Adv Syst Sci Appl 2022; **03**:156–162
Published online at https://ijassa.ipu.ru.

Solvability of Control Systems with Antiperiodic Boundary Constraints

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Abstract: In this paper, we consider control systems with implicit dynamics and antiperiodic boundary constraints ont the fixed time interval. We derive sufficient conditions for the existence of admissible control for the systems of this type and obtain estimates of the admissible controls. These results remain considerable in the partial case when the considered dynamics is explicit.

Keywords: Control system, implicit dynamics, antiperiodic boundary constraints, implicit function

1. INTRODUCTION

Before proceeding to the formulation of the problem, we introduce the notation used.

Let $\mathbb{R}^n$ stand for the n-dimensional real aritheoremetic space with the norm $|\cdot|$ and the inner product $\langle\cdot,\cdot\rangle$. For $\tau > 0$, for an absolutely continuous function $x : [0,\tau] \to \mathbb{R}^n$ and for a point $t \in [0,\tau]$ at which x is differentiable, we denote by $\dot{x}(t)$ the derivative of the function x at the point t. Denote by $\dot{x}$ the Lebesgue integrable function $t \mapsto \dot{x}(t)$, $t \in [0,\tau]$. Everywhere below $L^n_\infty[0,\tau]$ stands for the set of all Lebesgue measurable essentially bounded functions $v : [0,\tau] \to \mathbb{R}^n$ and $AC^n_\infty[0,\tau]$ stands for the set of all absolutely continuous functions $x : [0,\tau] \to \mathbb{R}^n$ such that $\dot{x} \in L^n_\infty[0,\tau]$. Here and below $\dot{\forall}$ stands for "for almost all".

Let a mapping $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ and a nonempty set $U \subset \mathbb{R}^m$ be given. For every $\tau > 0$ consider the control system

$$f(t, x, \dot{x}, u(t)) = 0, \quad u(t) \in U \quad \dot{\forall}\, t \in [0,\tau], \tag{1.1}$$

$$x(0) + x(\tau) = 0. \tag{1.2}$$

We will say that a pair of functions $(\bar{x}, \bar{u}) \in AC^n_\infty[0,\tau] \times L^m_\infty[0,\tau]$ is an admissible process to the problem (1.1), (1.2), if $\bar{u}(t) \in U$ for almost all $t \in [0,\tau]$, the functions $\bar{x}(\cdot)$ is a solution to the differential equation $f(t, x, \dot{x}, \bar{u}(t)) = 0$ on the segment $[0,\tau]$ and the equality $\bar{x}(0) + \bar{x}(\tau) = 0$ takes place. If a pair of functions $(\bar{x}, \bar{u}) \in AC^n_\infty[0,\tau] \times L^m_\infty[0,\tau]$ is an admissible process to the problem (1.1), (1.2), then we say that the function $\bar{x}$ is an admissible trajectory and the function $\bar{u}$ is an admissible control.

The main goal of our paper is to derive sufficient conditions for the existence of an admissible process to the problem (1.1), (1.2). The proof of the main result is based on the following idea. We pass from the control problem with implicit dynamics to the control problem with explicit dynamics using a nonlocal implicit function theorem; for a function

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$\bar{u}$ we consider a mapping φ that corresponds to the initial point x_0 the endpoint $\varphi(x_0)$ of a trajectory starting at x_0; than we use an analog of Brouwer's fixed point theorem to show the existence of x_0 such that $x_0 = -\varphi(x_0)$; and prove that the corresponding trajectory is admissible. The idea of reduction of implicit ODEs (ordinary differential equations) to explicit using implicit function theorems is well known. The usage of nonlocal implicit function theorems to control systems and ODEs was studied in [1, 3]. The usage of Brower's fixed point theorem and similar results for studying boundary value problems is a standard tool (see, for example, [4]). In this paper, we have combined and modified these known approaches and applied the for studying the problem (1.1), (1.2).

