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Dynamics of Evolutionary Equations with 1+2 Independent Variables

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Abstract: The paper is devoted to a method for constructing exact solutions of evolutionary differential equations with two space variables.The method uses the theory of symmetries of completely integrable distributions. As an example, we consider a linear parabolic equation that arises in filtration theory, thermodynamics, and mathematical biology. Dynamics are calculated for this equation and classes of their exact solutions are constructed.

Keywords: integrability, completely integrable distributions, symmetry, evolutionary equations, filtration, thermodynamics

1. INTRODUCTION

Consider the following partial differential equation:

$$\frac{\partial u}{\partial t} = f\left(x, u, \frac{\partial^{|\sigma|} u}{\partial x^{\sigma}}\right), \tag{1.1}$$

where $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a multi-index whose elements are non-negative integers, $|\sigma| = \sigma_1 + \cdots + \sigma_n$, $x = (x_1, \ldots, x_n)$,

$$\frac{\partial^{|\sigma|} u}{\partial x^{\sigma}} = \frac{\partial^{|\sigma|} u}{\partial x_1^{\sigma_1}, \ldots \partial x_n^{\sigma_n}}.$$

Here the variables $x_1, \ldots, x_n$ are called spatial, and the variable t is called temporal. For equations (1.1) with one spatial variable, the method of finite-dimensional dynamics was proposed in [1, 2]. This method was further developed in [3, 4].

It allows us to select finite-dimensional submanifolds of solutions from the infinite set of all solutions of evolutionary equations. These submanifolds are "numbered" by solutions of ordinary differential equations.

However, this method does not allow direct generalization to equations with several spatial variables. In this case, using of ordinary differential equations are not enough anymore.

In [5], a method was proposed in which systems of partial differential equations of finite type are used instead of ordinary differential equations.

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2. SYMMETRIES OF COMPLETELY INTEGRABLE DISTRIBUTIONS

Here we give the necessary information about a symmetry of distributions (see [6, 7]).

Let M be a smooth manifold and $\mathcal{P}$ a completely integrable distribution on M.

A vector field X on M is called an *infinitesimal symmetry* of a distribution $\mathcal{P}$ if the local group of translations Φ_t along trajectories of X preserves this distribution, i.e.

$$(\Phi_t)_* (\mathcal{P}) = \mathcal{P}.$$

In what follows, for brevity, an infinitesimal symmetry will be called *symmetry*.

The set of all symmetries of the distribution $\mathcal{P}$ forms the Lie $\mathbb{R}$-algebra $\mathrm{Sym}\mathcal{P}$ with respect to the Lie bracket. This Lie algebra contains the ideal of characteristic symmetries $\mathrm{Char}\mathcal{P}$ that consists of vector fields lying in the distribution $\mathcal{P}$.

The quotient Lie algebra

$$\mathrm{Shuf}\mathcal{P} = \mathrm{Sym}\mathcal{P}/\mathrm{Char}\mathcal{P}$$

is called the Lie algebra of *shuffling symmetries* of the distribution $\mathcal{P}$. Elements of this Lie algebra "shuffle" the maximal integral manifolds of the distribution.

3. SYMMETRIES OF FINITE TYPE DIFFERENTIAL EQUATIONS

A system of differential equations is called a system of *finite type* if the space of its solutions is finite-dimensional [6]. Examples of systems of finite type are systems of ordinary differential equations. A more important example is given by overdetermined systems of differential equations.

For simplicity, we consider the case $n = 2$ and finite type equations of the second order. Consider the following overdetermined system of three differential equations

$$\begin{cases} \dfrac{\partial^2 v}{\partial x^2} = P\left(x, y, v, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}\right), \\ \dfrac{\partial^2 v}{\partial x \partial y} = Q\left(x, y, v, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}\right), \\ \dfrac{\partial^2 v}{\partial y^2} = R\left(x, y, v, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}\right). \end{cases} \tag{3.2}$$

