\to\infty} p(\xi'_n) = p_*, \quad \frac{dp}{dx}(\xi_n) > 0, \quad \frac{dp}{dx}(\xi'_n) < 0. \tag{2.12}$$

See Fig. 2.1 (right). Substituting the solution $y(x)$ into (2.9), at the points $x = \xi_n$ and $x = \xi'_n$ we have the equalities

$$\Delta(\xi_n, y(\xi_n)) \frac{dp}{dx}(\xi_n) = M(\xi_n, y(\xi_n), p(\xi_n)), \tag{2.13}$$

$$\Delta(\xi'_n, y(\xi'_n)) \frac{dp}{dx}(\xi'_n) = M(\xi'_n, y(\xi'_n), p(\xi'_n)). \tag{2.14}$$

The right-hand sides of both equalities (2.13), (2.14) have the finite limit $M(x_0, y_0, p_*)$, whence their left-hand sides have the same limit $M(x_0, y_0, p_*)$. On the other hand, $\Delta(\xi_n, y(\xi_n))$ and $\Delta(\xi'_n, y(\xi'_n))$ are positive for every n, and from (2.12) it follows that the left-hand sides of equalities (2.13), (2.14) have different signs. Therefore, their limit

$$M(x_0, y_0, p_*) = 0.$$

Thus, we proved that $M(q_0, p_*) = 0$ for every $p_* \in (p', p'')$. Since the function $M(q_0, p)$ is analytic in p, this implies $M(q_0, p) = 0$ for every p. This completes the proof. □

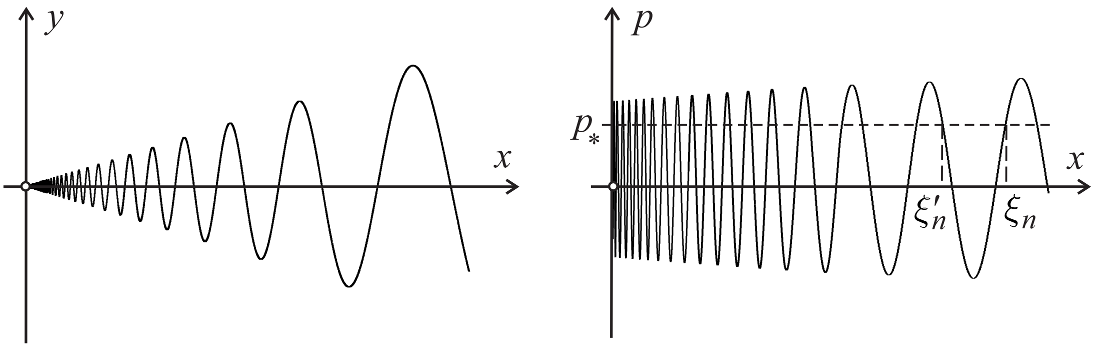

Fig. 2.1. An oscillating solution entering the origin (on the left) and its derivative (on the right).

Corollary 2.1:
Let q_0 be a singular point of equation (1.2), (1.3). *If the coefficients $\mu_0, \ldots, \mu_3$ of the cubic polynomial M do not vanish at q_0 simultaneously, there are no oscillating solutions entering q_0. Moreover, in the case of the geodesic equation, $d\Delta(q_0) = 0$ is a necessary condition for the existence of an oscillating solution entering q_0.*

Proof
The first statement is obvious, let us prove the second one.

The geodesic equation for the metric (1.4) has the form (1.2), (1.3), where $\Delta = ac - b^2$ and $\mu_0, \ldots, \mu_3$ are expressed through a, b, c by formula (1.5). By Theorem 2.1, for the non-existence of solutions oscillating at $q_0 \in \Gamma$, it is sufficient to prove that if all coefficients $\mu_0, \ldots, \mu_3$ simultaneously vanish at q_0, then $d\Delta(q_0) = 0$.