2. MAIN RESULTS

Denote by $B^n(r)$ the closed ball in $\mathbb{R}^n$ centered at zero with the radius $r \geq 0$. For an arbitrary linear operator $A : \mathbb{R}^n \to \mathbb{R}^k$, denote

$$\mathrm{cov}A := \max\{\alpha \geq 0 : \; B^k(\alpha) \subset AB^n(1)\}.$$

It is a straightforward task to ensure that $\mathrm{cov}A > 0$ if and only if A is surjective. As is known (see, for example, [2]), that the function cov is continuous.

Theorem 2.1:
Let

(i) *the mapping f be twice continuously differentiable;*

(ii) $\bar{\alpha} := \inf\big\{\mathrm{cov} f'_v(t, x, v, u) : \; (t, x, v, u) \in \mathbb{R}^1 \times \mathbb{R}^n \times \mathbb{R}^n \times U\big\} > 0.$

Given numbers $R > 0$, $\alpha \in (0, \bar{\alpha})$, $\tau > 0$ and functions $\bar{u} \in L^m_\infty[0, \tau]$ and

$$\bar{w}(t) := \max_{x \in B^n(R)} \Big| f(t, x, 0, \bar{u}(t)) \Big|, \quad t \in [0, \tau], \tag{2.3}$$

assume that

$$\frac{3}{2\alpha} \int_0^\tau \bar{w}(s)\, ds \leq R. \tag{2.4}$$

Then there exists a function $\bar{x} \in AC^n_\infty[0, \tau]$ such that the pair $(\bar{x}, \bar{u})$ is an admissible process to the problem (1.1), (1.2) and

$$|\bar{x}(t)| \leq R \quad \forall\, t \in [0, \tau] \quad \textit{and} \quad |\bar{x}(0)| \leq \frac{1}{2\alpha} \int_0^\tau \bar{w}(s)\, ds.$$

Note that the function $\bar{w}$ in Theorem 2.1 is Lebesgue integrable. Indeed, since f is continuous, then the function

$$(t, u) \mapsto \max_{x \in B^n(R)} \Big| f(t, x, 0, u) \Big|, \quad (t, u) \in [0, \tau] \times U$$

is continuous. Moreover, the function $\bar{u}$ is Lebesgue measurable and essentially bounded. Therefore, the function $\bar{w}$, defined by formula (2.3), is measurable and essentially bounded, hence $\bar{w}$ is integrable.

Note that the assumptions **(i)** and **(ii)** of Theorem 2.1 imply that there exist numbers $R > 0$, $\alpha \in (0, \bar{\alpha})$, $\tau > 0$ and a function $\bar{u} \in L^m_\infty[0, \tau]$ such that the inequality (2.4) holds for the function $\bar{w}$ defined by formula (2.3). Indeed, take arbitrary numbers $R > 0$, $\alpha \in (0, \bar{\alpha})$ and an arbitrary function $\widetilde{u} \in L_\infty^m([0, +\infty))$ such that $\widetilde{u}(t) \in U$ for almost all $t \geq 0$. Since the function

$$\tau \mapsto \int_0^\tau \max_{x \in B^n(R)} \Big| f(s, x, 0, \widetilde{u}(s)) \Big| \, ds, \quad \tau \geq 0$$

is continuous and vanishes zero at the point $\tau = 0$, then there exists $\tau > 0$ such that

$$\frac{3}{2\alpha} \int_0^\tau \max_{x \in B^n(R)} \Big| f(s, x, 0, \widetilde{u}(s)) \Big| \, ds < R.$$

Hence, (2.4) holds for the point τ, for the reduction $\bar{u} \in L_\infty^m[0, \tau]$ of $\widetilde{u}$ to the segment $[0, \tau]$ and for the function $\bar{w}$ defined by formula (2.3).

The regularity assumption **(ii)** is essential. Consider the corresponding example. Let $n = m = k = 1$, $f(t, x, \dot{x}, u) = u(u-1)\dot{x} + x - t^2$, $U = \{0, 1\}$, $\tau > 0$. Then $\bar{\alpha} = 0$ and therefore the assumption **(ii)** fails. Moreover, for every measurable essentially bounded function $u : [0, \tau] \to U$, the only solution to the ODE $f(t, x, \dot{x}, u(t)) = 0$ is the function $x(t) = t^2$. For this function x, we have $x(0) + x(\tau) > 0$. Therefore, in this example, there exists no admissible process.