In the space of 1-jets $J^1 = J^1(\mathbb{R}^2)$ with canonical coordinates $x_1 = x, x_2 = y, v, p_1, p_2$, this system defines the two-dimensional distribution

$$\mathcal{P} : a \ni J^1 \mapsto \mathcal{P}(a) = \bigcap_{i=0}^{2} \ker \omega_{i,a} \subset T_a J^1, \tag{3.3}$$

where differential 1-forms

$$\begin{aligned} \omega_0 &= dv - p_1 dx_1 - p_2 dx_2, \\ \omega_2 &= dp_1 - P(x_1, x_2, v, p_1, p_2)dx_1 - Q(x_1, x_2, v, p_1, p_2)dx_2, \\ \omega_2 &= dp_2 - Q(x_1, x_2, v, p_1, p_2)dx_1 - R(x_1, x_2, v, p_1, p_2)dx_2. \end{aligned}$$

This distribution is completely integrable if the conditions of the Frobenius theorem are satisfied:

$$d\omega_i \wedge \omega_0 \wedge \omega_1 \wedge \omega_2 = 0 \quad (i = 0, 1, 2). \tag{3.4}$$

If these conditions are satisfied, then system (3.2) is a system of finite type. In what follows, we assume that these conditions are satisfied. The distribution $\mathcal{P}$ can be given by two vector fields

$$\mathcal{D}_1 = \frac{\partial}{\partial x_1} + p_1\frac{\partial}{\partial v} + P\frac{\partial}{\partial p_1} + Q\frac{\partial}{\partial p_2},$$
$$\mathcal{D}_2 = \frac{\partial}{\partial x_2} + p_2\frac{\partial}{\partial v} + Q\frac{\partial}{\partial p_1} + R\frac{\partial}{\partial p_2}.$$

Let a vector field S be a shuffling symmetry of the distribution $\mathcal{P}$. It means that

$$\mathcal{L}_S(\omega_i) \wedge \omega_0 \wedge \omega_1 \wedge \omega_2 = 0 \quad (i = 0, 1, 2).$$

Here $\mathcal{L}$ is the symbol of Lie's derivative.

Theorem 3.1:
Each shuffling symmetry of distribution (3.3) *is uniquely determined by a function φ on the space J^1 and has the form*

$$S_\varphi = \varphi\frac{\partial}{\partial v} + \mathcal{D}_1(\varphi)\frac{\partial}{\partial p_1} + \mathcal{D}_2(\varphi)\frac{\partial}{\partial p_2}. \tag{3.5}$$

The function φ is called *generating* function of the vector field S_φ.

Theorem 3.2:
The generating function φ satisfies the following system of differential equations:

$$\begin{cases} \mathcal{D}_1^2(\varphi) - S_\varphi(P) = 0, \\ \mathcal{D}_2^2(\varphi) - S_\varphi(R) = 0, \\ \mathcal{D}_1\mathcal{D}_2(\varphi) - S_\varphi(Q) = 0, \\ \mathcal{D}_2\mathcal{D}_1(\varphi) - S_\varphi(Q) = 0. \end{cases} \tag{3.6}$$

4. DYNAMICS

Let Φ_t be a shift transformation along the trajectories of the vector field S_φ from $t = 0$ to t and L be an integral manifold of the distribution $\mathcal{P}$. Then the manifold $\Phi_t(L)$ is also integral manifold of this distribution.

A completely integrable distribution $\mathcal{P}$ is called a *dynamics* of the equation

$$\frac{\partial u}{\partial t} = f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}\right) \tag{4.7}$$

if

$$\varphi = f\left(x_1, x_2, v, p_1, p_2, \bar{P}, \bar{Q}, \bar{R}\right) \tag{4.8}$$

is a generating function of a shuffling symmetry of the distribution $\mathcal{P}$. Here $x_1 = x, x_2 = y$,

$$\begin{aligned} \bar{P} &= P(x_1, x_2, v, p_1, p_2), \\ \bar{Q} &= Q(x_1, x_2, v, p_1, p_2), \\ \bar{R} &= R(x_1, x_2, v, p_1, p_2) \end{aligned} \tag{4.9}$$

are restrictions of the functions to the distribution $\mathcal{P}$.

