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Bifurcation in the Model of Cargo Transportation Organization

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Abstract: This article is devoted to the study of the model of cargo transportation organization between two nodal stations. The main characteristic of an arbitrary station is the degree of inconsistency between receiving and sending cargo, which is the difference between the volume of incoming and outgoing cargo per unit of time. The initial node station accepts goods depending on the demand for them within its technical potential, which is determined by the maximum allowable increase in the degree of inconsistency between the reception and dispatch of goods per unit of time. The movement of goods from one station to another is carried out within the framework of their technical potentials. The distribution of goods from the final node station is carried out in a certain mode. Such a model is described by a system of differential equations with a number of parameters that define the characteristics of the demand for cargo transportation, the degree of use of the technical potential of the stations and the mode of cargo distribution from the final node station. When changing the parameters of the model, a bifurcation effect occurs and multiple stationary solutions appear, among which the most acceptable from the point of view of economic feasibility are identified.

Keywords: cargo transportation organization model, differential equations, stationary solutions, model parameters, stability

1. INTRODUCTION

The mathematical models used for the analysis of transport networks are diverse in terms of the tasks to be solved, the mathematical apparatus, the data used and the degree of detail of the description of traffic. Based on the functional role of models, i.e. on the tasks for which they are used, three main classes can be conditionally distinguished: predictive, simulation and optimization models [24].

Predictive models allow to determine what will be the traffic flows in the network with known geometry and properties of the transport network. The forecast of the load of the transport network includes the calculation of the average characteristics of traffic, such as the volume of inter-district movements, the intensity of the flow, the distribution of vehicles along the paths, etc. With the help of such models, it is possible to predict the consequences of changes in the transport network or in the placement of objects.

Simulation modeling aims to reproduce all the details of the movement, while the averaged value of the flows and the distribution along the paths are considered known and serve as the initial data for these models. Thus, flow forecasting and simulation modeling are complementary directions [9].

The third class of models is aimed at optimizing the functioning of transport networks. With their help the problems of optimization of transport routes, development of the optimal network configuration, etc. are solved [10, 23, 26, 27].

According to the method of describing traffic flows, all models of transport networks can be divided into classes: models-analogues, models-following the leader, models-probabilistic.

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In analog models, the movement of a vehicle is likened to some kind of physical flow (hydro- and gas-dynamic models) [6, 7, 9].

In models of following the leader, it is important to assume that there is a connection between the movement of the driven and the main vehicle [5].

In probabilistic models, the traffic flow is considered as a result of the interaction of vehicles on the elements of the transport network. Due to the rigid nature of network restrictions and the mass nature of traffic in the traffic flow, there are clear patterns of formation of queues, intervals, loads on the road lanes, etc. These patterns are significantly stochastic [25].

One of the most popular modes of transport for cargo transportation in Russia is rail. Publications devoted to railway logistics can be divided into the following main groups according to the type of tasks studied:

1) the tasks of designing the infrastructure of the railway network;
2) tasks of managing the fleet of locomotives and wagons;
3) tasks of railway planning.

In the first group we can highlight the works [8, 11, 19, 22]. The task of managing the fleet of locomotives and wagons is described in [4]. The third group, in particular, is represented by the tasks of forming the schedule of freight trains and the organization of freight flows [20, 21].

One of the approaches to the organization of cargo traffic is described in [1, 2, 12-16]. They present macroscopic dynamic models in which the process of organizing railway freight transportation is the formation of cargo traffic based on the interaction of neighboring stations. By their functional role, they are predictive, because they allow predicting the dynamics of loading of the stations and flows arising in the railway network, at the set procedure of the organization of a cargo flow. In the way of describing traffic flows, they are close to the models of following the leader, if the flow on a particular section of the railway network is identified with the congestion of the respective stations. At the same time, these models have a significant difference from the models of following the leader, which means that each station interacts not with one, but with the two nearest neighboring stations.

In [17] and [18], another approach to the organization of cargo traffic is presented, in which the process of its formation is determined by the demand for cargo transportation and the degree of use of the technical potential of stations.

This work is devoted to the study of the modification of the model described in [17], and unlike it, it is initially aimed at finding practically feasible modes of cargo transportation. Such models are characterized by the global stability of a stationary solution, which is sometimes economically impractical. At the same time, when the model parameters change, a bifurcation effect occurs and multiple stationary solutions appear, among which there are economically feasible ones. Thus, in this model, the presence of bifurcation makes it possible to obtain solutions that describe the optimal functioning of the system.

2. PROBLEM STATEMENT

Consider the movement of goods on a section of the railway network between two nodal stations connected by a set of intermediate stations. Denoting the number of intermediate stations by m , we get the following set of station numbers $\{0,1,..,m,m+1\}$, where 0 – is the number of the initial node station, and, a $m+1$ – is the number of the final node station. Denote by n_i the number of roads at the station with the number i. We assume that all paths are used with the same degree of efficiency.

