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Approximate Solutions and Estimate of Galerkin Method for Variable Third-Order Operator-Differential Equation

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Abstract: The paper considers a variable third-order operator-differential equation in a separable Hilbert space. Under certain assumptions, it is proved that this ODE has a unique solution. The proof is based on a classical Galerkin discretization of the separable Hilbert space in term of certain eigenfunctions. The approximation quality of the Galerkin approximations can be controlled in terms of the eigenvalues. We deduce estimates for the convergence rate of the approximate solutions to the exact one. An example provided as application to the investigated method.

Keywords: Galerkin method, operator-differential equation, self-adjoint operator, convergence rate, orthogonal projection

1. INTRODUCTION

An important direction in modern mathematics is the study of operator-differential equations. Using the operator method, it is possible to study a wide class of differential equations. The most different types of equations, such as linear and non-linear ordinary differential equations, partial differential equations, integral and integral differential equations can be present in operator form. With the help of the methods of the functional analysis and the theory of operators we can study the question related to the solvability of boundary-values problems and develop algorithms for finding the approximate solutions. It is known that the theory and methodology of operator-differential equations widely used in computational mathematics.

Among all the differential-operator equations, the most comprehensively studied are the first and the second-order differential-operator equations. In this regard, we can specify the works of A.G. Zarubin, P.V. Vinogradova [20], A.B.I. Ahmed [1] which investigate strong solutions of the Cauchy problem for linear and non-linear differential-operator equations of the first order. Existence, uniqueness, and continuous dependence of strong solutions of the Cauchy problem for various second-order linear differential equations with variable domains proved in the works of D.A. Lyakhov [14] and F. E. Lomovtsev [13]. The solvability of the third and higher-order linear differential-operator equations were studied in works [3], [4], [8]. From the standpoint of the spectral theory of linear operators in Hilbert space and the Fourier method for studying the differential-operator equations of the first and higher orders are represented in [6]. Through some assumptions, the solvability for differential-operator equations of certain orders with variable coefficients were considered in [1], [2], [3], [7], [8].

Note that some of the equations that arise during studying of the process soil and groundwater moisture dynamics, distribution of non-stationary external acoustic waves,

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relaxation processes during heat transfer are possible lead to the differential-operator equation of the third order in Hilbert space.

As a means of proving the theorems of the existence of solutions to non-stationary differential equations, Galerkin's method was used in the works [1], [7], [21] and other works.

L.A. Kantorovich [9] noted several problems that arise in the general theory of approximate methods for solving operator-differential equations and questions, leading to them, namely: the question of establishing the convergence of the algorithm, the process of studying the fast convergence of an approximate solution, and obtaining effective error estimates for the established approximate solution. The solution of the indicated tasks was devoted to many works. However, this area of research requires further development.

When studying Galerkin method, special attention is paid to the choice of the basis, since the properties of the basis functions significantly affect the rate of convergence of approximate solution of the differential equation to exact solution. P.E. Sobolevsky [19] proposed to choose the eigenfunctions of the differential operator as a basis, which does not depend on time and forms with the investigated operator, so-called, acute angle. This idea was used in [18] for the study of non-stationary differential-operator equations with subordinates operators.

At the present time, there is many works on the Galerkin method for solving operator-differential equations of a high order with an arbitrary basis. It should be noted that the dependence of the estimates for the rate of convergence of the behavior of the approximate solution on the type of key elements, properties of operators of the equation and its solution

2. PROBLEM STATEMENT AND AUXILIARY ASSERTIONS

Let $H-$ separable Hilbert space over the real scalar field with inner product $(.,\,.)_H$ and norm $\|.\|_H = \|.\|$ and a separable Hilbert space H_1 is compactly embedded into H. We denote the Hilbert space of all strongly measurable functions $f\colon\,[0,T]\to H$ by $\mathscr{B}_2(0,T;H)$, where

$$\mathscr{B}_2(0,T;H)=\left\{f:\|f\|_{\mathscr{B}_2(0,T;H)}=\left(\int_0^T\|f(t)\|_H^2dt\right)^{\frac{1}{2}}<+\infty\right\}.$$

In the space H, we will study the following variable third-order operator-differential equation with zero initial conditions:

$$\Phi(t)\frac{d^3u(t)}{dt^3}-S(t)u(t)=f(t),\,t\in[0,T], \tag{2.1}$$

$$u(0)=u(T)=u^{'}(0)=0, \tag{2.2}$$

where $u(t)$ is the required solution and $f(t)$ is a given scalar function.