Let us assume on the contrary that $\mu_i(q_0) = 0$ for all i and $d\Delta(q_0) \neq 0$. To simplify the calculations, we choose local coordinates centered at $q_0 \in \Gamma$ such that $b(q_0) = 0$, which is always possible to attain by an appropriate linear transformation. Then from the equalities $\Delta(q_0) = 0$, $d\Delta(q_0) \neq 0$ it follows that $|a(q_0)| + |c(q_0)| \neq 0$. Without loss of generality one can assume that $a(q_0) \neq 0$ and $c(q_0) = 0$. Then from the equalities $\mu_i(q_0) = 0$ and (1.5) it follows $c_x(q_0) = c_y(q_0) = 0$. Finally, from the equalities

$$b(q_0) = c(q_0) = c_x(q_0) = c_y(q_0) = 0$$

it follows $d\Delta(q_0) = 0$. The obtained contradiction completes the proof. □

3. SECOND-ORDER EQUATIONS CUBIC IN THE FIRST-ORDER DERIVATIVE

In this section, we shall briefly describe generic singularities of equation (1.2), whose right-hand side has the form (1.3). Here we shall assume that $\Delta, \mu_0, \ldots, \mu_3$ are smooth functions not connected with each other, in contrast with the geodesic equation, where all these functions are expressed through a, b, c and their derivatives. According to the genericity assumption, we shall assume that the coefficients $\mu_0, \dots, \mu_3$ of the cubic polynomial (1.3) do not vanish simultaneously and the set of singular points Γ is a regular curve, which locally separates the (x, y)-plane into two domains.

Then equation (1.2) has no oscillating solutions, and we consider only proper solutions passing through or entering a point $q_0 \in \Gamma$. Further we shall omit the word "proper".

Theorem 3.1:
Solutions of equation (1.2) *can enter a point* $q_0 \in \Gamma$ *at directions* p *that corresponds to the real roots of* $M(q_0, p)$ *only.*

The proof trivially follows from the qualitative analysis of the vector field

$$\dot{x} = \Delta(q), \;\; \dot{y} = p\Delta(q), \;\; \dot{p} = M(q, p) \tag{3.15}$$

generated by equation (1.2) in the 1-jets space.

Given point $q_0 \in \Gamma$, we define the *admissible directions* p that correspond to singular points (q_0, p) of the vector field (3.15), i.e., satisfy the conditions

$$\Delta(q_0) = 0, \;\; M(q_0, p) = 0.$$

The field (3.15) is a partial case of vector fields

$$\dot{x} = v, \;\; \dot{y} = w, \;\; \dot{z}_i = a_i v + b_i w, \;\; i = 1, \dots, n, \tag{3.16}$$

where $x \in \mathbb{R}^1$, $y \in \mathbb{R}^1$ and $z = (z_1, \dots, z_n) \in \mathbb{R}^n$. Here v, w and a_i, b_i are smooth functions on the variables (x, y, z).

The field (3.16) can be invariantly defined as follows: all its components belong to the ideal I generated by two of them (for instance, v and w). Obviously, the set of singular points of this field is given by two equations $v = 0$, $w = 0$. Therefore, the spectrum of its linear part at every singular point is

$$(\lambda_1, \lambda_2, 0, \dots, 0),$$

where the eigenvalues $\lambda_{1,2}$ (either real or complex) continuously depend on the point. If $\mathrm{Re}\lambda_{1,2} \neq 0$, then the center manifold W^c of the field (3.16) coincides with the set of its singular point, and the Reduction Principle (see [1] or [7]) states that the germ of (3.16) is topologically equivalent to

$$\dot{x} = \sigma_1 x, \;\; \dot{y} = \sigma_2 y, \;\; \dot{z}_i = 0, \;\; i = 1, \dots, n,$$

where $\sigma_i = \mathrm{sign}\,(\mathrm{Re}\lambda_i)$. We recall that the topological equivalence of two vector fields means that there exists a homeomorphism that sends the phase portrait of the first field to the phase portrait of the second field.