Consider discrete moments of time $t_0,\ t_1,\ t_2,\dots;\ t_k = t_{k-1} + \Delta t,\ k = 1,2,\dots$

Let $\bar{V}_{ij}(t_k)$ is the volume of cargo received on the jth road of the ith station for a period of time $[t_{k-1}, t_k]$, and $\bar{\bar{V}}_{ij}(t_k)$ is the volume of cargo sent from the ith road of the ith station for a period of time $[t_{k-1}, t_k]$. Denote

$$x_{ij}(t_k) = \begin{cases} \dfrac{\bar{V}_{ij}(t_k) - \bar{\bar{V}}_{ij}(t_k)}{\bar{V}_{ij}(t_k)}, & if \quad \bar{V}_{ij}(t_k) > \bar{\bar{V}}_{ij}(t_k) \\ 0, & if \quad \bar{V}_{ij}(t_k) \le \bar{\bar{V}}_{ij}(t_k). \end{cases}$$

It's obvious that

$$0 \le x_{ij}(t_k) \le 1, \quad i = 0,1,\dots,m+1; \quad j = 1,2,\dots,n_i$$

and characterize the degree of inconsistency between receiving and sending on the jth road of the ith station at time t_k. Denote

$$z_i(t_k) = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij}(t_k).$$

It is also obvious that $0 \le z_i(t_k) \le 1$ and characterizes the degree of inconsistency between receiving and sending at the ith station at time t_k.

The technical potential of the station is determined by the maximum allowable increase in the degree of inconsistency between the reception and dispatch of goods per unit of time and is given by a non-negative decreasing function $\varphi(z)$ defined on the segment [0,1] and satisfying the condition $\varphi(1) = 0$.

The initial node station accepts cargo depending on the demand for transportation within its technical potential and sends it to the next station within its technical potential. Each of the intermediate stations receives cargo within its technical potential and sends it within the technical potential of the next station. The final node station accepts cargo within its technical potential and distributes it in a certain mode. Taking into account the above, we will write down a system of finite-difference equations describing the change in the degree of inconsistency between the reception and dispatch of goods at stations.

$$z_0(t_k) = z_0(t_{k-1}) + \left[\min\left(d_0, \varphi\big(z_0(t_{k-1})\big)\right) - \lambda\varphi\big(z_1(t_{k-1})\big)\right]\Delta t, \quad k = 1,2,3\dots \tag{2.1}$$

$$z_i(t_k) = z_i(t_{k-1}) + \left[\lambda\varphi\big(z_i(t_{k-1})\big) - \lambda\varphi\big(z_{i+1}(t_{k-1})\big)\right]\Delta t, \quad i = \overline{1,m}, \quad k = 1,2,3\dots \tag{2.2}$$

$$z_{m+1}(t_k) = z_{m+1}(t_{k-1}) + \left[\lambda\varphi\big(z_{m+1}(t_{k-1})\big) - d_{m+1}\right]\Delta t, \quad k = 1,2,3\dots \tag{2.3}$$

$$0 \le z_i(t_k) \le 1, \quad i = 0,1,\dots,m+1, \quad k = 0,1,2,\dots \tag{2.4}$$

Here $d_0 > 0$, $0 < \lambda \le 1$, $d_{m+1} > 0$ are the model parameters:

d_0 is a characteristics of the demand for transportation;

λ is a characteristics of the degree of use of the technical potential of the stations;

d_{m+1} is a characteristics of the cargo distribution mode from the final node station.

Let's move on to the continuous analog of the system of discrete-difference equations (2.1)–(2.4), presented below

$$\dot{z}_0(t) = \min(d_0, \varphi(z_0(t))) - \lambda\varphi(z_1(t)), \quad t \in [t_0, +\infty), \tag{2.5}$$

$$\dot{z}_i(t) = \lambda[\varphi(z_i(t)) - \varphi(z_{i+1}(t))], \quad i = 1,2,\dots,m, \quad t \in [t_0, +\infty), \tag{2.6}$$

$$\dot{z}_{m+1}(t) = \lambda\varphi(z_{m+1}(t)) - d_{m+1}, \quad t \in [t_0, +\infty), \tag{2.7}$$

$$0 \le z_i(t) \le 1, \quad i = 0, 1, \dots, m+1, \quad t \in [t_0, +\infty). \tag{2.8}$$

Next, consider the function that sets the technical potential of the stations, of the following type

$$\varphi(z) = a(1 - z), \quad a > 0. \tag{2.9}$$

The parameter $a > 0$, which participates in the definition of the function $\varphi(z)$, is a characteristic of the ability of stations to increase cargo traffic. Since $\varphi(z) \leq \mathrm{a}$ for all $0 \leq z \leq 1$ then the parameter d_0, which is a characteristic of the demand for transportation and is involved in equation (1.1), can be represented as follows:

$$d_0 = \mu a, \quad 0 < \mu \leq 1. \tag{2.10}$$

The parameter d_{m+1}, which is a characteristic of the cargo distribution mode from the final node station, is represented as follows:

$$d_{m+1} = \gamma a, \quad \gamma > 0. \tag{2.11}$$

Let's rewrite the system (2.5)-(2.8), in which the function $\varphi(z)$ is defined according to (2.9), and the parameters d_0 and $d_{\mathrm{m+1}}$ are defined according to (2.10) and (2.11), respectively.