The following theorem allows us to obtain solutions of equation (4.7) from the known integral manifolds of the distribution $\mathcal{P}$.

Theorem 4.1:
Let L be a maximal integral manifold of the distribution $\mathcal{P}$. Then the three-dimensional manifold $\Phi_t(L)$ is the 1-graph of a solution of evolutionary equation (4.7).

Let us show how we can calculate the dynamics.

1. First, we construct function (4.8) with unknown P, Q and R.
2. Second, we construct vector field (3.5) and the differential 1-forms $\omega_0, \omega_1, \omega_2$.
3. Third, we construct equations (3.4) and add to them equations (3.6). Since equation (3.4) with $i = 0$ is trivial, we obtain a system of six equations for the unknown functions P, Q, R.
4. Solving it we find dynamics (3.2).

Since system (3.2) is involutive, it has a three-parameter family of solutions. In order to solve this system, we can use the Lie–Bianchi theorem (see [7]). According to this theorem, a completely integrable distribution $\mathcal{P}$ is integrable by quadratures if the Lie algebra $\mathrm{Sym}\mathcal{P}$ is solvable and, moreover, $\dim \mathrm{Sym}\mathcal{P} = \mathrm{codim}\mathcal{P}$.

In our case, $\mathrm{codim}\mathcal{P} = 3$, and the vector field S_φ is a symmetry of the distribution $\mathcal{P}$. Therefore, we need two more symmetries. For example, if the function f in equation (4.7) does not explicitly depend on variables x, y. Then we obtain these two symmetries automatically. Namely, these are vector fields with generating functions p_1 and p_2.

Due to Theorem 3, to construct an explicit solution of equation (3.2) we should calculate the flow Φ_t of the vector field S_φ. Suppose that an integral manifold of the distribution $\mathcal{P}$ is defined by the following system:

$$\begin{cases} H_1(x_1, x_2, v, p_1, p_2) = 0, \\ H_2(x_1, x_2, v, p_1, p_2) = 0, \\ H_3(x_1, x_2, v, p_1, p_2) = 0. \end{cases} \tag{4.10}$$

Acting to this system by the transformation Φ_t^{-1}, we get the system

$$\begin{cases} U_1(t, x_1, x_2, v, p_1, p_2) = 0, \\ U_2(t, x_1, x_2, v, p_1, p_2) = 0, \\ U_3(t, x_1, x_2, v, p_1, p_2) = 0. \end{cases} \tag{4.11}$$

Solving this system concerning the function $v(t, x)$, we get an explicit solution $u = v(t, x)$ of equation (3.2). If the flow Φ_t cannot be found explicitly, then we can construct solutions numerically.

5. DYNAMICS AND SOLUTIONS OF LINEAR PARABOLIC EQUATIONS

The linear equation

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(k_i(x) \frac{\partial u}{\partial x_i} \right) \tag{5.12}$$

is among the most commonly used equations in mathematical physics. Such equations are often encountered in biology [8], thermodynamics, diffusion theory, and filtration theory. These equations describe, for example, linear conservation laws.

5.1. Case $n = 1$

For $n = 1$ equation (5.12) takes the form (see [9])

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x_i}\right). \tag{5.13}$$

Theorem 5.1:
Suppose that the function k is not vanishing. Then equation (5.13) has a finite dimension second-order dynamics

$$(k(x)v')' + \alpha v - \beta\int\frac{dx}{k(x)} - \gamma = 0, \tag{5.14}$$

where α, β, and γ are constants.

If $\alpha = 0$ then general solution of equation (5.13) is

$$v(x) = \int\frac{\int\left(\beta\int\frac{dx}{k(x)} + \gamma\right)dx + \delta}{k(x)}dx. \tag{5.15}$$

Here δ is an arbitrary constant. The corresponding vector field

$$S_\varphi = \left(\beta\int\frac{dx}{k(x)} + \gamma\right)\frac{\partial}{\partial v}.$$

Applying translations along its trajectories to solution (5.15), we obtain a three-parameter family of solutions of equation (5.13):

$$u(t, x) = \left(\beta\int\frac{dx}{k(x)} + \gamma\right)t + \int\frac{\int\left(\beta\int\frac{dx}{k(x)} + \gamma\right)dx + \delta}{k(x)}dx.$$

Note that this solution was obtained for any non-zero function $k(x)$.