Now let us derive the sufficient conditions for the existence of an admissible control. The following assertion follows directly from Theorem 2.1.

Corollary 2.1:
Let the assumptions **(i)** *and* **(ii)** *of Theorem 2.1 hold, the set U be closed. Given numbers $R > 0$, $\alpha \in (0, \bar{\alpha})$, $\tau > 0$ and a function*

$$\bar{w}_U(t) := \min_{u \in U} \max_{x \in B^n(R)} \Big| f(t, x, 0, u) \Big|, \quad t \in [0, \tau],$$

assume that

$$\frac{3}{2\alpha} \int_0^\tau \bar{w}_U(s) \, ds \leq R.$$

Then there exist functions $\bar{x} \in AC_\infty^n[0, \tau]$ and $\bar{u} \in L_\infty^n[0, \tau]$ such that the pair $(\bar{x}, \bar{u})$ is an admissible process to the problem (1.1), (1.2) and

$$|\bar{x}(t)| \leq R \quad \forall\, t \in [0, \tau] \quad \textit{and} \quad |\bar{x}(0)| \leq \frac{1}{2\alpha} \int_0^\tau \bar{w}_U(s) \, ds.$$

Note that both Theorem 2.1 and Corollary 2.1 remain considerable in the partial case when the dynamics in (1.1) is explicit, i.e. for the problem

$$\dot{x} = \widetilde{f}(t, x, u(t)), \quad u(t) \in U \quad \dot{\forall}\, t \in [0, \tau], \quad x(0) + x(\tau) = 0. \tag{2.5}$$

In this case the following assertion is valid. *Let $\widetilde{f}$ be twice continuously differentiable and U be closed. Given numbers $R > 0$ and $\tau > 0$ and a function*

$$w_U(t) := \min_{u \in U} \max_{x \in B^n(R)} \Big| \widetilde{f}(t, x, u) \Big|, \quad t \in [0, \tau],$$

assume that

$$\frac{3}{2}\int\limits_0^\tau w_U(s)\,ds < R.$$

Then there exist functions $\bar{x} \in AC^n_\infty[0,\tau]$ *and* $\bar{u} \in L^n_\infty[0,\tau]$ *such that the pair* $(\bar{x},\bar{u})$ *is an admissible process to the problem (2.5) and*

$$|\bar{x}(t)| \le R \quad \forall\, t \in [0,\tau], \quad |\bar{x}(0)| \le R/3.$$

3. PROOF OF THE MAIN RESULT

In the proof of Theorem 2.1, along with classical theorems on the properties of solutions to ODEs and fixed point theorems, we will use the global implicit function theorem from [2]. Let us recall it.

Theorem 3.1:
(see [2, Theorem 5]) Given a positive integer s, *an open nonempty set* $\Sigma \subset \mathbb{R}^s$ *and a twice continuously differentiable mapping* $F : \mathbb{R}^n \times \Sigma \to \mathbb{R}^k$, *assume that*

$$\alpha_0 := \inf\{\mathrm{cov} F'_v(v,\sigma) > 0 :\ v \in \mathbb{R}^n, \quad \sigma \in \Sigma\} > 0.$$

Then there exists a continuously differentiable mapping $G : \Sigma \times \mathbb{R}^k \to \mathbb{R}^n$ *such that*

$$F(G(y,\sigma),\sigma) = y, \quad |G(y,\sigma)| \le \frac{|y - F(0,\sigma)|}{\alpha_0} \quad \forall\, y \in \mathbb{R}^k, \quad \forall\, \sigma \in \Sigma. \tag{3.6}$$

In what follows, we use the following corollary of this theorem.