$$\dot{z}_0(t) = \min(\mu a,\ a(1 - z_0(t))) - \lambda a(1 - z_1(t)), \quad t \in [t_0, +\infty), \tag{2.12}$$

$$\dot{z}_i(t) = \lambda a\ (z_{i+1}(t) - z_i(t)), \quad i = 1,2,\dots,m, \quad t \in [t_0, +\infty), \tag{2.13}$$

$$\dot{z}_{m+1}(t) = \lambda a(1 - z_{m+1}(t)) - \gamma a\,, \quad t \in [t_0, +\infty), \tag{2.14}$$

$$0 \leq z_i(t) \leq 1\,, \quad i = 0,1,\dots,m+1, \quad t \in [t_0, +\infty). \tag{2.15}$$

Here $\mu, \ \ a, \ \ \lambda, \ \ \gamma$ are the model parameters:

μ $(0 < \mu \leq 1)$ is a characteristics of the range of demand for transportation, which can be satisfied with the existing technical potential of the stations;

a $(a > 0)$ is a characteristic of the ability of stations to increase cargo traffic;

λ $(0 < \lambda \leq 1)$ is a characteristics of the degree of use of the technical potential of the stations;

γ $(\gamma > 0)$ is a characteristics of the cargo distribution mode from the final node station.

Here are the main objectives of the study:

- to determine the ranges of change of parameters $\mu, \ \ a, \ \ \lambda, \ \ \gamma$, in which the system of cargo transportation can function smoothly, i.e. system (2.12)-(2.15) has a solution, describe the qualitative behavior of solutions depending on the parameters.
- for a given value of the demand characteristics for cargo transportation (parameter μ) set the most acceptable achievable levels of the degree of inconsistency between the reception and dispatch of goods at all stations, by controlling the values of the following characteristics: the ability of stations to increase cargo traffic (parameter a), the degree of use of the technical potential of stations (parameter λ) and the mode of cargo distribution from the final node station (parameter γ).

3. INVESTIGATION OF SYSTEM SOLUTIONS (2.12)-(2.15)

The study of the set of solutions of the system (2.12)–(2.15) begins with the study of all solutions of the system of differential equations (2.12)–(2.14).

First of all, we will highlight the stationary solutions of the system (2.12)–(2.14). With the help of direct verification, you can verify the validity of the following statement

Proposition 3.1:

System (2.12)–(2.14) for any parameter values

$$0 < \mu \leq 1, \quad a > 0, \quad 0 < \lambda \leq 1, \quad \gamma > 0$$

such that $\gamma \leq \mu$ has stationary solutions:

$$z_0(.) \equiv 1 - \gamma, \qquad z_i(.) \equiv 1 - \tfrac{\gamma}{\lambda}, \quad i = 1,\dots,m+1, \text{ for } \gamma < \mu; \tag{3.1}$$

$$z_0(.) \leq 1 - \mu, \ \ z_i(.) \equiv 1 - \tfrac{\mu}{\lambda}, \quad i = 1,\dots,m+1, \text{ for } \gamma = \mu. \tag{3.2}$$

For $\gamma > \mu$ the system (2.12)–(2.14) has no stationary solutions.

Let's proceed to the study of the remaining solutions of the system (2.12)–(2.14).

Theorem 3.1:

An arbitrary solution of the system (2.12)–(2.14) at $\gamma < \mu$ eventually goes to a stationary solution (3.1), and at $\gamma = \mu$ to one of the stationary solutions (3.2). For $\gamma > \mu$ coordinates $z_i(.)$, $i = 1, \ldots, m+1$ of solutions of the system (2.12)–(2.14) eventually enter the stationary mode specified in (3.1), and the function $z_0(.)$ becomes linearly decreasing.

Proof.