5.2. Case $n = 2$

For $n = 2$ equation (5.12) takes the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_1}\left(k_1(x)\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(k_2(x)\frac{\partial u}{\partial x_2}\right). \tag{5.16}$$

Let us look for dynamics (3.2) in the form

$$\begin{cases} \dfrac{\partial^2 v}{\partial x^2} = P(x, y), \\ \dfrac{\partial^2 v}{\partial x\partial y} = Q(x, y), \\ \dfrac{\partial^2 v}{\partial y^2} = R(x, y). \end{cases} \tag{5.17}$$

The distribution $\mathcal{P}$ integrability condition has the form

$$P(x_1, x_2) = \int \frac{\partial Q}{\partial x_1} dx_2 + p(x_1),$$
$$R(x_1, x_2) = \int \frac{\partial Q}{\partial x_2} dx_1 + q(x_2),$$

where p, q are arbitrary functions. Here $x_1 = x, x_2 = y$. Note that in this case $[\mathcal{D}_1, \mathcal{D}_2] = 0$.
Consider the case when the functions are quadratic:

$$k_1 = k_{120}x_1^2 + k_{111}x_1x_2 + k_{102}x_2^2 + k_{110}x_1 + k_{101}x_2 + k_{100},$$
$$k_2 = k_{220}x_1^2 + k_{211}x_1x_2 + k_{202}x_2^2 + k_{210}x_1 + k_{201}x_2 + k_{200},$$

where k_{ins} are constants. The index i means the number of the function, and the indices n and s mean the powers of x_1 and x_2, respectively.

5.2.1. Trivial dynamics $P = Q = R = 0$ Equation (5.16) admits trivial dynamics when the functions P, Q, and R are zero. We can construct the corresponding explicit solution to equation (5.16). However, the form of this solution is very cumbersome, and we will consider the special case when

$$k_{211} \neq 0 \quad \text{and} \quad k_{120}^2 - 2k_{202}k_{120} + k_{202}^2 + k_{111}k_{211} = 1.$$

The differential 1-forms are

$$\omega_1 = dp_1, \quad \omega_2 = dp_2.$$

Linear functions

$$v = C_1x_1 + C_2x_2 + C_0 \tag{5.18}$$

form a 3-parameter family of solutions to equation (3.2). The vector field

$$\begin{aligned} S_\varphi = {} & (2k_{120}x_1p_1 - k_{211}^{-1}(k_{120}^2 - 2k_{202}k_{120} + k_{202}^2 - 1)x_2p_1 \\ & + p_1k_{110} + 2p_2k_{202}x_2 + p_2k_{211}x_1 + p_2k_{201})\frac{\partial}{\partial v} \\ & + (2k_{120}p_1 + k_{211}p_2)\frac{\partial}{\partial p_1} \\ & + (-k_{211}^{-1}(k_{120}^2 - 1 - 2k_{202}k_{120} + k_{202}^2)p_1 + 2k_{202}p_2)\frac{\partial}{\partial p_2} \end{aligned}$$

is a shuffling symmetry.