In this work, we introduce some assumptions which concerning the operators $\Phi(t)$ and $S(t)$ as follows:

(i) $\Phi(t)$ is self-adjoint operator defined only on H_1, $(\Phi(t)=\Phi^*(t)\geq\sigma_0E,\sigma_0>0)$, $E-$ is the unit operator and σ_0- is the spectrum lower bound $(\sigma_0\in\sigma(\Phi(t))$ [1], [8]. $\Phi(t)$ is a positive definite operator where, $\|u\|_{H_1}=\|\Phi(t)u\|_H$.

(ii) $\Phi(t)$ and $S(t)$ are three-times strongly continuously differentiable operators on the interval $[0,T]$ [11].

(iii) There exist some positive constants $C_{00},\,C_{01},\,C_{03}\geq 0$ for all $v\in H_1$ such that

$$(\Phi(t)v,v)_H\geq C_{00}\left\|\Phi^{1/2}(0)v\right\|^2,\;\;(\Phi^{'}(t)v,v)_H\leq -C_{01}\left\|\Phi^{1/2}(0)v\right\|^2,$$

$$(\Phi^{'''}(t)v,v)_H\geq C_{03}\left\|\Phi^{1/2}(0)v\right\|^2.$$

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(iv) $S(t)$ is $\alpha-$subordinate (with order $\alpha:\ 0\leq\alpha<1$) to $\Phi(0)$ [5], [11] i.e., For any $v\in H_1$ there exists a constant $k_2\geq 0$ independent on $t,\ v$, the $S(t)$ satisfies the following inequality:

$$\|S(t)v\|\leq k_2\frac{\|\Phi(0)v\|^{\alpha}}{\|v\|^{\alpha-1}}. \tag{2.3}$$

(v) There exists a linear operator Ψ is similar to $\Phi(0)$ [16] i.e., Ψ is a self-adjoint positive definite operator. Moreover, the domain of (Ψ) is equal to the domain of $(\Phi(0))$ and $\Phi^{-1}(t),\Psi^{-1}:H\rightarrow H_1$ are compact in H. Both that operators are forming an acute angle in H [19], that is there exists a positive constant k_3 independent of the choice $v\in H_1$ and t such that

$$\big(\Phi(t)v,\ \Psi v\big)\geq k_3\|\Phi(0)v\|\|\Psi v\|. \tag{2.4}$$

Suppose that $u(t)$ and $\Phi(t)$ have continuous derivatives $\dfrac{d^iu(t)}{dt^i},\ \dfrac{d^i\Phi(t)}{dt^i},\ i=1,\,2,\,3$ respectively in H. Moreover, $\big(\Phi^{(i)}(t)u,v\big)_H=\big(u,\Phi^{(i)}(t)v\big)_H,\ i=1,\,2,\,3$ for all $u,v\in H_1$ and for almost $t\in(0,T)$ [4] such that

$$W_2^3(H,H_1)=\left\{u(t)\in\mathscr{B}_2(0,T;H_1):\frac{d^3}{dt^3}\Big(\Phi(t)u(t)\Big)\in\mathscr{B}_2(0,T;H)\right\},$$

with norm

$$\|u(t)\|^2_{W_2^3(H,H_1)}=\int_0^T\left(\|u(t)\|^2_{H_1}+\left\|\frac{d^3}{dt^3}\Big(\Phi(t)u(t)\Big)\right\|^2_H\right)dt.$$

Define the subspace which contains the strong solution $u(t)$ of problem (2.1)-(2.2) by

$$\overset{\circ}{W}{}_2^3(H,H_1)=\left\{u(t)\in W_2^3(H,H_1):\ u(0)=u(T)=u^{'}(0)=0\right\}.$$

According to [11], for all t under assumptions (i) and (ii) in the space H and for some constants $c_1,\ c_2>0$ that do not depend on t, we have

$$\|\Phi(t)\Phi^{-1}(0)\|_{H\rightarrow H}\leq c_1,\quad \|\Phi(0)\Phi^{-1}(t)\|_{H\rightarrow H}\leq c_2. \tag{2.5}$$

3. GALERKIN METHOD

Let $e_1,\,\ldots,\,e_n,\ldots$ be eigenvectors of the spectral problem $\Psi e_r=\lambda_r e_r,\ r=1,\ldots,n,\ldots,$ which form complete orthonormal system and $\lambda_1,\,\ldots\,\lambda_n,\,\ldots$ be the corresponding eigenvalues, in which $0<\lambda_1\leq\,\ldots\,\leq\lambda_n\leq\,\ldots,\ \lambda_n$ approaches infinity as n approaches infinity.