Smooth local normal forms of vector fields (3.16) are also known; see., e.g. [6] or [24] (Appendix). Here we present only one result of smooth classification.

Let $q_0 \in \Gamma$ and $M(q_0, p_*) = 0$, i.e., p_* be an admissible direction at q_0. The number and the behaviour of solutions of equation (1.2) that enter q_0 with the direction p_* is determined by the ratio of the non-zero eigenvalues of the field (3.15):

$$\lambda_1(q_0, p_*) = (\Delta_x + p\Delta_y)(q_0, p_*),$$
$$\lambda_2(q_0, p_*) = M_p(q_0, p_*).$$

Further we shall always assume that $\lambda_{1,2}(q_0, p_*) \neq 0$. Define the value

$$\lambda = \lambda_2(q_0, p_*) : \lambda_1(q_0, p_*)$$

and the set $\mathbb{N}^{-1} = \{1/n, \; n \in \mathbb{N}\}$.

Theorem 3.2:
If $\lambda > 0$, the germ of (3.16) is C^∞-smoothly equivalent to

$$\dot{\xi} = a_1(\zeta)\xi, \ \ \dot{\eta} = a_2(\zeta)\eta, \ \ \dot{\zeta}_j = 0, \ \ \lambda \notin \mathbb{N} \cup \mathbb{N}^{-1},$$
$$\dot{\xi} = a_1(\zeta)\xi, \ \ \dot{\eta} = a_2(\zeta)\eta + b_2(\zeta)\xi^n, \ \ \dot{\zeta}_j = 0, \ \ \lambda \in \mathbb{N} \cup \mathbb{N}^{-1} \backslash \{1\},$$
$$\dot{\xi} = a_1(\zeta)\xi + b_1(\zeta)\eta, \ \ \dot{\eta} = a_2(\zeta)\eta + b_2(\zeta)\xi, \ \ \dot{\zeta}_j = 0, \ \ \lambda = 1.$$

Using Theorem 3.2, one can get the following result:

Theorem 3.3:
If $\lambda < 0$, then equation (1.2) has only one solutions passing though the point q_0 with the tangential direction p_*.

If $\lambda > 0$, then equation (1.2) has an infinite number of solutions entering the point q_0 with the direction p_*. In appropriate local coordinates centered at q_0, these solutions have one of two following forms:

$$y = F(x, c|x|^\lambda), \ \textit{if} \ \lambda \notin \mathbb{N},$$
$$y = F(x, x^\lambda(c + \varepsilon \ln|x|)), \ \ \varepsilon \in \{0, 1\}, \ \textit{if} \ \lambda \in \mathbb{N},$$

where F is a smooth function, $c = \text{const}$.

The proof of this theorem is quite similar to those for analogous theorem for the equation of geodesics in signature varying metrics (see [22]– [19]) although it has a much more general character. Indeed, for a generic cubic polynomial $M(q_0, p)$, there are four (up to rearrangements of its real roots p_i) possible cases:

- **c1.** 1 admissible direction p_0 with $\lambda(q_0, p_0) > 0$,
- **c2.** 3 admissible directions p_0, p_1, p_2 with $\lambda(q_0, p_0) > 0$ and $\lambda(q_0, p_i) < 0$ for $i = 1, 2$,
- **c3.** 1 admissible direction p_0 with $\lambda(q_0, p_0) < 0$,
- **c4.** 3 admissible directions p_0, p_1, p_2 with $\lambda(q_0, p_0) < 0$ and $\lambda(q_0, p_i) > 0$ for $i = 1, 2$.

Here negative λ corresponds to a unique solution passing through q_0 with given tangential direction, while positive λ corresponds to an infinite family of solutions passing through q_0 with given tangential direction. For equations of geodesics only the cases c1 and c2 can be realized.