Corollary 3.1:
Let f *satisfy the assumptions* **(i)** *and* **(ii)** *of Theorem 2.1. Then for any positive* $\alpha < \bar{\alpha}$ *there exists a continuously differentiable mapping* $g : \mathbb{R}^1 \times \mathbb{R}^n \times U \to \mathbb{R}^n$ *such that*

$$f(t,x,g(t,x,u),u) = 0, \quad |g(t,x,u)| \le \frac{|f(t,x,0,u)|}{\alpha} \quad \forall\, (t,x,u) \in \mathbb{R}^1 \times \mathbb{R}^n \times U. \tag{3.7}$$

Proof
Take an arbitrary $\alpha < \bar{\alpha}$. Since **(i)** and **(ii)** holds, f is sufficiently smooth and the function cov is continuous, then there exists an open neighbourhood $U_0 \subset \mathbb{R}^m$ of U such that

$$\alpha \le \inf\big\{\mathrm{cov} f'_v(t,x,v,u) :\ (t,x,v,u) \in \mathbb{R}^1 \times \mathbb{R}^n \times \mathbb{R}^n \times U_0\big\}. \tag{3.8}$$

Put $\Sigma := \mathbb{R}^1 \times \mathbb{R}^n \times U_0$. Define the mapping $F : \mathbb{R}^n \times \Sigma \to \mathbb{R}^k$ by formula

$$F(v,\sigma) := f(t,x,v,u) \quad \sigma := (t,x,u) \in \Sigma, \quad v \in \mathbb{R}^n.$$

It is a straightforward task to ensure that F is twice continuously differentiable and

$$F'_v(v,\sigma) \equiv f'_v(t,x,v,u). \tag{3.9}$$

Thus,

$$\begin{aligned} \alpha_0 := \inf\{\mathrm{cov} F'_v(v,\sigma) > 0 :\ v \in \mathbb{R}^n, \quad \sigma \in \Sigma\} \overset{(3.9)}{=} \\ \overset{(3.9)}{=} \inf\big\{\mathrm{cov} f'_v(t,x,v,u) :\ (t,x,v,u) \in \mathbb{R}^1 \times \mathbb{R}^n \times \mathbb{R}^n \times U_0\big\} \overset{(3.8)}{\ge} \alpha > 0. \end{aligned} \tag{3.10}$$

Therefore, the mapping F satisfies the assumptions of Corollary 3.1. Hence, there exists a continuously differentiable mapping $G:\mathbb{R}^k\times\Sigma\to\mathbb{R}^n$ such that (3.6) holds.

Put

$$g(t,x,u):=G(0,\sigma),\quad \sigma=(t,x,u)\in\mathbb{R}^1\times\mathbb{R}^n\times U\subset\Sigma.$$

The mapping g is well-defined, since $U\subset U_0$.

Let us show that g is the desired mapping. Obviously g is continuously differentiable, since G is. Moreover, we have

$$f(t,x,g(t,x,u),u)=F(G(0,\sigma),\sigma)\overset{(3.6)}{=}0,$$

$$|g(t,x,u)|=|G(0,\sigma)|\overset{(3.6)}{\le}\frac{|F(0,\sigma)|}{\alpha_0}\overset{(3.10)}{\le}\frac{|F(0,\sigma)|}{\alpha}=\frac{|f(t,x,0,u)|}{\alpha}$$

for any $\sigma=(t,x,u)\in\mathbb{R}^1\times\mathbb{R}^n\times U$. □

Proof of Theorem 2.1

Take arbitrary numbers $R>0$, $\alpha\in(0,\bar{\alpha})$, $\tau>0$, a function $\bar{u}\in L^m_\infty[0,\tau]$ and the function $\bar{w}$ defined by the equality (2.3) such that (2.4) holds. It follows from Corollary 3.1 that there exists a continuously differentiable mapping $g:\mathbb{R}^1\times\mathbb{R}^n\times U\to\mathbb{R}^n$ such that (3.7) holds.

Put

$$r:=\frac{1}{2\alpha}\int_0^\tau \bar{w}(s)\,ds.$$

Then (2.4) implies

$$0\le r<R,\quad \frac{1}{\alpha}\int_0^\tau \bar{w}(s)\,ds\le R-r\quad\text{and}\quad -2r+\frac{1}{\alpha}\int_0^\tau \bar{w}(s)\,ds\le 0. \tag{3.11}$$