The Galerkin solution (approximate) of (2.1) - (2.2) is considered to be in the discrete form $u_n(t)=\sum_{i=1}^n\mathcal{G}_i(t)e_i,\ i=1,2,\ldots,n$ where the functions $\mathcal{G}_i(t)$ assumed to be unknown and represent the exact solutions to the following Cauchy problem:

$$\Phi(t)\frac{d^3\mathcal{G}_j(t)}{dt^3}-\sum_{i=1}^n\mathcal{G}_j(t)\big(S(t)e_i,e_j\big)=\big(f(t),e_j\big)_H, \tag{3.6}$$

$$\mathcal{G}_j(0)=\mathcal{G}_j(T)=\mathcal{G}_j^{'}(0)=0,\ j=1,2,\ldots,n. \tag{3.7}$$

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Next we denote the orthogonal projection in H onto H^n by P_n, where H^n is the linear span of $\{e_1,\, e_2,\, \dots,\, e_n\}$. Then (3.6) - (3.7) is equivalent to the following problem:

$$\Phi(t)\frac{d^3u_n(t)}{dt^3} - P_nS(t)u_n(t) = P_nf(t), \tag{3.8}$$

$$u_n(0) = u_n(T) = u_n^{'}(0) = 0. \tag{3.9}$$

Let us establish the unique solvability of problem (2.1) - (2.2) and obtain the necessary estimates for the rate of convergence of the approximate solutions by using the Galerkin method. Now and then, c is used to denote different positive constants independent of t and n.

Theorem 3.1:
Under the preceding assumptions, there exists a unique solution $u_n(t)$ *of* (3.8)-(3.9) *in* $\overset{\circ}{W}{}_2^3(H,H_1)$ *for each* n*. Moreover, the sequence* $\{u_n(t)\}_{n=1,2,\dots}$ *is convergent in* $W_2^3(H,H_1)$ *and the solution of* (2.1) - (2.2) *is unique.*

Proof
Look into the Cauchy problem:

$$\Phi(t)\frac{d^3\mathscr{O}_n(t)}{dt^3} = P_ng(t), \tag{3.10}$$

$$\mathscr{O}_n(0) = \mathscr{O}_n(T) = \mathscr{O}_n^{'}(0) = 0. \tag{3.11}$$

If $g(t) \in \mathscr{B}_2(0,T;H)$, then problem (3.10) - (3.11) has a unique solution $\mathscr{O}_n(t) \in \overset{\circ}{W}{}_2^3(H,H_1)$ and hence

$$\int_0^T \left\|\Phi(t)\frac{d^3\mathscr{O}_n(t)}{dt^3}\right\|^2 dt = \int_0^T \|P_ng(t)\|^2 dt.$$

From assumption (i) it follows that the operator $\Phi(t)$ has an inverse $\Phi^{-1}(t) : H \to H_1$. Let the operator $\left(\Phi(t)\frac{d^3}{dt^3}\right)^{-1} : \mathscr{B}_2(0,T;H) \to \overset{\circ}{W}{}_2^3(H,H_1)$ be a homeomorphism, then holds the following inequality:

$$\left\|\left(\Phi(t)\frac{d^3}{dt^3}\right)^{-1}\right\|_{\mathscr{B}_2(0,T;H)\to\mathscr{B}_2(0,T;H)} \le c. \tag{3.12}$$

Multiply equation (3.10) by $\mathscr{O}_n(t)$ in H and then integrate with respect to t, where t varies from 0 to T, we get

$$\begin{aligned}\int_0^T \left(\Phi(t)\frac{d^3\mathscr{O}_n(t)}{dt^3}, \mathscr{O}_n(t)\right) dt &= -\frac{3}{2}\int_0^T \left(\frac{d\Phi(t)}{dt}\frac{d\mathscr{O}_n(t)}{dt}, \frac{d\mathscr{O}_n(t)}{dt}\right) dt \\ + \frac{1}{2}\int_0^T \left(\frac{d^3\Phi(t)}{dt^3}\mathscr{O}_n(t), \mathscr{O}_n(t)\right) dt &= \int_0^T (P_ng(t), \mathscr{O}_n(t)) dt.\end{aligned} \tag{3.13}$$

As the operator $\Phi(0)\Phi^{-1}(t)$ is uniformly bounded. From the Heinz inequality [11] yields that the operator $\Phi^{\frac{1}{2}}(0)\Phi^{-\frac{1}{2}}(t)$ is also uniformly bounded. From assumption (iii), it follows that

$$\frac{c}{2}\left\|\Phi^{\frac{1}{2}}(0)\mathscr{O}_n(t)\right\|^2 \le \|P_ng(t)\|_{\mathscr{B}_2(0,T;H)} \left(\int_0^T \|\mathscr{O}_n(t)\|^2\right)^{1/2}.$$

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Furthermore,

$$\left\|\Phi^{\frac{1}{2}}(0)\mathscr{O}_n(t)\right\| \leq c\|P_n g(t)\|_{\mathscr{B}_2(0,T;H)}.$$

Hence,

$$\left\|\Phi^{\frac{1}{2}}(0)\left(\Phi(t)\frac{d^3}{dt^3}\right)^{-1}P_n g(t)\right\| \leq c\left\|P_n g(t)\right\|_{\mathscr{B}_2(0,T;H)}. \tag{3.14}$$

