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Binary Opinion Space in the SCARDO Model

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Abstract: The paper presents the analysis of the SCARDO model in the case of the binary opinion space. The model itself and the conditions under which the analysis is performed are described and discussed in details. Analytical solutions are found for the mean field approximation. The fixed points as well as their stability properties are characterized. Furthermore, we precisely describe the hyperplane in the parameter space that defines which opinion will gather more supporters. Extensive computational experiments are performed to demonstrate the applicability of our theoretical results. Experiments suggest that the mean-field approximation can fairly accurately predict the behavior of the model provided there is a nonzero probability of anticonformity-type opinion changes. At the end of the paper, we outline the possible direction for future studies.

Keywords: opinion dynamics models, mean-field approximation, differential equations

1. INTRODUCTION

To mitigate the spread of fake news, stop opinion polarization, and convince citizens to wear face masks and vaccinate during the Covid-19 pandemic [19], it is essential to understand key mechanisms that underlie the processes of opinion formation [4, 7, 11]. However, as was pointed out in Ref. [18],"... understanding human behavior remains a grand challenge across disciplines." In recent decades, the way information flows between individuals has dramatically changed from one-directional messages that come from the mass media and political elites upon citizens (one-step information paths) to complex communications whereby even ordinary people can play a crucial role and affect the macroscopic properties of the social system. This issue has become possible due to proliferation of online social networks that provide a low-cost way of communication without social, geographical, and economic borders. In online networks, complex information paths can be established, with one or more users serving as intermediaries between mass media and other people.

A promising way to analyze these social dynamics is to use agent-based opinion formation models (aka social influence models) [22]. Stipulating how agents (artificial entities that model real people or mass media) communicate with each other and how their opinions evolve following these interactions, scholars can analyze the outcomes that this or that opinion formation mechanism will lead to [2].

In this paper, we advance the recently introduced opinion formation model [13]. In the following we will refer to it as the Stochastic Conditional ARranged Discrete Opinions model, abbreviated as the SCARDO model. In the SCARDO model, the agents update their opinions within a probabilistic sequential process whereby two randomly chosen neighboring

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agents (the influence recipient and the influence source) communicate at each iteration. In these local consecutive interactions, the agents' new opinions are determined stochastically, according to a probability distribution, which is a function of the recipient's and the source's opinions (opinions are encoded by discrete variables). Assuming that there are m possible opinion values, one can exhaustively describe opinion dynamics processes in the model via m^3 probabilities standing for all possible pairwise interactions and outcomes. Together, these quantities form what the author of [13] calls *the transition matrix*. However, strictly speaking, this mathematical 3-D construction is not a matrix. For this reason, in the current paper, we adopt a different terminology and call it by "transition table".

By tuning the values of the transition table, one can use the SCARDO model to approximate a broad range of opinion formation models, including those with discrete opinions [3, 9, 16] and those with continuous ones [5, 8, 17] (to achieve the latter purpose, one should assume that m is large enough). On this occasion, the SCARDO model can be used to compare opinion formation models against each other in terms of the transition table. Further, the SCARDO model can be relatively easily calibrated on real data upon which it puts forward only a few requirements. Thus, the SCARDO model can act as a link between theoretical opinion formation model and empirical studies of social dynamics processes [13].

In Ref. [13], for the SCARDO model, a mean-field approximation was derived. This approximation describes successfully the dynamics of the populations of m opinion camps with the system of ordinary differential equations. The system was thoroughly investigated under empirically-calibrated settings (with the transition table calibrated on the data from Refs. [12, 14]) for different configurations of the opinion spectrum $m = 2, m = 3$, and $m = 10$. In this paper, we apply a different strategy and focus on a low-dimensional case of this model ($m = 2$), for which analytical derivations are feasible and a general description of the mean-field approximation system of equations can be obtained for any values of the models' parameters. For this low-dimensional case, we find exact solutions of the mean-field equations, characterize fixed points, and investigate their stability properties. We exemplify our analytical analysis with computational simulations.