Example 3.1:
Consider the differential equation

$$x\frac{dp}{dx} = \alpha p(p^2 - 1), \ \ \alpha \neq 0, \tag{3.17}$$

whose singular points fill the curve $\Gamma = \{x = 0\}$. At every $q_0 \in \Gamma$ the cubic polynomial

$$M(p) = \alpha p(p^2 - 1)$$

has three prime roots: $p_0 = 0$ and $p_{1,2} = \pm 1$. Here $\lambda(q_0, 0) = -\alpha$ and $\lambda(q_0, \pm 1) = 2\alpha$.

Therefore, if $\alpha > 0$ then equation (3.17) has a single solution passing through the point q_0 with the tangential direction $p_0 = 0$ (the bold line in Fig. 3.2, left) and two infinite families of solutions entering q_0 with the directions $p_{1,2} = \pm 1$ (solid lines in Fig. 3.2, left).

If $\alpha < 0$ then equation (3.17) has an infinite family of solutions entering the point q_0 with the tangential direction $p_0 = 0$ (solid lines in Fig. 3.2, right) and two single solutions passing through the point q_0 with the directions $p_{1,2} = \pm 1$ (the bold line in Fig. 3.2, right).

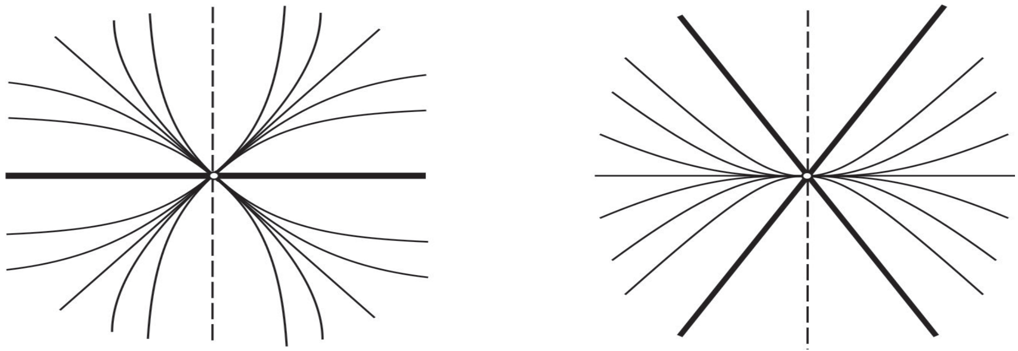

Fig. 3.2. Solutions of equation (3.17). On the left: $\alpha > 0$. On the right: $\alpha < 0$.

Example 3.2:
Consider the differential equation

$$x\frac{dp}{dx} = \alpha p(p^2 + 1), \quad \alpha \neq 0. \tag{3.18}$$

whose singular points fill the curve $\Gamma = \{x = 0\}$. At every $q_0 \in \Gamma$ the cubic polynomial

$$M(p) = \alpha p(p^2 + 1)$$

has only one real root $p_0 = 0$, whose $\lambda(q_0, 0) = \alpha$.

Therefore, if $\alpha > 0$ then equation (3.18) has an infinite family of solutions entering the point q_0 with the tangential direction $p_0 = 0$ (solid lines in Fig. 3.3, left). If $\alpha < 0$ then equation (3.18) has a single solution entering the point q_0 with the tangential direction $p_0 = 0$ (the bold line in Fig. 3.3, right).