Then using the following substitution $\Phi(t)\dfrac{d^3\mathscr{O}_n(t)}{dt^3} \equiv \omega_n(t)$ in (3.8) - (3.9), we get

$$\omega_n(t) - P_n S(t)\Big(\Phi(t)\frac{d^3}{dt^3}\Big)^{-1}\omega_n(t) = P_n f(t). \tag{3.15}$$

In the coming step we need to prove that the operator $P_n S(t)\Big(\Phi(t)\frac{d^3}{dt^3}\Big)^{-1}$ is compact in $\mathscr{B}_2(0,T;H)$.
According (2.3) and Hölder inequality for all $\nu \in H_1$, we get

$$\begin{aligned}&\left\|P_n S(t)\Big(\Phi(t)\frac{d^3}{dt^3}\Big)^{-1}\nu\right\|_{\mathscr{B}_2(0,T;H)}\\ &\leq c\left\|P_n\Phi(0)\Big(\Phi(t)\frac{d^3}{dt^3}\Big)^{-1}\nu\right\|^{\alpha}_{\mathscr{B}_2(0,T;H)}\left\|P_n\Big(\Phi(t)\frac{d^3}{dt^3}\Big)^{-1}\nu\right\|^{1-\alpha}_{\mathscr{B}_2(0,T;H)}.\end{aligned} \tag{3.16}$$

Taking into account the uniform boundedness of $\Phi(0)\Phi^{-1}(t)$ and inequalities (3.12) (3.16), yields that

$$\left\|P_n S(t)\Big(\Phi(t)\frac{d^3}{dt^3}\Big)^{-1}\nu\right\|_{\mathscr{B}_2(0,T;H)} \leq c\left\|P_n\Big(\Phi(t)\frac{d^3}{dt^3}\Big)^{-1}\nu\right\|^{1-\alpha}_{\mathscr{B}_2(0,T;H)}\|\nu\|^{\alpha}_{\mathscr{B}_2(0,T;H)}. \tag{3.17}$$

As a result of the compact embedding of H_1 into H and based on the lemma of compactness [15], yields that the space $W_2^3(H, H_1)$ is compactly embedded in $\mathscr{B}_2(0,T;H)$. Consequently, $\Big(\Phi(t)\frac{d^3}{dt^3}\Big)^{-1}$ is actually compact in $\mathscr{B}_2(0,T;H)$. Hence, from inequality (3.17) yields that the operator $P_n S(t)\Big(\Phi(t)\frac{d^3}{dt^3}\Big)^{-1}$ is compact in $\mathscr{B}_2(0,T;H)$.

The second step, we shall prove that problem (3.15) is resolvable.
In (3.15), let the right-hand side be zero, then (3.15) will be equivalent to the following Cauchy problem:

$$\Phi(t)\frac{d^3u_n(t)}{dt^3} - P_n S(t)u_n(t) = 0, \tag{3.18}$$

$$u_n(0) = u_n(T) = u_n^{'}(0) = 0, \tag{3.19}$$

with a solution $u_n(t) \in \overset{\circ}{W}{}_2^3(H, H_1)$. Using assumption (2.3), yields

$$\begin{aligned}&\int_0^T\left\|\Phi(t)\frac{d^3u_n(t)}{dt^3}\right\|^2 dt \leq k_2^2\int_0^T\left\|P_n\Phi(0)u_n(t)\right\|^{2\alpha}\left\|P_n u_n(t)\right\|^{2(1-\alpha)}dt\\ &\leq k_2^2\|\Phi(0)\Phi^{-1}(t)\|^{2\alpha}_{H\to H}\int_0^T\|P_n\Phi(t)u_n(t)\|^{2\alpha}\|P_n u(t)_n\|^{2(1-\alpha)}dt.\end{aligned}$$

Thus,

$$\int_0^T \left\| \Phi(t)\frac{d^3u_n(t)}{dt^3} \right\|^2 dt \leq c\int_0^T \|P_n\Phi(t)u_n(t)\|^{2\alpha}\|P_nu_n(t)\|^{2(1-\alpha)}dt.$$

Apply Hölder inequality, yields that

$$\int_0^T \left\| \Phi(t)\frac{d^3u_n(t)}{dt^3} \right\|^2 dt \leq c\left(\int_0^T \|P_n\Phi(t)u_n(t)\|^2\right)^{\alpha}\left(\int_0^T \|P_nu_n(t)\|^2 dt\right)^{(1-\alpha)}.$$