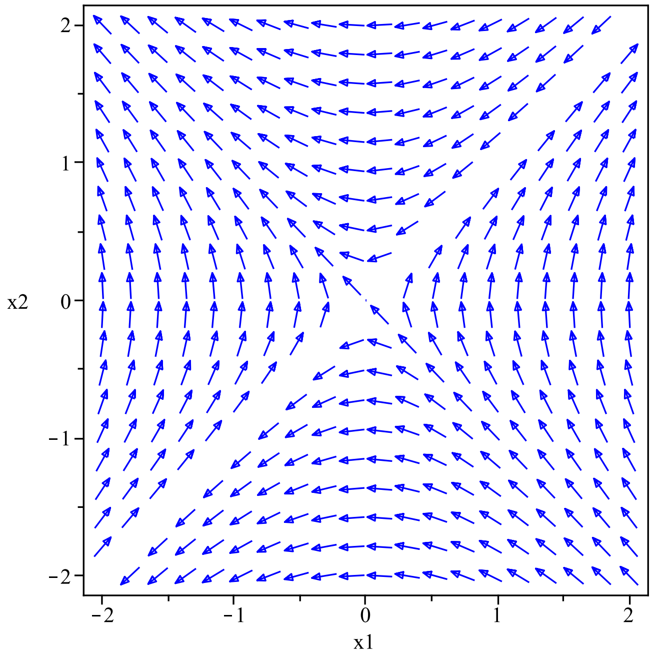

Fig. 5.1. Vector field (k_1, k_2).

Acting to function (5.18) by the transformation, we obtain the following explicit solution to equation (5.16):

$$
\begin{aligned}
u = &- [(bk_{211} + a(k_{120} - 1 - k_{202})(k_{120} + k_{202} + 1) \\
&(k_{120} + k_{202} + 1)((k_{120}\, x_1 + k_{110} + x_1\, k_{202} - x_1)k_{211} \\
&- (k_{120} + 1 - k_{202})(x_2\, k_{120} + k_{201} - x_2 + k_{202}\, x_2)) \\
&\times e^{(k_{120}+k_{202}-1)t} - (k_{120} + k_{202} - 1) \\
&(bk_{211} + a(k_{120} + 1 - k_{202}))((k_{120}\, x_1 + k_{110} \\
&+ x_1\, k_{202} + x_1)k_{211} - (k_{120} - 1 - k_{202})(x_2\, k_{120} + k_{201} \\
&+ x_2 + k_{202}\, x_2))\mathrm{e}^{(k_{120}+k_{202}+1)t} - 2\, bk_{211}^2 k_{110} \\
&+ (4\, ak_{110}\, k_{202} + 4\, bk_{120}\, k_{201})k_{211} \\
&+ 2\, ak_{201}\, (k_{120} + 1 - k_{202})(k_{120} - 1 - k_{202})] \\
&\times (2k_{211}(k_{120} + k_{202} + 1)(k_{120} + k_{202} - 1))^{-1},
\end{aligned}
$$

where $a = C_1, b = C_2$ are arbitrary constant. For brevity, we assume $C_0 = 0$.

5.2.2. Dynamics $P = Q = R = 1$ Consider the following partial case of equation (5.16) with

$$
\begin{aligned}
k_1(x_1, x_2) &= x_1 x_2 - x_2^2, \\
k_2(x_1, x_2) &= -x_1 x_2 + x_1^2
\end{aligned}
$$

(see Fig. 5.1). Then equation (5.16) admits the dynamics with $P = Q = R = 1$. The corresponding solution of equation (3.2) is

$$
v = \frac{1}{2}(x_1^2 + x_2^2) + x_1 x_2 + a x_1 + b x_2 + c, \qquad (5.19)
$$

where a, b and c are arbitrary constants. The generating function is

$$\varphi = x_1^2 - x_2^2 + x_2 p_1 - x_1 p_2.$$

Therefore,

$$S_\varphi = (x_1^2 - x_2^2 + x_2 p_1 - x_1 p_2)\frac{\partial}{\partial v} + (x_1 + x_2 - p_2)\frac{\partial}{\partial p_1} + (p_1 - x_1 - x_2)\frac{\partial}{\partial p_2}.$$

The translation along this vector field is

$$\Phi_t : \begin{cases} x_1 \mapsto x_1, \\ x_2 \mapsto x_2, \\ v \mapsto v + (-x_2^2 + (-2x_1 + p_2)x_2 \\ \qquad - x_1(-p_1 + x_1)) \cos t \\ \qquad + (-x_2^2 + x_2 p_1 + x_1(x_1 - p_2)) \sin t \\ \qquad + x_2^2 + (2x_1 - p_2)x_2 + x_1^2 - x_1 p_1, \\ p_1 \mapsto (x_1 + x_2) \cos^2 t + (p_1 - x_2 - x_1) \cos t \\ \qquad + ((x_2 + x1) \sin t + x_1 + x_2 - p_2) \sin t, \\ p_2 \mapsto x_2 + x_1 + (p_2 - x_1 - x_2) \cos t \\ \qquad + (p_1 - x_2 - x_1) \sin t. \end{cases}$$