Let is find the general solution of the system (2.12)–(2.14). Let is start with the last equation, which contains one variable (z_{m+1}). Let 's rewrite it in the following form

$$\dot{z}_{m+1}(t) + \lambda a z_{m+1}(t) = (\lambda - \gamma)a. \tag{3.3}$$

It is not difficult to verify that linear equation (3.3) has the following general solution

$$z_{m+1}(t) = 1 - \frac{\gamma}{\lambda} + c_{m+1}e^{-\lambda at}. \tag{3.4}$$

Substituting the expression for z_{m+1} from (3.4) into the penultimate equation of the system (2.12)–(2.14), we find its general solution. It has the following form

$$z_m(t) = 1 - \frac{\gamma}{\lambda} + e^{-\lambda at}(\lambda a c_{m+1} t + c_m). \tag{3.5}$$

Similarly, we will find general solutions to all other equations of the system (2.12)-(2.14) except the initial one:

$$z_{m-1}(t) = 1 - \frac{\gamma}{\lambda} + e^{-\lambda at}\left(\frac{\lambda a c_{m+1}}{2} t^2 + c_m t + c_{m-1}\right);$$

$$\ldots$$

$$z_{m-k}(t) = 1 - \frac{\gamma}{\lambda} + e^{-\lambda at}\left(\frac{\lambda a c_{m+1}}{(k+1)!} t^{k+1} + c_m t^k + \cdots + c_2 t + c_1\right); \tag{3.6}$$

$$z_1(t) = 1 - \frac{\gamma}{\lambda} + e^{-\lambda at}\left(\frac{\lambda a c_{m+1}}{m!} t^m + c_m t^{m-1} + \cdots + c_2 t + c_1\right).$$

It follows from (3.5) and (3.6) that

$$\lim_{t\to+\infty} z_i(t) = 1 - \frac{\gamma}{\lambda}, \quad i = 1, 2, \ldots, m.$$

Let's move on to solving the first equation of the system (2.12)-(2.14). Let 's rewrite it in the following form

$$\dot{z}_0(t) = \begin{cases} \mu a - \lambda a\big(1 - z_1(t)\big), & \text{if } z_0(t) < 1 - \mu, \ \ t \in [t_0, +\infty), \\ a\big(1 - z_0(t)\big) - \lambda a\big(1 - z_1(t)\big), & \text{if } z_0(t) \geq 1 - \mu, \ \ t \in [t_0, +\infty). \end{cases} \tag{3.7}$$

Consider the following two equations

$$\dot{z}_0(t) = \mu a - \lambda a\big(1 - z_1(t)\big) \ , \ t \in [\,\bar{t}, +\infty), \tag{3.8}$$

$$\dot{z}_0(t) = a\big(1 - z_0(t)\big) - \lambda a\big(1 - z_1(t)\big), \quad t \in [\,\bar{t}, +\infty), \tag{3.9}$$

where $\bar{t} \geq t_0$.

Using the expression for $z_1(t)$ from (3.6), we obtain the solution of equations (3.8) and (3.9). They can be represented as follows

$$z_0(t) = a(\mu - \gamma)t + F_1(t) + c_o, \text{ where } F_1(t) \in C^\infty[\,\bar{t}, +\infty), \ \lim_{t\to+\infty} F_1(t) = 0, \tag{3.10}$$

$$z_0(t) = 1 - \gamma + F_2(t), \text{ where } F_2(t) \in C^\infty[\,\bar{t}, +\infty), \ \lim_{t\to+\infty} F_2(t) = 0 \tag{3.11}$$

These solutions allow us to investigate the asymptotic behavior of the solution of equation (3.7). It is easy to see that when $\gamma < \mu$ the asymptotics of the solution of equation (3.7) is determined by the relation (3.11), i.e. $\lim_{t\to+\infty} z_0(t) = 1-\gamma$. For $\gamma > \mu$ the asymptotics of equation (3.7) is determined by the relation (3.10), i.e. $z_0(.)$ decreases linearly and $\lim_{t\to+\infty} z_0(t) = -\infty$. For $\gamma = \mu$ the asymptotics of equation (3.7), depending on the initial conditions, can be determined by both equation (3.10) and equation (3.11), i.e. either $\lim_{t\to+\infty} z_0(t) = c_0$, where $c_0 < 1-\mu$ or $\lim_{t\to+\infty} z_0(t) = 1-\mu$. □

Let us proceed to the study of solutions of system (2.12)-(2.14) satisfying constraints (2.15).

Lemma 3.1:

For all values of parameters $a > 0$, $\gamma > 0$ *and* λ , $\gamma \le \lambda \le 1$ *components* $z_1(.)$, $z_2(.)$, ... $z_{m+1}(.)$ *of an arbitrary solution of the system (2.12)-(2.14) satisfying the constraints (2.15) at the initial moment of time will satisfy them at subsequent moments of time.*

Proof.

Let's start by considering the last component of the solution of the system (2.12)-(2.14), i.e. $z_{m+1}(.)$. It has the form (3.4), where c_{m+1} is determined from the condition

$$1-\frac{\gamma}{\lambda}+c_{m+1}e^{-\lambda a t_0} = \overline{z}_{m+1}, \text{ where } 0 \le \overline{z}_{m+1} \le 1,$$

i.e.

$$c_{m+1} = (\frac{\gamma}{\lambda}-1+\overline{z}_{m+1})e^{\lambda a t_0}, \text{ where } 0 \le \overline{z}_{m+1} \le 1. \quad (3.12)$$

From (3.12) follows

$$(\frac{\gamma}{\lambda}-1)e^{\lambda a t_0} \le c_{m+1} \le \frac{\gamma}{\lambda}e^{\lambda a t_0} .$$