The rest of the paper is organized as follows. Section 2 provides the description of the SCARDO model. In Section 3, we present the mean-field approximation derived for this model. Section 4 presents the main results of the paper. Subsection 4.2 obtains and analyzes the solutions of the mean-field equations in the case of the binary opinion space while Subsection 4.3 complemente these results with illustrative examples that compare theoretical predictions against simulation data. Section 5 ends the paper with concluding remarks.

2. OPINION DYNAMICS MODEL

In this section, We present the SCARDO model that was first introduced in Ref. [13]. In the model, N agents communicate via a (by default, static) connected (undirected and unweighted) social network G and discuss a socially important topic [21]. The time in the model is discrete. At each iteration, one chooses a random agent i (by default, all agents have the equal probability $1/N$ to be chosen). Then, one selects at random one of their friends j in the network G (again, according to the uniform probability distribution). Then, the agent j (influence source) influences the agent i (recipient) during a one-directional influence procedure. As a result, the recipient's opinion changes, depending on the current opinions of the agents i and j.

Agent opinions (denoted by o) belong to a discrete opinion space with m elements:

$$X = \{x_1, \dots, x_m\}.$$

The elements $x_1, \dots, x_m$ represent an opinion alphabet of the model and may encode a broad spectrum of opinion systems, such as competing alternatives (as in the Voter model

[16]) or arranged attitudes that may approximate continuous opinion spaces for large m—see Ref. [1]:

$$x_1 \prec \ldots \prec x_m.$$

After receiving influence from j's opinion $o_j(t) = x_l$, the agent i's opinion $o_i(t) = x_s$ may change or may remain the same. The outcome of the communication is determined stochastically, according to the following probability distribution:

$$\{p_{s,l,1}, \ldots, p_{s,l,m}\},$$

with the element $p_{s,l,k}$ standing for the probability of selecting the opinion x_k. One can think of this quantity as the conditional probability that is defined as follows:

$$p_{s,l,k} = \text{Pr}\,\{o_i(t+1) = x_k \mid o_i(t) = x_s, o_j(t) = x_l\}. \tag{2.1}$$

Stuck together, these conditional probabilities are the elements of a 3-D object $P = [p_{s,l,k}]_{s,l,k=1}^{m}$, which is called the transition table [13]. Note that this table can be safely expressed as a list of matrices:

$$P_1 = \begin{bmatrix} p_{1,1,1} & \cdots & p_{1,1,m} \\ \cdots & \cdots & \cdots \\ p_{1,m,1} & \cdots & p_{1,m,m} \end{bmatrix}, \ldots, P_m = \begin{bmatrix} p_{m,1,1} & \cdots & p_{m,1,m} \\ \cdots & \cdots & \cdots \\ p_{m,m,1} & \cdots & p_{m,m,m} \end{bmatrix} \tag{2.2}$$

Within these shorthands, $P_1, \ldots, P_m$ are 2-D matrices with m rows and m columns that encode opinion change strategies of individuals espousing opinions $x_1, \ldots, x_m$ correspondingly. Note that all these m matrices are row-stochastic because $p_{s,l,1} + \ldots + p_{s,l,m} = 1$ for each s and l.

3. MEAN-FIELD APPROXIMATION

For the model presented in the previous section, the following mean-field approximation was obtained in Ref. [13]. First, let us denote the population of agents espousing the opinion x_s at a time moment t by $Y_s(t)$:

$$Y_s(t) = \{j \mid o_j(t) = x_s\}.$$

Correspondingly, the fraction of such individuals among the whole population is symbolized by $y_s(t) = Y_s(t)/N$. Using the scaled time τ, which is defined as $\tau = t/N, \delta\tau = 1/N$, one can ends up with the following system of differential equations (see Ref. [13] for details):

$$\begin{cases} \frac{dy_1(\tau)}{d\tau} = \sum_{s,l=1}^{m} y_s(\tau) y_l(\tau) p_{s,l,1} - y_1(\tau), \\ \cdots \\ \frac{dy_i(\tau)}{d\tau} = \sum_{s,l=1}^{m} y_s(\tau) y_l(\tau) p_{s,l,i} - y_i(\tau), \\ \cdots \\ \frac{dy_m(\tau)}{d\tau} = \sum_{s,l=1}^{m} y_s(\tau) y_l(\tau) p_{s,l,m} - y_m(\tau), \end{cases} \tag{3.3}$$

That is, from the micro-level description of the model we come to the macro-level [20].