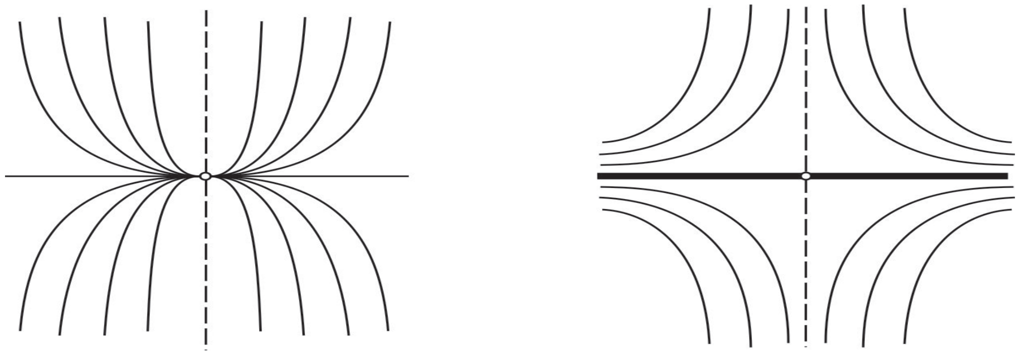

Fig. 3.3. Solutions of equation (3.18). On the left: $\alpha > 0$. On the right: $\alpha < 0$.

Example 3.3:
Three more examples taken from [18] present families of geodesics in signature varying metrics entering a point q_0 (or passing through q_0) where the metrics degenerates. In the series [22]– [19], it is shown that for equations of geodesics only the cases c1 and c2 can be realized.

Three such examples are presented in Fig. 3.4. On the left and center: one admissible direction $p_0 = \infty$ with $\lambda > 0$, an infinite family of geodesics entering q_0. On the right: three admissible directions $p_0 = \infty$ with $\lambda > 0$ and $p_{1,2}$ with $\lambda < 0$, which give an infinite family of geodesics entering q_0 with the direction p_0 and two single geodesics (bold lines) passing trough q_0 with the directions $p_{1,2}$.

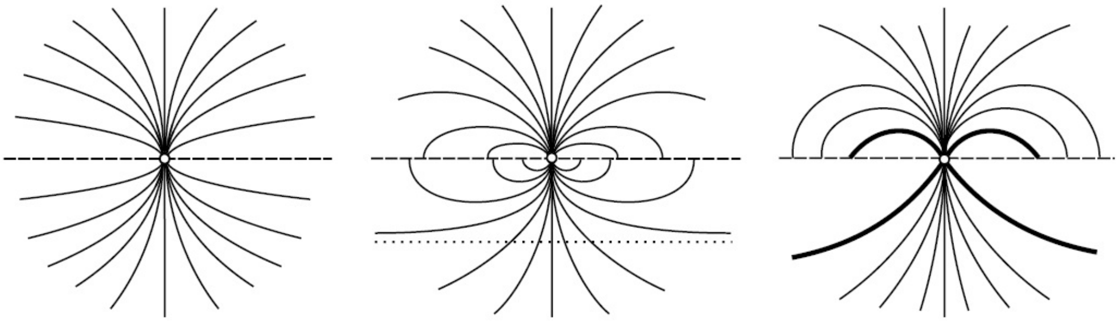

Fig. 3.4. Three families of geodesics in degenerate metrics with varying signature.

4. GENERALIZATION FOR HIGHER ORDER EQUATIONS

Here we present some generalizations of the previous results. Consider the quasi-linear differential equation of the order $n \geq 2$ using the notations similar to (2.9):

$$\Delta(x, \bar{y})\frac{dp}{dx} = M(x, \bar{y}, p), \;\; \bar{y} = (y^{(0)}, \ldots, y^{(n-2)}), \;\; p = y^{(n-1)}, \tag{4.19}$$

where $y^{(i)}$ denotes the i-th derivative of y, and $\Delta(x, \bar{y})$, $M(x, \bar{y}, p)$ are smooth functions. It is important to emphasize that the coefficient Δ standing with the higher derivative depends on $x, y^{(0)}, \ldots, y^{(n-2)}$ and it does not depend on $y^{(n-1)}$. This plays an important role for the further study.