Consequently, apply the Young inequality :

$$ab \leq \varepsilon a^{1/\alpha} + \left(\frac{\alpha}{\varepsilon}\right)^{\alpha/1-\alpha}(1-\alpha)b^{1/1-\alpha},$$

yields that

$$\int_0^T \left\| \Phi(t)\frac{d^3u_n(t)}{dt^3} \right\|^2 dt$$
$$\leq c\left(\varepsilon\int_0^T \|P_n\Phi(t)u_n(t)\|^2 dt + \left(\frac{\alpha}{\varepsilon}\right)^{\frac{\alpha}{1-\alpha}}(1-\alpha)\int_0^T \|P_nu_n(t)\|^2 dt\right).$$

Take $\varepsilon = \frac{1}{2c}$, then

$$\int_0^T \left\| \Phi(t)\frac{d^3u_n(t)}{dt^3} \right\|^2 dt \leq c\int_0^T \|P_nu_n(t)\|^2 dt. \tag{3.20}$$

Whenever we multiply the above-mentioned equation (3.18) by $u_n(t)$ and integrate the outcome, we get

$$\frac{c_{03}}{2}\|\Phi^{\frac{1}{2}}(0)u_n(t)\|^2 \leq \int_0^T \|u_n(t)\|\|P_nS(t)u_n(t)\|dt \leq c\int_0^s \|P_n\Phi(0)u_n(t)\|^{\alpha}\|u_n(t)\|^{2-\alpha}dt$$
$$\leq c\left(\int_0^T \|P_n\Phi(0)u_n(t)\|^2 dt\right)^{\frac{\alpha}{2}}\left(\int_0^T \|u_n(t)\|^2 dt\right)^{\frac{2-\alpha}{2}}.$$

Using (3.20), yields that

$$\|u_n(t)\|^2 \leq c\int_0^T \|u_n(t)\|^2 dt. \tag{3.21}$$

From the Bellman–Gronwall theorem [5] it follows that this inequality is possible only in the case when $u_n(t) \equiv 0$. Thus, by using the the Fredholm alternative [17] yields that problem (3.8)-(3.9) has a unique solution $u_n(t) \in \overset{\circ}{W}{}_2^3(H, H_1)$.

Let $\delta_n(t) = \Phi(t)\dfrac{d^3u_n(t)}{dt^3} - S(t)u_n(t) - f(t)$, follows that

$$\|\delta_n(t)\|_{\mathscr{B}_2(0,T;H)} \leq \|(I-P_n)f(t)\|_{\mathscr{B}_2(0,T;H)} + \|(I-P_n)S(t)u_n(t)\|_{\mathscr{B}_2(0,T;H)}.$$

From (3.21), yields that $\{u_n(t)\}$ is bounded in $W_2^3(H, H_1)$. Since the subspace H_1 is compactly embedded in the space H, yields that $S(t) : W_2^3(H, H_1) \to \mathscr{B}_2(H, H_1)$ is compact. Consequently, $\{S(t)u_n(t)\}$ is also compact in $\mathscr{B}_2(H, H_1)$. As in [10], the sequence $\{I - P_n\}$ uniformly converges to zero. Therefore, $\|\delta_n(t)\|_{\mathscr{B}_2(0,T;H)} \to 0$ as $n \to \infty$.
Now, we need to prove that $\{u_n(t)\}$ in $W_2^3(H, H_1)$ is a Cauchy sequence.

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From (3.13) we have

$$\left\|\Phi^{'''}(0)\left(u_n(t)-u_{n+p}(t)\right)\right\|_{\mathscr{B}_2(0,T;H)} \le c\left\|\Phi(t)\left(\frac{d^3u_n(t)}{dt^3}-\frac{d^3u_{n+p}(t)}{dt^3}\right)\right\|_{\mathscr{B}_2(0,T;H)}$$
$$\le c\left(\|\delta_n(t)\|_{\mathscr{B}_2(0,T;H)}+\|\delta_{n+p}(t)\|_{\mathscr{B}_2(0,T;H)}+\|S(t)(u_n(t)-u_{n+p})\|_{\mathscr{B}_2(0,T;H)}\right).$$

Applying Young inequality, we get

$$\begin{aligned}&\left\|\Phi^{'''}(0)\left(u_n(t)-u_{n+p}(t)\right)\right\|_{\mathscr{B}_2(0,T;H)}\\ &\le c\left(\|\delta_n(t)\|_{\mathscr{B}_2(0,T;H)}+\|\delta_{n+p}(t)\|_{\mathscr{B}_2(0,T;H)}+\|(u_n(t)-u_{n+p})\|_{\mathscr{B}_2(0,T;H)}\right).\end{aligned} \tag{3.22}$$

In equation (3.8), replace $u_n(t)$ by $u_{n+p}(t)$ and then subtract the resulting equation from equation (3.8), we get the following relation

$$\begin{aligned}&\Phi(t)\left(\frac{d^3u_{n+p}(t)}{dt^3}-\frac{d^3u_n(t)}{dt^3}\right)-S(t)(u_{n+p}-u_n)\\ &=(I-P_n)S(t)u_n(t)-(I-P_n)S(t)u_{n+p}(t)+(P_{n+p}-P_n)f(t).\end{aligned}$$