One can equip system (3.3) with the initial condition

$$y_1(0) = q_1, \ldots, y_m(0) = q_m \tag{3.4}$$

whereby $q_1 + \ldots + q_m = 1$ and all initial values are no less than zero. The Cauchy problem (3.3), (3.4) features the following property (see [13]).

Corollary 3.1:
The function $u = y_1(\tau) + \ldots + y_m(\tau)$ is the first integral of the autonomous system (3.3).

Remember that the quantities $y_1, \ldots, y_m$ stand for the populations of all the opinion camps $x_1, \ldots, x_m$ and thus these variables should be nonnegative and sum up to one. The following theorem ensures that solutions of (3.3) meet this property (see [15] for proofs).

Theorem 3.1:
Assume that $q_1 \geq 0, \ldots, q_m \geq 0$ and $q_1 + \ldots + q_m = 1$. Then the Cauchy problem (3.3), (3.4) *has a unique solution $\vec{y}(\tau)$, which can be extended on the whole τ-axis (in fact, we are interested only in $\tau \geq 0$). The components of $\vec{y}(\tau)$ are nonnegative and sum up to one for each τ.*

Theorem 3.1 indicates that the mean-field approximation is a reliable mathematical construction which gives the outputs consistent with the physical definitions of the models' parameters.

4. BINARY OPINION SPACE

4.1. Problem formulation

Let us now concentrate on the case of the binary opinion space $m = 2$. In this situation, the table matrix can be safely presented by only two row-stochastic matrices:

$$P_1 = \begin{bmatrix} p_{1,1,1} & p_{1,1,2} \\ p_{1,2,1} & p_{1,2,2} \end{bmatrix}, P_2 = \begin{bmatrix} p_{2,1,1} & p_{2,1,2} \\ p_{2,2,1} & p_{2,2,2} \end{bmatrix}.$$

For now, the mean-field predictions (3.3) will have the following form:

$$\begin{cases} \frac{dy_1}{d\tau} = p_{1,1,1}y_1^2 + (p_{1,2,1} + p_{2,1,1})y_1y_2 + p_{2,2,1}y_2^2 - y_1, \\ \frac{dy_2}{d\tau} = p_{1,1,2}y_1^2 + (p_{1,2,2} + p_{2,1,2})y_1y_2 + p_{2,2,2}y_2^2 - y_2 \end{cases}$$

or

$$\begin{cases} \frac{dy_1}{d\tau} = -p_{1,1,2}y_1^2 + (p_{2,1,1} - p_{1,2,2})y_1y_2 + p_{2,2,1}y_2^2, \\ \frac{dy_2}{d\tau} = p_{1,1,2}y_1^2 - (p_{2,1,1} - p_{1,2,2})y_1y_2 - p_{2,2,1}y_2^2 \end{cases} \tag{4.5}$$

For the sake of simplicity, let us introduce notations $\alpha = p_{1,1,2}, \beta = p_{1,2,2}, \gamma = p_{2,1,1}$, and $\delta = p_{2,2,1}$. Within this notation strategy, α and β represent the probabilities of anticonformity-type opinion changes for both the opinion camps (note that anticonformity refers to the situation when individuals change opinions after interacting with those holding similar positions [16]). In turn, β and γ represent the probabilties of the conformity-type opinion shuffles. These four variables represent the transition table exhaustively:

$$P_1 = \begin{bmatrix} 1-\alpha & \alpha \\ 1-\beta & \beta \end{bmatrix}, P_2 = \begin{bmatrix} \gamma & 1-\gamma \\ \delta & 1-\delta \end{bmatrix}. \tag{4.6}$$