Denote the locus of singular points of equation (4.19) by Γ i.e.,

$$\Gamma = \{(x, \bar{y}) : \Delta(x, \bar{y}) = 0\}.$$

Generically, Γ is a hypersurface in the $(x, \bar{y})$-space. We start with the following definitions that generalize Definitions 2.1 and 2.2:

Definition 4.1:
Let $q_0 = (x_0, \bar{y}_0) \in \Gamma$. A solution $y(x)$ of equation (4.19) *enters* the point q_0 if the following conditions hold true:

1. The vector-function $\bar{y}(x)$ is continuous on a non-empty segment $\overline{I}_\varepsilon$ with endpoints x_0 and $x_0 + \varepsilon$, where ε can be either positive or negative, $\bar{y}(x_0) = \bar{y}_0$.
2. The vector-function $\bar{y}(x)$ is differentiable and it satisfies (4.19) on the open interval I_ε.
3. The graph of $\bar{y}(x)$ has no common points with Γ except for q_0.

If in addition to the above conditions, the function $p(x)$ has a (finite or infinite) limit as $x \to x_0$, the solution $y(x)$ is called *proper*. Otherwise the solution $y(x)$ is called *oscillating* at the point q_0.

Definition 4.2:
A solution $y(x)$ is called *passing through* the point q_0 if the vector-function $\bar{y}(x)$ is differentiable and it satisfies (4.19) at all points of an open interval I that contains x_0, $\bar{y}(x_0) = \bar{y}_0$, and the graph of $\bar{y}(x)$, $x \in I$, intersects the hypersurface Γ at the point q_0 only.

Finally, we introduce one more notion. If $y(x)$ is a solution of equation (4.19), the corresponding vector-function $\bar{y}(x)$ we shall call an integral curve of this equation.

Theorem 4.1:
Let $q_0 \in \Gamma$ and $M(q_0, p)$ be an analytic function not identically zero. Then equation (4.19) *has no oscillating solutions entering q_0.*

The proof of this theorem repeats the proof of Theorem 2.1.

Further we shall assume that the conditions of Theorem 4.1 hold true. Moreover, we assume that Γ is a regular hypersurface, which locally separates the $(x, \bar{y})$-plane into two domains denoted by D and D'. Then equation (4.19) has no oscillating solutions, and we shall study its proper solutions (omitting the adjective *proper*).

Similarly to what was done in Section 3, equation (4.19) generates a direction field in the $(n-1)$-jet space J^{n-1} with the coordinates $(x, \bar{y}, p)$. This direction field is determined by the vector field

$$\dot{x} = \Delta(x, \bar{y}), \;\; \dot{y}^{(i)} = y^{(i+1)}\Delta(x, \bar{y}), \;\; \dot{y}^{(n-2)} = p\Delta(x, \bar{y}), \;\; \dot{p} = M(x, \bar{y}, p). \tag{4.20}$$

The projection $\pi : J^{n-1} \to J^{n-2}$ (along the p-direction called vertical) of an integral curve of the field (4.20) different from a straight vertical line, is an integral curve of equation (4.19).

Here we generalize the main statements of Section 3 except for those of them that are connected with the direction $p = \infty$. The case $p = \infty$ is excluded from consideration, since it cannot be reduced to a finite value similarly to Section 3 and it requires a special detail research for various types of equations.

Theorem 4.2:
Solutions of equation (4.19) *can enter a singular point* $q_0 \in \Gamma$ *only with admissible values of* p *that correspond to zeros of the function* $M(q_0, p)$.

The proof of this theorem repeats the proof of Theorem 3.1.