Using this estimate for c_{m+1} and expression (3.4), we get an estimate for $z_{m+1}(.)$. It will take the following form

$$(1-\frac{\gamma}{\lambda})(1-e^{\lambda a t_0}e^{-\lambda a}) \le z_{m+1}(t) \le 1-\frac{\gamma}{\lambda}(1-e^{\lambda a t_0}e^{-\lambda a t}). \quad (3.13)$$

It follows from (3.13) that for all $\gamma \le \lambda$ there is an inequality

$$0 \le z_{m+1}(t) \le 1, \quad t \in [t_0, +\infty) . \quad (3.14)$$

We show that for the other components of the solution of the system (2.12)-(2.14), inequalities similar to inequality (3.14) are fulfilled. Let's start with the component $z_m(.)$. To do this, consider equation (2.13) for $i = m$:

$$\dot{z}_m(t) = \lambda a\left(z_{m+1}(t) - z_m(t)\right), \quad t \in [t_0, +\infty).$$

We show that the function $z_m(.)$ cannot take a value greater than 1. Indeed, otherwise, due to the continuity of the function $z_m(.)$ there must be a point $t^* > t_0$, such that $z_m(t^*) = 1$. Then it follows from (3.14) that $\dot{z}_m(t^*) \le 0$. Similarly, the function $z_m(.)$ cannot take a value less than 0. Thus, it is proved that for all $\gamma \le \lambda$ function $z_m(.)$ also satisfies an inequality similar to inequality (3.14). Similarly, the satisfiability of all other inequalities is proved. □

We formulate a similar lemma for the zero component of the solution of system (2.12)-(2.14).

Lemma 3.2:

For all values of parameters $a > 0$, $\gamma > 0$ *and* μ, $\gamma \le \mu \le 1$ *there exists* $\tilde{\lambda}(\mu,\ z_0(t_0),\ z_1(t_0),\ \ldots,\ z_{m+1}(t_0))$, $\mu \le \tilde{\lambda} \le 1$ *such that for any the value of the parameter* λ *from the segment* $[\mu,\ \tilde{\lambda}]$ *the zero component* $z_0(.)$ *of an arbitrary solution of the system (2.12)-(2.14) satisfying the constraint (2.15) at the initial moment of time, satisfies it at subsequent moments of time.*

Proof.

As for the other components, we show that the function $z_0(.)$ cannot take a value greater than 1. Indeed, otherwise, due to the continuity of the function $z_0(.)$ there must be a point $t^{**} > t_0$, such that $z_0(t^{**}) = 1$. Then it follows from (2.12) that

$$\dot{z}_0(t^{**}) = \lambda a(z_1(t^{**}) - 1),$$

that is, according to lemma 1, $\dot{z}_0(t^{**}) \leq 0$.

Let's move on to evaluating the function $z_0(.)$ from below. To do this, we investigate the behavior of its derivative at $z_0(.) \to 0+$. According to (2.12), it is described by the equation

$$\dot{z}_0(t) = a(\mu - \lambda(1 - z_1(t))).$$

Let 's investigate the inequality

$$\mu - \lambda(1 - z_1(t)) \geq 0.$$

Let 's rewrite it in the form

$$z_1(t) \geq 1 - \frac{\mu}{\lambda}. \tag{3.15}$$

According to lemma 1, for arbitrary $\gamma > 0$, λ satisfying the condition $\gamma \leq \lambda \leq 1$ there is an inequality

$$0 \leq z_1(t) \leq 1, \quad t \in [t_0, +\infty). \tag{3.16}$$

It follows from (3.16) that an arbitrary μ satisfying the condition $\gamma \leq \mu \leq 1$ exists $\tilde{\lambda}$, $\mu \leq \tilde{\lambda} \leq 1$ such that for any value of the parameter λ from the segment $[\mu, \tilde{\lambda}]$ inequality (3.15) will hold for all $t \in [t_0, +\infty)$, i.e. $\dot{z}_0(t) \geq 0$ for $z_0(t) \to 0+$, which shows that the function $z_0(.)$ is limited from below by the value 0. Obviously $\tilde{\lambda}$ depends on both μ, and initial conditions, so we denote it $\tilde{\lambda}(\mu, z_0(t_0), z_1(t_0), \ldots, z_{m+1}(t_0))$. □

We formulate the main result of this paragraph.