One of the equations in (4.5) is redundant. After making use of $y_2 = 1 - y_1$ and applying encoding strategy (4.6), one can end up with the following differential equation:

$$\frac{dy_1}{d\tau} = (\delta - \alpha - \theta)y_1^2 + (\theta - 2\delta)y_1 + \delta,$$

where $\theta = \gamma - \beta$. For simplicity, we denote $y = y_1$ and focus on the analysis of the following differential equation:

$$\frac{dy}{d\tau} = (\delta - \alpha - \theta)y^2 + (\theta - 2\delta)y + \delta, \tag{4.7}$$

4.2. Equation (4.7)*: fixed points, solutions, and stability*

Let us now concentrate on solving equation (4.7). For this equation, the fixed points can be found from the following quadratic trinomial:

$$(\delta - \alpha - \theta)y^2 + (\theta - 2\delta)y + \delta = 0.$$

The discriminant of this equation is $D = \theta^2 + 4\delta\alpha \geq 0$, and one can easily obtain the following expression for fixed points:

$$y^*_{-,+} = \frac{2\delta - \theta \pm \sqrt{\theta^2 + 4\delta\alpha}}{2(\delta - \alpha - \theta)}. \tag{4.8}$$

Depending on how θ, δ, and α, the denominator in (4.8) may be positive, negative, or may be equal to zero. Let us consider these three situations separately.

Case 1. Let $\theta = \delta - \alpha$.

In this case, equation (4.7) simplifies to

$$y' = -(\delta + \alpha) \cdot y + \delta. \tag{4.9}$$

Thus, we end up with the following equation on the fixed points :

$$-(\delta + \alpha) \cdot y^* + \delta = 0. \tag{4.10}$$

Taking into account the non-negativity of the parameters δ and α we obtain that for $\alpha = \delta = 0$ equation (4.10) has infinitely many solutions: any y^* that lies in the interval $[0;\ 1]$ is a fixed point. For other values of α and δ, we have one fixed point given by the expression

$$y^* = \frac{\delta}{\delta + \alpha}.$$

If $\alpha = \delta = 0$, equation (4.9) turns out to be $y' = 0$, i.e. the solution of the Cauchy problem is $y = y(0)$. If $(\alpha, \delta) \neq (0, 0)$, then we obtain

$$\frac{dy}{-(\delta + \alpha) \cdot y + \delta} = d\tau.$$

After integrating this equality, we obtain the solution of the Cauchy problem

$$y = \frac{\delta - C \cdot \exp -(\alpha + \delta)\tau}{\delta + \alpha},$$

where C is determined from the initial condition $y(0) = \frac{\delta - C}{\delta + \alpha}$, whence $C = \delta - (\delta + \alpha) \cdot y(0)$.

Case 2. Let $\theta \in [-1;\ \delta - \alpha)$. In this case, equation (4.7) has the form

$$y' = (\delta - \alpha - \theta)(y - y_-^*)(y - y_+^*), \tag{4.11}$$

where y_- and y_+ are defined in (4.8). For fixed points, we have the following equation:

$$(\delta - \alpha - \theta)(y^* - y_-^*)(y^* - y_+^*) = 0. \tag{4.12}$$

Let's see in which cases the solutions of (4.12) satisfy the condition of staying within $[0; 1]$. The denominator of (4.8) is positive, and we can rewrite the condition $0 \leq y_{-,+}^* \leq 1$ as

$$0 \leq 2\delta - \theta \pm \sqrt{\theta^2 + 4\delta\alpha} \leq 2(\delta - \alpha - \theta) \tag{4.13}$$

Consider the inequality on the left in (4.13). For the sign "+" it is true because the root is positive and the inequality

$$2\delta - \theta > \delta + \alpha \geq 0,$$

where θ is estimated using the initial condition, is true. We proceed to the inequality

$$2\delta - \theta \geq \sqrt{\theta^2 + 4\delta\alpha}. \tag{4.14}$$

for the sign "$-$" .