The inverse transformation is

$$\Phi_t^{-1} : \begin{cases} x_1 \mapsto x_1, \\ x_2 \mapsto x_2, \\ v \mapsto v + (x_1^2 + (2x_2 - p_1)x_1 + x_2(x_2 - p_2)) \cos^2 t \\ \qquad + (-x_1^2 + (p_1 - 2x_2)x_1 - x_2(x_2 - p_2)) \cos t \\ \qquad + (x_1^2 + (2x_2 - p_1)x_1 + x_2(x_2 - p_2)) \sin^2 t \\ \qquad + (-x_1^2 + x_1 p_2 + x_2(-p_1 + x_2)) \sin t, \\ p_1 \mapsto (p_1 - x_2 - x_1) \cos t + (p_2 - x_1 - x_2) \sin t \\ \qquad + x_1 + x_2, \\ p_2 \mapsto (p_2 - x_1 - x_2) \cos t + (x_1 + x_2 - p_1) \sin t \\ \qquad + x_1 + x_2. \end{cases}$$

Applying the transformations Φ_t^{-1} to the vector function

$$[\, v - \frac{1}{2}(x_1^2 + x_2^2) + x_1 x_2 + a x_1 + b x_2 + c, p_1 - x_1 - x_2 - a, p_2 - x_2 - x_1 - b \,]$$

we get system (4.11):

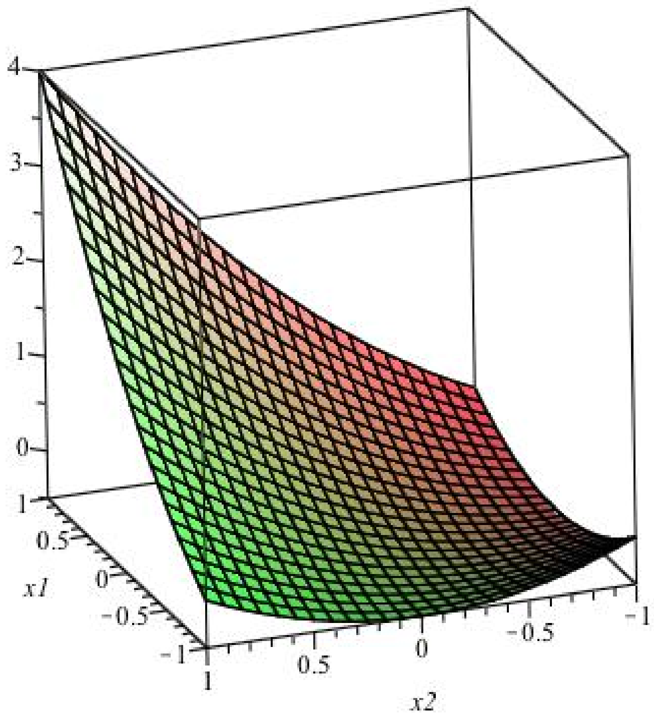

Fig. 5.2. The graph of solution (5.20) at $t = 0$.

$$\begin{cases} (x_1^2 + (2x_2 - p_1)x_1 + x_2(x_2 - p_2)) \cos^2 t \\ + (-x_1^2 + (p_1 - 2x_2)x_1 - x_2(x_2 - p_2)) \cos t \\ + (x_1^2 + (2x_2 - p_1)x_1 + x_2(x_2 - p_2)) \sin^2 t \\ + (-x_1^2 + x_1 p_2 + x_2(-p_1 + x_2)) \sin t \\ + v - \dfrac{x_1^2 + x_2^2}{2} - x_1 x_2 - a x_1 - b x_2 - c = 0, \\ - a + (p_1 - x_2 - x_1) \cos t + (p_2 - x_1 - x_2) \sin t = 0, \\ - b + (p_2 - x_1 - x_2) \cos t + (x_1 + x_2 - p_1) \sin t = 0. \end{cases}$$