Theorem 3.2:

For any initial values $0 \leq z_i(t_0) \leq 1$, $i = 0, 1, \ldots, m+1$, *parameters* $a > 0$, $\gamma > 0$ *and* μ, $\gamma \leq \mu \leq 1$ *there exists* $\tilde{\lambda}(\mu, z_0(t_0), z_1(t_0), \ldots, z_{m+1}(t_0))$, $\mu \leq \tilde{\lambda} \leq 1$ *such that for any value of the parameter* λ *from the segment* $[\mu, \tilde{\lambda}]$ *the solution of the system (2.12)-(2.15) exists and converges to the stationary solution (3.1) (for* $\gamma < \mu$*) or to one of the stationary solutions (3.2), the same for all* γ *and* a *(for* $\gamma = \mu$*).*

Proof. The proof follows directly from Theorem 1, Lemma 1 and Lemma 2. □

Corollary 3.1:

The system of differential equations (2.12)–(2.15) has a globally stable stationary solution (3.1) and a family of stable solutions of the form (3.2).

Proof. The proof follows directly from Theorem 2. □

4. EXAMPLES OF SOLUTIONS OF THE SYSTEM (2.12)–(2.15)

Let's consider some examples of solutions of the system (2.12)–(2.15) that clearly demonstrate the results of the previous paragraph. In all the examples below, the number of stations is 10 (the initial node station, eight intermediate stations, the final node station), the values of parameters, μ, a and the initial conditions are fixed and take the following values:

$$\mu = 0.6, \quad a = 1.5 \tag{4.1}$$

$$z_0(t_0) = 0.4, \quad z_1(t_0) = 0.2, \quad z_2(t_0) = 0.1, \quad z_3(t_0) = 0.4, \quad z_4(t_0) = 0.7,$$
$$z_5(t_0) = 0.1, \quad z_6(t_0) = 0.2, \quad z_7(t_0) = 0.3, \quad z_8(t_0) = 0.5, \quad z_9(t_0) = 0.3. \tag{4.2}$$

According to theorem 3.2, for any initial values satisfying condition (2.15), parameters $a > 0$, $\gamma > 0$ and μ, $\gamma \leq \mu \leq 1$ there exists $\tilde{\lambda}$ ($\mu \leq \tilde{\lambda} \leq 1$) depending on the initial conditions and the parameter μ, such that for all λ from the segment $[\mu, \tilde{\lambda}]$ the solution of the system (2.12)–(2.15) exists. Calculations have shown that for the above initial conditions (4.2) and parameter μ (4.1), $\tilde{\lambda}$ takes the value equal to 0.898 ($\tilde{\lambda} = 0.898$). The nature of the solution of the system (2.12)–(2.15) depends on the value of the parameter γ. Depending on whether it is less than the value of the parameter μ or equal to it, there are two types of solutions of the system (2.12)–(2.15).

Let's start with the first case ($\gamma < \mu$). In fig. 4.1, fig. 4.2 and fig. 4.3. graphs of solutions of the system (2.12)–(2.15) with a fixed value of parameter γ ($\gamma = 0.5$) and three different values of parameter λ are given: two at the ends of the segment $[\mu, \tilde{\lambda}]$ and one at an internal point.

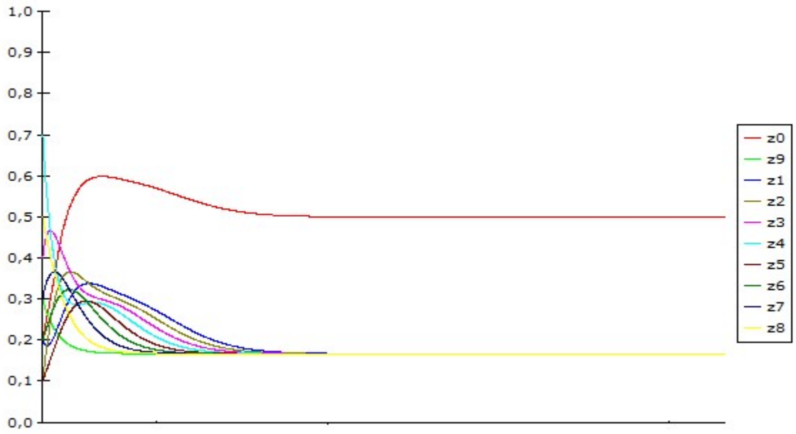

Fig. 4.1. Graph of the solution of the system (2.12)–(2.15) ($\gamma = 0.5$, $\mu = \lambda = 0.6$).

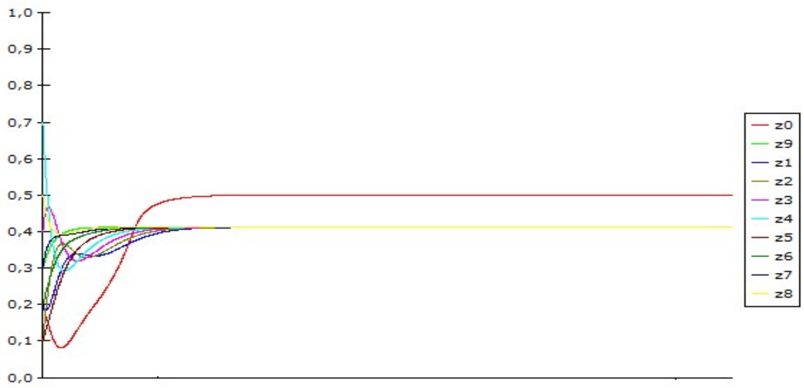

Fig. 4.2. Graph of the solution of the system (2.12)–(2.15) ($\gamma = 0.5$, $\mu = 0.6$, $\lambda = 0.85$)