Since its left side is always positive, we square both sides and come to

$$4\delta \cdot (\delta - \theta - \alpha) \geq 0.$$

Applying the initial condition to θ we obtain that the inequality is true.

Consider the right inequality in (4.13): For the sign "$-$" we pass to the inequality

$$2\alpha + \theta \leq \sqrt{\theta^2 + 4\delta\alpha}. \tag{4.15}$$

Note that the right side is positive, and therefore, squaring both sides, we arrive at a more strict inequality:

$$4\alpha \cdot (\alpha + \theta - \delta) \leq 0.$$

Nevertheless, it is still satisfied, because $\alpha \geq 0$, and $\theta \leq \delta - \alpha$ from the initial condition. And since the more strict condition is satisfied, the original condition is also true.

For the sign " $+$ " we get

$$2\alpha + \theta + \sqrt{\theta^2 + 4\delta\alpha} \leq 0,$$

which is only possible for $\theta \leq -2\alpha$. Let's take this fact into account and move $2\alpha + \theta$ to the right. After that, we square the inequality. The resulting inequality is satisfied only if $\alpha = 0, \theta \leq 0$. Thus, there is a unique fixed point in the zone $\theta \in [-1;\ \delta - \alpha)$ given by the following expression:

$$y^* = \frac{2\delta - \theta - \sqrt{\theta^2 + 4\delta\alpha}}{2(\delta - \alpha - \theta)}. \tag{4.16}$$

Further, if the condition $\alpha = 0, \theta \leq 0$ is satisfied, then the second fixed point arises – $y^* = 1$.

Let us now focus on solving equation (4.11). We rewrite it as

$$\frac{dy}{(\delta - \alpha - \theta)(y - y_-^*)(y - y_+^*)} = d\tau.$$

We need to mention the case when $y_-^* = y_+^*$ which happens when $\theta^2 + 4\alpha\delta = 0$. Due to parameter conditions, it occurs only if $\theta = 0, \alpha = 0, \delta > 0$.

In that case, the equation has the following form

$$y' = \delta \cdot (y - 1)^2$$

with a solution

$$y = 1 - \frac{1}{\delta\tau + C},$$

whence $C = \frac{1}{1-y(0)}$. It is worth pointing out that this notation is correct only in case of $y(0) < 1$. In case of $y(0) = 1$ the solution simply stays for $y(0) = 1$.

After that, we end up with

$$-\frac{1}{\sqrt{\theta^2 + 4\delta\alpha} \cdot (y - y_-^*)} + \frac{1}{\sqrt{\theta^2 + 4\delta\alpha} \cdot (y - y_+^*)} = d\tau$$

After integrating both parts and simplifying them, we get

$$\ln |\frac{y - y_+^*}{y - y_-^*}| = \sqrt{\theta^2 + 4\delta\alpha} \cdot (\tau + C),$$

whence, after conversion, we get

$$y = \frac{y_+^* - C \cdot \exp\left(\sqrt{\theta^2 + 4\delta\alpha} \cdot \tau\right) \cdot y_-^*}{1 - C \cdot \exp\left(\sqrt{\theta^2 + 4\delta\alpha} \cdot \tau\right)}, \tag{4.17}$$

In turn, C can be easily found from the initial condition: $y(0) = \frac{y_+^* - C \cdot y_-^*}{1-C}$, whence $C = \frac{y_+^* - y(0)}{y_-^* - y(0)}$. It is worth pointing out that this notation is correct only in case of $y(0) \neq y_-^*$. In case of $y(0) = y_-^*$ the solution simply stays for $y(0) = y_-^*$.

Case 3. It is simple to demonstrate that the only fixed point in the zone $\theta \in (\delta - \alpha;\ 1]$ is

$$y^* = \frac{2\delta - \theta - \sqrt{\theta^2 + 4\delta\alpha}}{2(\delta - \alpha - \theta)}.$$

In the case $\delta = 0, \theta \geq 0$, the second fixed point $y^* = 0$ appears.