For arbitrary $x_0\in B^n(r)$, consider the Cauchy problem

$$\dot{x}=g(t,x,\bar{u}(t)),\quad t\in[0,\tau],\quad x(0)=x_0. \tag{3.12}$$

Since

$$\int_0^\tau \max_{|x-x_0|\le R-r}|g(s,x,\bar{u}(s))|\,ds\le\int_0^\tau\max_{x\in B_n(R)}|g(s,x,\bar{u}(s))|\,ds\overset{(3.7)}{\le}$$

$$\overset{(3.7)}{\le}\frac{1}{\alpha}\int_0^\tau\max_{x\in B_n(R)}|f(s,x,0,\bar{u}(s))|\,ds\overset{(2.3)}{=}\frac{1}{\alpha}\int_0^\tau\bar{w}(s)\,ds\overset{(3.11)}{\le}R-r,$$

it follows from [6, Theorem II.4.1] that there exists a solution $\varphi(\cdot,x_0):[0,\tau]\to B^n(r)$ to the problem (3.12). It follows from [6, Theorem II.4.5] that this solution is unique. It follows from [6, Theorem II.4.11] that the mapping $\varphi(\tau,\cdot):B^n(r)\to B^n(R)$ is continuous.

Define the mapping $\psi:B^n(r)\to\mathbb{R}^n$ by formula

$$\psi(x_0):=-\varphi(\tau,x_0),\quad x_0\in B^n(r).$$

This mapping is continuous, since $\varphi(\tau,\cdot)$ is continuous. Moreover, the following relation takes place

$$\langle\psi(x_0)-x_0,x_0\rangle\le 0\quad\forall\,x_0:\quad|x_0|=r. \tag{3.13}$$

Indeed,

$$\psi(x_0) = -\varphi(\tau, x_0) = -x_0 - \int_0^\tau g(s, \varphi(s, x_0), \bar{u}(s))\, ds \tag{3.14}$$

and, hence, for $|x_0| = r$ we obtain

$$\langle \psi(x_0) - x_0, x_0 \rangle \overset{(3.14)}{=} -2|x_0|^2 - \left\langle \int_0^\tau g(s, \varphi(s, x_0), \bar{u}(s))\, ds, x_0 \right\rangle \leq$$

$$\leq -2|x_0|^2 + |x_0| \int_0^\tau |g(s, \varphi(s, x_0), 0, \bar{u}(s))|\, ds = -2r^2 + r \int_0^\tau |g(s, \varphi(s, x_0), \bar{u}(s))|\, ds \overset{(3.7)}{\leq}$$

$$\overset{(3.7)}{\leq} -2r^2 + \frac{r}{\alpha} \int_0^\tau |f(s, \varphi(s, x_0), 0, \bar{u}(s))|\, ds \overset{(2.3)}{\leq} -2r^2 + \frac{r}{\alpha} \int_0^\tau \bar{w}(s)\, ds \overset{(3.11)}{\leq} 0.$$

So, it follows from (3.13) that the mapping ψ has a fixed point $\bar{x}_0 \in B^n(r)$, i.e. $\bar{x}_0 = \psi(\bar{x}_0)$ (see, for example, [5, §1.6]).

Put $\bar{x}(t) := \varphi(t, \bar{x}_0)$, $t \in [0, \tau]$. Let us show that the function $\bar{x}$ is a desired admissible trajectory. We have

$$f(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{u}(t)) = f(t, \bar{x}(t), \dot{\varphi}(t, \bar{x}_0), \bar{u}(t)) = f(t, \bar{x}(t), g(t, x, \bar{u}(t)), \bar{u}(t)) \overset{(3.7)}{=} 0$$

for almost all $t \in [0, \tau]$. Moreover,

$$\bar{x}(0) + \bar{x}(\tau) = \varphi(0, \bar{x}_0) + \varphi(\tau, \bar{x}_0) = \bar{x}_0 - \psi(\bar{x}_0) = 0.$$

Thus, $(\bar{x}, \bar{u})$ is an admissible process. Finally,

$$|\bar{x}(t)| = |\varphi(t, \bar{x}_0)| \leq R \quad \forall\, t \in [0, \tau];$$

$$|\bar{x}(0)| = |\bar{\varphi}(0, \bar{x}_0)| = |\bar{x}_0| \leq r.$$

Therefore, the function $\bar{x}$ is a desired admissible trajectory. □

ACKNOWLEDGEMENTS

The research is supported by the Russian Science Foundation No 22-11-00042, https://rscf.ru/project/22-11-00042/.

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