Fig. 4.3. Graph of the solution of the system (2.12)–(2.15)
$(\gamma = 0.5, \quad \mu = 0.6, \quad \lambda = \tilde{\lambda} = 0.898)$

Note that due to the global stability of the stationary solution (3.1), the asymptotic behavior of the solutions shown in figures 4.1–4.3 does not change when the initial values change. These figures clearly show that the steady – state mode for the zero component of the solution of the system (2.12)–(2.15) does not depend on the parameter λ, and for the remaining components, an increase in the parameter λ leads to their asymptotic increase. Recall that the components of the solution of the system (2.12)–(2.15) determine the dynamics of the degree of inconsistency between the reception and dispatch of goods at stations. Therefore, in this case, the question of choosing a parameter λ is uniquely determined, its value should be taken equal to the value of the parameter μ.

Let's move on to the second case ($\gamma = \mu$). Fig. 4.4, fig. 4.5 and fig. 4.6 show graphs of solutions of the system (2.12)–(2.15) at $\gamma = 0.6$ and the same values of parameter λ, as for the previous case.

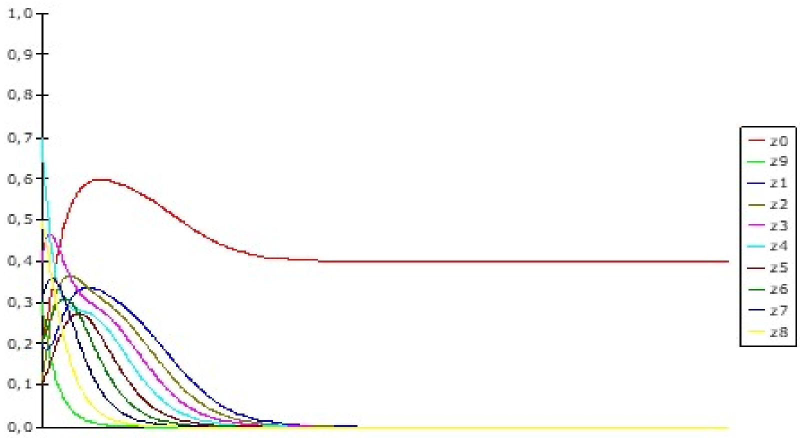

Fig. 4.4. Graph of the solution of the system (2.12)–(2.15)
$\gamma = \lambda = \mu = 0.6$

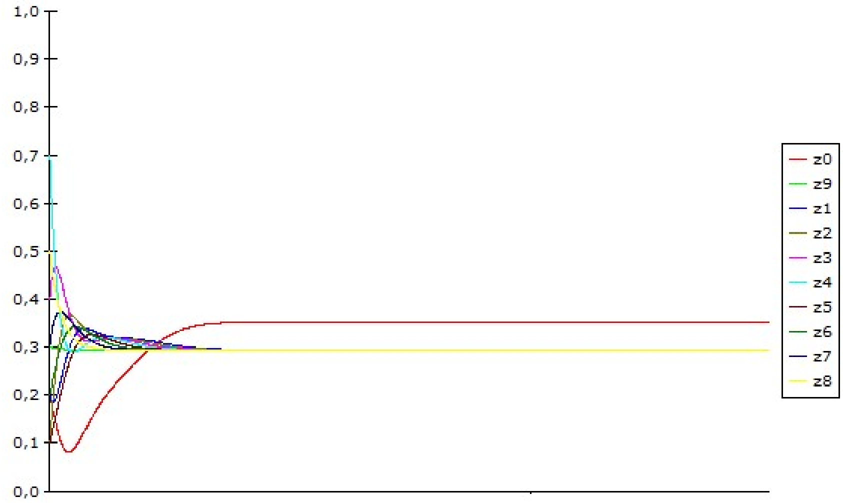

Fig. 4.5. Graph of the solution of the system (2.12)–(2.15)
$\gamma = \mu = 0.6, \quad \lambda = 0.85$