It is also easy to show that the solution of the differential equation (4.7) in the zone $\theta \in (\delta - \alpha;\ 1]$ is

$$y = \frac{y_+^* - C \cdot \exp\left(\sqrt{\theta^2 + 4\delta\alpha} \cdot t\right) \cdot y_-^*}{1 - C \cdot \exp\left(\sqrt{\theta^2 + 4\delta\alpha} \cdot \tau\right)}, \tag{4.18}$$

where C is determined from the initial conditions as $C = \frac{y^*_+ - y(0)}{y^*_- - y(0)}$.

It is worth pointing out that this notation is correct only in case of $y(0) \neq y^*_-$. In case of $y(0) = y^*_-$ the solution simply stays for $y(0) = y^*_-$.

We also need to mention the case when $y^*_- = y^*_+$ which happens when $\theta^2 + 4\alpha\delta = 0$. Due to parameters conditions it occurs only if $\theta = 0, \alpha > 0, \delta = 0$.

In that case, the equation has the following form

$$y' = -\alpha \cdot y^2$$

with a solution

$$y = \frac{1}{\alpha\tau + C}$$

whence $C = \frac{1}{y(0)}$ from the initial solution. It is worth pointing out that this notation is correct only in case of $y(0) > 0$. In case of $y(0) = 0$ the solution simply stays for $y(0) = 0$.

Table 4.1 summarizes our findings. Note that stability properties of the fixed points can be easily obtained given we know how the solution of the Cauchy problem looks like in each particular situation. As one can note from table 4.1, if there is only one fixed point, then it is always a global attractor. If there are two of them, then one fixed point is always unstable and the other one is a global attractor excluding the case the initial point is exactly the unstable fixed point. If $\theta = \delta = \alpha = 0$, then something unusual appears. In this case, we end up with the following configuration of the transition table (note that $\beta = \gamma$ because $\theta = \beta - \gamma$):

$$P_1 = \begin{bmatrix} 1 & 0 \\ 1-\beta & \beta \end{bmatrix}, P_2 = \begin{bmatrix} \beta & 1-\beta \\ 0 & 1 \end{bmatrix}.$$

In other words, we have symmetric opinion dynamics with no anticonformity. The mean-field prediction says that in such settings, the opinion camps have no advantage over each other, so the populations of the camps should not change.

From the fixed points depicted in table 4.1, one can easily compute the outcome of the competition between opinions x_1 and x_2. We will focus on three scenarios: (i) opinion x_1 obtains a total victory ($y^* = 1$—recall that within our notations, the variable y depicts the proportion of the opinion x_1's backers); (ii) a draw - when both the opinions have the same number of supporters ($y^* = 1/2$); (iii) opinion x_2 obtains a total victory ($y^* = 0$). In table 4.2, we depict how the outcomes of the competition depend on the transition table (we do not consider the trivial case $\theta = \delta = \alpha = 0$). Among other things, from table 4.2 one can conclude that a draw between competing opinions is ensured by the equality of

$$\theta = \alpha - \delta$$

or

$$\gamma - \beta = \alpha - \delta. \tag{4.19}$$

Intuitively, equation (4.19) is quite clear because the quantity $\gamma - \beta$ describes the advantage of the opinion x_1 over the opinion x_2 with respect to how these opinions attract the individuals via the conformity mechanism whereas $\alpha - \delta$ in (4.19) demonstrates the relative outcome of x_1-backers caused by anticonformity. Therefore, if the right side of (4.19) outweighs the left one, then we should expect that x_2 wins. Correspondingly, if $\gamma - \beta > \alpha - \delta$, then x_1 will have more supporters at the equilibrium for nearly all initial points.