Multiply the preceding relation by $(u_{n+p}(t)-u_n(t))$ and then integrate the outcome. Further, employ Hölder inequality, yields that

$$\begin{aligned}&\left\|\Phi^{1/2}(0)(u_{n+p}(t)-u_n(t))\right\|^2_{\mathscr{B}_2(0,T;H)}\\ &\le (\|(I-P_n)S(t)u_n(t)\|_{\mathscr{B}_2(0,T;H)}+\|(I-P_n)S(t)u_{n+p}(t)\|_{\mathscr{B}_2(0,T;H)}\\ &+\|(P_{n+p}-P_n)f(t)\|_{\mathscr{B}_2(0,T;H)})\times\|\Phi^{-1/2}(0)\|_{H\to H}\left\|\Phi^{1/2}(0)(u_{n+p}(t)-u_n(t))\right\|_{\mathscr{B}_2}.\end{aligned}$$

Consequently,

$$\begin{aligned}&\left\|\Phi^{1/2}(0)(u_{n+p}(t)-u_n(t))\right\|_{\mathscr{B}_2(0,T;H)}\\ &\le (\|(I-P_n)S(t)u_n(t)\|_{\mathscr{B}_2(0,T;H)}+\|(I-P_n)S(t)u_{n+p}(t)\|_{\mathscr{B}_2(0,T;H)}\\ &+\|(P_{n+p}-P_n)f(t)\|_{\mathscr{B}_2(0,T;H)})\times\|\Phi^{-1/2}(0)\|_{H\to H}.\end{aligned}$$

From the compactness of $\{S(t)u_n(t)\}$ in $\mathscr{B}_2(0,T;H)$, yields that the right side of the preceding inequality approaches to zero. Therefore, according to (3.22), the $\{u_n(t)\}$ in $W_2^3(H,H_1)$ is a Cauchy sequence.

Let $\{u_n(t)\}\to u(t)$ as $n\to\infty$. Then

$$\begin{aligned}&\left\|-f(t)+\Phi(t)\frac{d^3u(t)}{dt^3}-S(t)u(t)\right\|_{\mathscr{B}_2(0,T;H)}\\ &\le\left\|\Phi(t)\left(\frac{d^3u(t)}{dt^3}-\frac{d^3u_n(t)}{dt^3}\right)-S(t)\left(u(t)-u_n(t)\right)\right\|_{\mathscr{B}_2(0,T;H)}+\|\delta_n\|_{\mathscr{B}_2(0,T;H)}\\ &\le c\left(\Phi(0)(\|u_n(t)-u(t))\|_{\mathscr{B}_2(0,T;H)}+\delta_n\|_{\mathscr{B}_2(0,T;H)}\right).\end{aligned}$$

Therefore, there exists a strong solution $u(t)\in W_2^3(H,H_1)$ of problem (2.1)-(2.2).

Now we will show that the solution of problem (2.1)-(2.2) is unique.
Assume that problem (2.1), (2.2) has two solutions $\overline{u}$ and $\widehat{u}$. Then

$$\Phi(t)\frac{d^3}{dt^3}\left(\overline{u}-\widehat{u}\right)-S(t)\left(\overline{u}-\widehat{u}\right)=0.$$

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Multiply the preceding equation by $(\overline{u} - \widehat{u})$ and integrate with respect to t, then we can simply show that $\|\overline{u} - \widehat{u}\|_{W_2^3(H,H_1)} \leq 0$, which means that the solution of problem (2.1), (2.2) is unique. The theorem is proved. □

After the question of the solvabilty of problem (2.1) - (2.2) is proven by Galerkin method, we shall use also the same method to estimate the rate of convergence of the approximate solutions of problem (2.1) - (2.2).

Theorem 3.2:
Suppose that all of assumptions of Theorem 1 be fulfilled. Then

$$\left\|\Phi^{1/2}(0)\left(u_n(t) - u(t)\right)\right\|_{\mathscr{B}_2(0,T;H)} \leq c\lambda_{n+1}^{-1/2}. \tag{3.23}$$

Proof
Let $\mathcal{U}_n(t) = u(t) - u_n(t)$, where $u(t)$ is the exact solution of problem (2.1) - (2.2) and $u_n(t)$ is the approximate solutions of problem (3.8) - (3.9). Then

$$\Phi(t)\frac{d^3\mathcal{U}_n}{dt^3} - S(t)\mathcal{U}_n(t) = (I - P_n)(f(t) - S(t)u(t)).$$