Definition 4.3:
We shall call a singular point q_0 *generic*, if $d\Delta(q_0) \neq 0$, the function $M(q_0, p)$ has a finite number of zeros, all prime, and the corresponding admissible value p_i determines the direction $(y^{(1)}, \dots, y^{(n-2)}, p)$ in the $(x, \bar{\;})$-space that is transversal to the hypersurface Γ, i.e.,

$$\Delta_x + y^{(1)}\Delta_{y^{(0)}} + y^{(2)}\Delta_{y^{(1)}} + \cdots + p\Delta_{y^{(n-2)}} \neq 0.$$

Singular points of the field (4.20) have the form (q, p_i), where $q \in \Gamma$ and p_i is one of zeros of the function $M(q, p)$. The spectrum of the linear part of the field (4.20) at its singular point (q_0, p_i) has the form $(0, \dots, 0, \lambda_1, \lambda_2)$, where

$$\lambda_1 = \Delta_x + y^{(1)}\Delta_{y^{(0)}} + y^{(2)}\Delta_{y^{(1)}} + \cdots + p\Delta_{y^{(n-2)}}, \;\; \lambda_2 = M_p$$

evaluated at (q_0, p_i). If $q_0 \in \Gamma$ is generic, then the both eigenvalues $\lambda_{1,2}$ are non-zero. Denote

$$\lambda(q_0, p_i) = \lambda_2(q_0, p_i)/\lambda_1(q_0, p_i).$$

The sign of $\lambda(q_0, p_i)$ determines the number of solutions entering the point q_0 with the given admissible direction p_i and their rough local properties:

Theorem 4.3:
Let $q_0 \in \Gamma$ *be a generic singular point of equation* (4.19) *and* p_i *be an admissible value of* p*, that is,* $M(q_0, p_i) = 0$.

1. If $\lambda < 0$*, then the equation has a* C^∞*-smooth solution passing through* q_0 *and it has no other solutions entering* q_0 *with the value* p_i.

2. If $\lambda > 0$*, then the equation has an infinite number of solutions entering* q_0 *with the admissible value* p_i*. Moreover, the family of corresponding integral curves contains an infinite number of curves entering* q_0 *from the domain* D *and an infinite number of curves entering* q_0 *from the domain* D'*. There exist local coordinates centered at* q_0 *such that solutions mentioned above have one of two forms:*

$$y = F(x, c|x|^\lambda), \quad \textit{if } \lambda \notin \mathbb{N},$$
$$y = F(x, x^n(c + \varepsilon \ln|x|)), \quad \textit{if } \lambda = n \in \mathbb{N},$$

where ε is 0 or 1, F is a smooth function on two variables, $c = \mathrm{const}$.

Moreover, in the case that equation (4.19) *is analytic, in the above statements C^∞ can be replaced with C^ω. Then the function F is analytic, and the latter two formulas give Newton–Puiseux series for solutions.*

Theorem 4.3 is similar to Theorem 3.3, the main difference is that now we do not bring the hypersurface Γ in the $(x, \bar{y})$-space to a simple form (for instance, the hyperplane $x = 0$). In general, it is impossible, since the coordinates of the $(x, \bar{y})$-space with $n > 2$ are not independent variables. The proof of Theorem 4.3 repeats the proof of Theorem 3.3 with obvious changes.

5. CONCLUSION

First, we established a necessary condition for quasi-linear equations (1.2) to have an oscillating solution, which shows that generically such equations have no oscillating solution. However, the proof of this result (Theorem 2.1) is not appropriate for quasi-linear systems (1.8), and the question of the existence of oscillating solution for generic systems (1.8) is still open. The interest to this question is motivated, in particular, by the studying of the dynamics generated by singular Lagrangians, which goes back to the works of P.A.M. Dirac and P. G. Bergmann (middle of 20th century).

Second, we described the phenomenon of admissible directions for proper solutions of equations (1.2), (1.3) and considered the case that such directions correspond prime roots of the cubic polynomial M. The case of double roots was previously studied in the partial case if equation (1.2), (1.3) is the geodesic equation generated by a pseudo-Riemannian metric with varying signature; see the survey [18] or the original works [19, 22, 24]. The study of double roots for equation (1.2) with a generic cubic polynomial (1.3) is more complicated and it is not done yet.

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