Solving this system, we get the following 3-parameter solutions family to evolutionary equation (5.16):

$$u = \frac{1}{2}\left(x_1^2 + x_2^2\right) + (a x_1 + b x_2) \cos t + (a x_2 - b x_1) \sin t + x_1 x_2 + c, \tag{5.20}$$

where a, b and c are arbitrary constants. The evolution of a graph of function (5.20) in time, see Fig. 5.2, 5.3. Here $a = b = 1, c = 0$.

REFERENCES

1. Kruglikov, B.S. & Lychagina, O.V. (2005) Finite dimensional dynamics for Kolmogorov – Petrovsky – Piskunov equation. *Lobachevskii Journal of Mathematics*, **19**, 13–28.

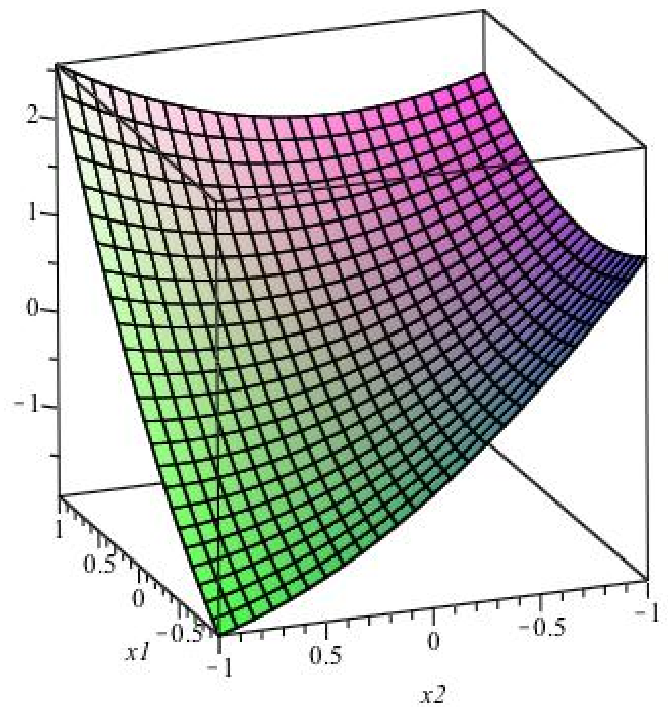

Fig. 5.3. The graph of solution (5.20) at $t = 5$.

2. Lychagin, V.V. & Lychagina, O.V. (2007) Finite Dimensional Dynamics for Evolutionary Equations. *Nonlinear Dyn.*, **48**, 29–48.
3. Akhmetzyanov, A.V., Kushner, A.G. & Lychagin, V.V. (2017) Attractors in Models of Porous Media Flow. *Doklady. Mathematics*, **472**:6, 627–630.
4. Kushner, A.G. & Matviichuk, R.I. (2020) Exact solutions of the Burgers – Huxley equation via dynamics. *Journal of Geometry and Physics*, **151**, 103615.
5. Kushner, A.G. (2022) Dynamics of evolutionary equations with multiple spatial variables. *Proc. of Conf. "Symmetries: theoretical and methodological aspects*, **9**(33), Astrakhan, 15-16 Sept. 2022, 37-43, [in Russian].
6. Duzhin, S.V, Lychagin, V.V. (1991) Symmetries of distributions and quadrature of ordinary differential equations. *Acta Appl. Math.* **24**(1), 29–57.
7. Kushner, A.G., Lychagin, V.V. & Rubtsov, V.N. (2007) Contact geometry and nonlinear differential equations. Cambridge: Cambridge University Press, xxii+496 pp.
8. Pethame, B. (2015) Parabolic Equations in Biology. Springer, Heidelberg, 203 p.
9. Kushner, A.G., Kushner, E.N. & Tao, S. (2022) Finite-dimensional dynamics of filtration equation. *Differential equations and mathematical modeling*, **4**, 62–63, [in Russian].