Multiply the preceding equation by $\mathcal{U}_n(t)$ and hence integrate the outcome taking into account that for each n, we have that $\mathcal{U}_n(t)$ belongs to domain $\Phi(t)$, we get

$$\begin{aligned}&\frac{3c}{2}\left\|\Phi^{1/2}(0)\mathcal{U}_n^{'}(t)\right\|_{\mathscr{B}_2(0,T;H)} + \frac{3c}{2}\int_0^T \left\|\Phi^{1/2}(0)\mathcal{U}_n(t)\right\| \\ &\leq \int_o^T \mid f(t)((I - P_n)\mathcal{U}_n(t))(S(t)u_n(t), (I - P_n)\mathcal{U}_n(t)) \mid dt.\end{aligned}$$

From assumption (iv), we get

$$\begin{aligned}&\left\|\Phi^{1/2}(0)\mathcal{U}_n(t)\right\|^2_{\mathscr{B}_2(0,T;H)} \leq c\left(\|f(t)\|_{\mathscr{B}_2(0,T;H)} + \|S(t)u_n(t)\|_{\mathscr{B}_2(0,T;H)}\right) \\ &\times \left\|\Phi^{-1/2}(0)\Psi^{1/2}\right\|_{H\to H}\left\|\Psi^{-1/2}(I - P_n)\Phi^{1/2}(0)\mathcal{U}_n(t)\right\|.\end{aligned}$$

Consequently,

$$\left\|\Phi^{1/2}(0)\mathcal{U}_n(t)\right\|_{\mathscr{B}_2(0,T;H)} \leq c\lambda_{n+1}^{-1/2}.$$

The proof is complete. □

4. APPLICATION

As an applied example on Galerkin method we will investigate an initial-boundary value problem for non-classical higher order differential equations of composite-mixed type with smooth coefficients. The theory of equations of mixed type is one of the most important branches of the theory of non-classical differential equations of mathematical physics.These types of problems have many applications in gas dynamics [12] and in other branches of physics
In the rectangle domain $\overline{Q} = [0, 1] \times [0, T]$, consider the initial-boundary value problem

$$\begin{aligned}\chi(x,t)\frac{\partial^5 u(x,t)}{\partial x^2\partial t^3} + \chi_x(x,t)\frac{\partial^4 u(x,t)}{\partial x\partial t^3} - \zeta_1(x,t)\frac{\partial u(x,t)}{\partial x}& \\ - \zeta_0(x,t)u(x,t) = f(x,t),\ (x,t) \in \overline{Q}&\end{aligned} \tag{4.24}$$

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$$u(x,0)=u_t(x,0)=u(x,T)=0,\;\; 0\le x\le 1, \tag{4.25}$$

$$u(0,t)=u_0(t),\;\; 0\le t\le T, \tag{4.26}$$

$$u(1,t)=u_1(t),\;\; 0\le t\le T. \tag{4.27}$$

where $f(x,t)\in \mathscr{B}_2(Q),\ \zeta_i(x,t),\ \dfrac{\partial \zeta_i(x,t)}{\partial t}\in C(\overline{Q}),\ u_i(t)\in C^3[0,T],\ i=0,1.$
Assume that

$$\chi(x,t)\ge \chi_0>0,\ \frac{\partial^{s+1}}{\partial x\partial t^s}\chi(x,t)\in C(\overline{Q}), s=1,2,3,$$

and the functions $u_0(t),\ u_1(t)$ fulfill the following matching conditions:

$$u_0(0)=u_0(T)=u_1(0)=u_1(T)=u_0^{'}(0)=u_1^{'}(0)=0.$$

By using the substitution $\mathcal{V}(x,t)=u(x,t)-(1-x)u_0(t)-xu_1(t)$, problem (4.24) - (4.27) will be equivalent to the following problem:

$$\begin{aligned}\chi(x,t)\frac{\partial^5\mathcal{V}(x,t)}{\partial x^2\partial t^3}+\chi_x(x,t)\frac{\partial^4\mathcal{V}(x,t)}{\partial x\partial t^3}-\zeta_1(x,t)\frac{\partial \mathcal{V}(x,t)}{\partial x}&\\ -\zeta_0(x,t)\mathcal{V}(x,t)=h(x,t),\ (x,t)\in\overline{Q}&\end{aligned} \tag{4.28}$$

$$\mathcal{V}(x,0)=\mathcal{V}_t(x,0)=\mathcal{V}(x,T)=0,\;\; 0\le x\le 1, \tag{4.29}$$

$$\mathcal{V}(0,t)=\mathcal{V}(1,t)=0,\;\; 0\le x\le 1, \tag{4.30}$$

where,

$$\begin{aligned}h(x,t)=f(x,t)-\zeta_1(x,t)u_0(t)+\zeta_1(x,t)u_1(t)+(1-x)\zeta_0(x,t)u_0(t)\\ +x\zeta_0(x,t)u_1(t)-(1-x)u_0^{'''}(t)-xu_1^{'''}(t).\end{aligned}$$