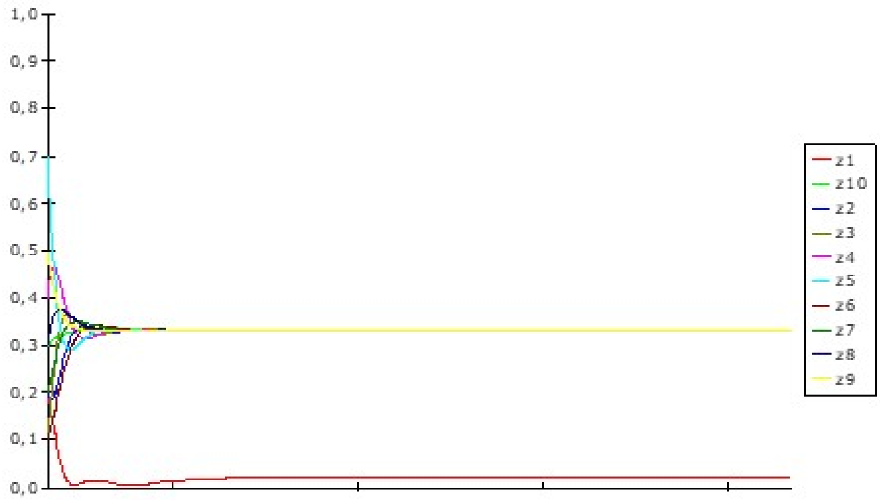

Fig. 4.6. Graph of the solution of the system (2.12)–(2.15)
$\gamma = \mu = 0.6, \quad \lambda = \tilde{\lambda} = 0.898$

Comparing figures 4.4, 4.5 and 4.6, it can be seen that an increase in the value of parameter λ leads to an asymptotic decrease in the zero component of the solution of the system (2.12)–(2.15), in contrast to the first case. The remaining components of the solution, as in the first case, increase asymptotically with an increase in the value of the parameter λ. This means that in this case, by controlling the parameter λ you can set the desired degree of inconsistency between the reception and dispatch of goods at all stations (including at the zero station). If we take λ equal to μ, then over time the degree of inconsistency between the reception and dispatch of goods at all stations except zero will become zero, and at zero station $1 - \mu$. If the value of μ is close to one, then this choice of the parameter λ will be optimal. Otherwise, everything will depend on the specific value of the parameter μ, as well as on how important it is in a given situation to reduce the degree of inconsistency between receiving and sending goods at the initial node station by increasing this characteristic at other stations.

Comparing figures 4.1 and 4.4, figures 4.2 and 4.5, as well as figures 4.3 and 4.6, it can be concluded that in terms of minimizing the degree of inconsistency between the reception and dispatch of goods at stations, it is advisable to take the parameter γ equal to the parameter μ. This means that the mode of cargo distribution from the final node station must be coordinated with the characteristics of the demand for transportation.

5. INVESTIGATION OF THE ATTRACTION AREA OF STATIONARY SOLUTIONS OF THE SYSTEM (2.12)–(2.15) OF THE FORM (3.2)

According to consequence 1, the system of differential equations (2.12)–(2.15) has a globally stable stationary solution (3.1) and a family of stable solutions of the form (3.2). We select the following stationary solution from this family

$$z_0(.) \equiv 1 - \mu, \quad z_i(.) \equiv 1 - \frac{\mu}{\lambda}, \quad i = 1, \dots, m+1. \tag{5.1}$$

As numerical experiments have shown, this stationary solution is in contrast to other stationary solutions of the form (3.2) presented below

$$z_0(.) < 1 - \mu, \quad z_i(.) \equiv 1 - \frac{\mu}{\lambda}, \quad i = 1, \dots, m+1 \tag{5.2}$$

and being locally stable, has a certain area of attraction.

Denote

$$\tilde{c}(\mu, \lambda) = \frac{(m+1)(1-\mu/\lambda)+1-\mu}{m+2}.$$

The analysis of a large number of numerical experiments made it possible to describe the region of attraction of the stationary solution (5.1). The results of this analysis are given in the following proposition.

Proposition 5.1:

For any parameter values $0 < \mu \leq 1$, $a > 0$, $\mu \leq \lambda \leq \tilde{\lambda}$, $\gamma = \mu$ *the solution of the system (2.12)–(2.15) converges to the stationary solution (5.1) if the initial values satisfy the condition*

$$z_i(t_0) \geq \tilde{c}(\mu, \lambda), \quad i = 0, 1, \dots, m+1.$$

Otherwise, i.e. if the condition$z_i(t_0) \leq \tilde{c}(\mu, \lambda)$, $i = 0, 1, \dots, m+1$ *and* $\exists\, \bar{\iota} \in \{0, 1, \dots, m+1\}$ *such that*

$$z_{\bar{\iota}}(t_0) < \tilde{c}(\mu, \lambda)$$

is met, then the solution of the system (2.12)–(2.15) converges to one of the stationary solutions (5.2). □

6. CONCLUSION

This article presents a model of organization of cargo transportation between two nodal stations, described by a system of differential equations with a number of parameters that define the characteristics of demand for cargo transportation, the degree of use of the technical potential of the stations and the mode of cargo distribution from the final nodal station. The ranges of parameter changes under which the cargo transportation system can function smoothly are determined. For a given value of the demand characteristics for cargo transportation, by controlling the degree of use of the technical potential of the stations and the mode of cargo distribution from the final node station, the most acceptable achievable levels of the degree of inconsistency between the reception and dispatch of goods at all stations are established.

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