Table 4.1. Equilibrium points and exact solutions

Parameters	Fixed points	Exact solution
$(\theta = \delta - \alpha)$ & $(\alpha = \delta = 0)$ – or, simply, $\theta = \delta = \alpha = 0$	y^* is any number (in [0; 1]) – Lyapunov stable	$y = y(0)$
$(\theta = \delta - \alpha)$ & $\neg(\alpha = \delta = 0)$	$y^* = \frac{\delta}{\delta+\alpha}$ – asymptotic stable (global attractor)	$y = \frac{\delta - C\cdot\exp(-(\alpha+\delta)\tau)}{\delta+\alpha}$, where $C = \delta - (\delta+\alpha)\cdot y(0)$
$(\theta < \delta - \alpha)$ & $(\alpha = 0, \theta = 0)$	$y^* = 1$ – asymptotic stable (global attractor)	If $y(0) \neq 1$, then $y = 1 - \frac{1}{\delta\tau + C}$, where $C = \frac{1}{1-y(0)}$. If $y(0) = 1$, then $y = 1$
$(\theta < \delta - \alpha)$ & $(\alpha = 0, \theta < 0)$	$y^* = 1$ – unstable, $y^* = \frac{\delta}{(\delta-\theta)}$ – asymptotic stable (global attractor excepting the initial point $y(0) = 1$)	If $y(0) \neq y^*_-$, then $y = \frac{y^*_+ - C\cdot\exp(-\theta\cdot\tau)\cdot y^*_-}{1 - C\cdot\exp(-\theta\cdot\tau)}$, where $C = \frac{y^*_+ - y(0)}{y^*_- - y(0)}$. If $y(0) = y^*_- : y = y^*_-$
$(\theta < \delta - \alpha)$ & $\neg(\alpha = 0, \theta \leq 0)$	$y^* = y^*_-$ – asymptotic stable (global attractor)	If $y(0) \neq y^*_-$, then $y = \frac{y^*_+ - C\cdot\exp(\sqrt{\theta^2+4\delta\alpha}\cdot\tau)\cdot y^*_-}{1 - C\cdot\exp(\sqrt{\theta^2+4\delta\alpha}\cdot\tau)}$, where $C = \frac{y^*_+ - y(0)}{y^*_- - y(0)}$. If $y(0) = y^*_- : y = y^*_-$
$(\theta > \delta - \alpha)$ & $(\delta = 0, \theta = 0)$	$y^* = 0$ – asymptotic stable (global attractor)	If $y(0) \neq 0$, then $y = \frac{1}{\alpha\tau + C}$, where $C = \frac{1}{y(0)}$. If $y(0) = 0 : y = 0$
$(\theta > \delta - \alpha)$ & $(\delta = 0, \theta > 0)$	$y^* = 0$ – unstable, $y^* = \frac{\theta}{(\alpha+\theta)}$ – asymptotic stable (global attractor excepting the initial point $y(0) = 0$)	If $y(0) \neq y^*_- : y = \frac{y^*_+ - C\cdot\exp(\theta\cdot\tau)\cdot y^*_-}{1 - C\cdot\exp\theta\cdot\tau)}$, where $C = \frac{y^*_+ - y(0)}{y^*_- - y(0)}$. If $y(0) = y^*_-$, then $y = y^*_-$
$(\theta > \delta - \alpha)$ & $\neg(\delta = 0, \theta \geq 0)$	$y^* = y^*_-$ – asymptotic stable (global attractor)	If $y(0) \neq y^*_-$, then $y = \frac{y^*_+ - C\cdot\exp(\sqrt{\theta^2+4\delta\alpha}\cdot\tau)\cdot y^*_-}{1 - C\cdot\exp(\sqrt{\theta^2+4\delta\alpha}\cdot\tau)}$, where $C = \frac{y^*_+ - y(0)}{y^*_- - y(0)}$. If $y(0) = y^*_- : y = y^*_-$

4.3. Experimental data compared with theoretical results

We complement our theoretical results with computational experiments with the SCARDO model. In this section, we present some ideal-typical simulation runs, with a special focus on those cases where theoretical results and experimental data can be very different from each other (see Figure 4.1).