Now, we need to redefine some spaces as follows:
Assign $H=\mathscr{B}_2(0,1), H_1=W_2^2(0,1)\cap \overset{\circ}{W}{}_2^1(0,1)$. Notice that the $W_2^2(0,1)$ is denoted to the vector norm Sobolev type space, the subspace $\overset{\circ}{W}{}_2^1(0,1)=\{\mathcal{V}(x)\in W_2^2(0,1),\mathcal{V}(0)=\mathcal{V}(1)=0\}$ and $W_2^3(H,H_1)=W_2^{2,3}(Q)$. As $\frac{\partial}{\partial x}\chi(x,t)\in C(\overline{Q})$, provided that $\mid \frac{\partial}{\partial x}\chi(x,t)\mid\le \chi_1$, where χ_1 is a constant greater than zero.
On H_1 we identify the following operators

$$\Phi(t)=\frac{\partial}{\partial x}\left(\chi(x,t)\frac{\partial}{\partial x}\right)-\Theta I,\quad S(t)=\zeta_1(x,t)\frac{\partial}{\partial x}+\left(\zeta_0(x,t)-\Theta\right)I,$$

where the constant $\Theta<\frac{\chi_1^2}{\chi_0}$ and domain $S(t)\supset$ domain $\Phi(t)$.
For problem (4.28) - (4.30), we take $\Psi=\frac{d^2}{dx^2}$. Distinctly, It can be seen that the assumptions $(i)-(v)$ are fully satisfied. Consequently, simply we can verify that $S(t)$ is $\alpha-$ subordinate to $\Phi(0)$ with $\alpha=\frac{1}{2}$.

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In the following step,, we'll show that the operators $\Phi(t)$ and Ψ fulfill the acute-angle inequality (2.4). For each element $y(x) \in H_1$, we have

$$\begin{aligned}
&(\Phi(t)y(x), \Psi y(x))_{\mathscr{B}_2(0,1)} \\
&\geq \chi_0 \int_0^1 \left(\frac{d^2y(x)}{dx^2}\right)^2 dx - \Theta \int_0^1 \left(\frac{dy(x)}{dx}\right)^2 dx + \chi_1 \int_0^1 \mid \frac{dy(x)}{dx} \mid\mid \frac{d^2y(x)}{d^2x} \mid dx \\
&\geq \chi_0 \int_0^1 \left(\frac{d^2y(x)}{dx^2}\right)^2 dx + \frac{\chi_1 \varepsilon}{2} \int_0^1 \left(\frac{d^2y(x)}{dx^2}\right)^2 dx \\
&+ \frac{\chi_1}{2\varepsilon} \int_0^1 \left(\frac{dy(x)}{dx}\right)^2 dx - \Theta \int_0^1 \left(\frac{dy(x)}{dx}\right)^2 dx.
\end{aligned}$$

We choose $\varepsilon = \frac{\chi_0}{2\chi_1}$. Then we get

$$(\Phi(t)y(x), \Psi y(x))_{\mathscr{B}_2(0,1)} \geq \left(\frac{\chi_1^2}{\chi_0} - \Theta\right) \int_0^1 \left(\frac{dy(x)}{dx}\right)^2 dx \geq c\|y(x)\|^2_{W_2^2(0,1)}, \quad (4.31)$$

Comparing (4.31) with (2.4), we get the fact that $\Phi(t)$ and Ψ verified the acute-angle condition.
As well, Ψ satisfies the eigenvalue equation $\Psi e_r(x) = \lambda_r e_r(x)$, $r = 1, 2, \ldots$, where $e_r(x) = \dfrac{\sin r\pi x}{\sqrt{2}}$ are eigenfunction solutions, which are normalizable and $\lambda_r = (r\pi)^2$, $r = 1, 2, \ldots$ are the corresponding eigenvalues.
All hypotheses of theorems 3.1 and 3.2 are exactly fulfilled. Thus, for the Galerkin solution of problem (4.28) - (4.30) and hence problem (4.24) - (4.27), holds the following estimate

$$\max_{0 \leq t \leq T} \|u_n(x,t) - u(x,t)\|_{\mathscr{B}_2(0,1)} \leq c\frac{1}{\sqrt{n}}.$$

5. CONCLUSION

For a third-order multi-variable operator differential equations with initial-boundary conditions, the existence and uniqueness theorem of a strong solution was proved using Galerkin method. The rate of convergence of the Galerkin solution (approximate solutions) to the exact one was estimates. The results of Galerkin method enabled us to present an application and prove the solvability of a boundary value problem for non-classical higher order differential equations of composite-mixed type with smooth coefficients.

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