Table 4.2. The outcome of the opinion competition as a function of transition table organization

Parameters	Total victory of x_1 ($y^* = 1$)	Draw ($y^* = 1/2$)	Total victory of x_2
$(\theta = \delta - \alpha)$ & $\neg(\alpha = \delta = 0)$	$\alpha = 0$	$\theta = 0$	$\delta = 0$
$(\theta < \delta - \alpha)$ & $(\alpha = 0, \theta = 0)$	Any transition table	None	None
$(\theta < \delta - \alpha)$ & $(\alpha = 0, \theta < 0)$	$y(0) = 1$	$(\theta = -\delta)$ & $(y(0) \neq 1)$	$(\delta = 0)$ & $(y(0) \neq 1)$
$(\theta < \delta - \alpha)$ & $\neg(\alpha = 0, \theta \leq 0)$	$\alpha = 0$	$\theta = \alpha - \delta$	$\delta = 0$
$(\theta > \delta - \alpha)$ & $(\delta = 0, \theta = 0)$	None	None	Any transition table
$(\theta > \delta - \alpha)$ & $(\delta = 0, \theta > 0)$	$(\alpha = 0)$ & $(y(0) \neq 0)$	$(\theta = \alpha)$ & $(y(0) \neq 0)$	$y(0) = 0$
$(\theta > \delta - \alpha)$ & $\neg(\delta = 0, \theta \geq 0)$	$\alpha = 0$	$\theta = \alpha - \delta$	$\delta = 0$

First, we should say that the equilibrium points obtained within the framework of mean-field approximation are not necessarily of the same status from the perspective of the model's dynamics. To be more specific, only states $y = 0$ and $y = 1$ can be *pure* equilibriums, and this occurs if and only if $\delta = 0$ or $\alpha = 0$ correspondingly. Otherwise, if $\delta \neq 0$ and $\alpha \neq 0$, the system cannot reach an equilibrium.

We report that if the transition table has no anticonformity ($\alpha > 0$ and $\delta > 0$), then the model behavior is nearly in agreement with the theoretical predictions (see Panels C, D, and F) – the populations of opinion camps *fluctuate* around theoretical solutions, with the oscillation magnitude disappearing if $N \to \infty$. Note that in this case, the system cannot reach a pure equilibrium state because there is always a nonzero probability that the populations of the opposing opinion camps will change in the next few moments. Further, we report that in this case the theoretical predictions are still valid even if considering structured populations that are characterized by non-complete social graphs. Note that the mean-field predictions were derived under assumption that the underlying network is a complete graph whereby each two agents have the same probability of interaction. However, our numerical simulations indicate that this assumption is redundant if considering transition tables with non-zero anticonformity effects.

If, however, at least one of the anticonformity parameters is equal to zero, then the model features deviations from the mean-field predictions (see Panels A, B, and E). In some situations, these deviations can be understood as oscillations around the theoretically predicted solutions again (Panel A). However, the same transition table configurations can result in qualitively different behaviors – see Panels A and B. Instead of Panel A, on Panel B we see that the system substantially deviates from the theoretical solution, with no sign of fluctuations around it. Finally, on Panel E, for $t \leq 250,000$ the system oscillates around the fixed point $y^* = 5/6$, which, according to the mean-field predictions, should be a global attractor for all the trajectories that start from $y \neq 1$. But because in the underlying simulation run there were only $N = 200$ agents, the magnitude of the fluctuations was quite huge and there is a high chance that the trajectory finds itself in the fixed point $y^* = 1$, from which there is no coming back (because the probability of anticonformity for individuals espousing opinion x_1 is equal to zero in this case). As a result, the system ends up in the unstable fixed point, which is a pure equilibrium.

In a nutshell, if $\alpha > 0$ and $\delta > 0$, then the model behavior is well-predictable, with both the opinions having nonzero populations of supporters. In this case, the balance between populations is defined by the hyperplane (4.19) – if this equality is true, then both opinion camps will have the same number of supporters ($N/2$ agents). If the left side dominates, then the position x_1 will have an advantage over x_2. The reverse is also true.

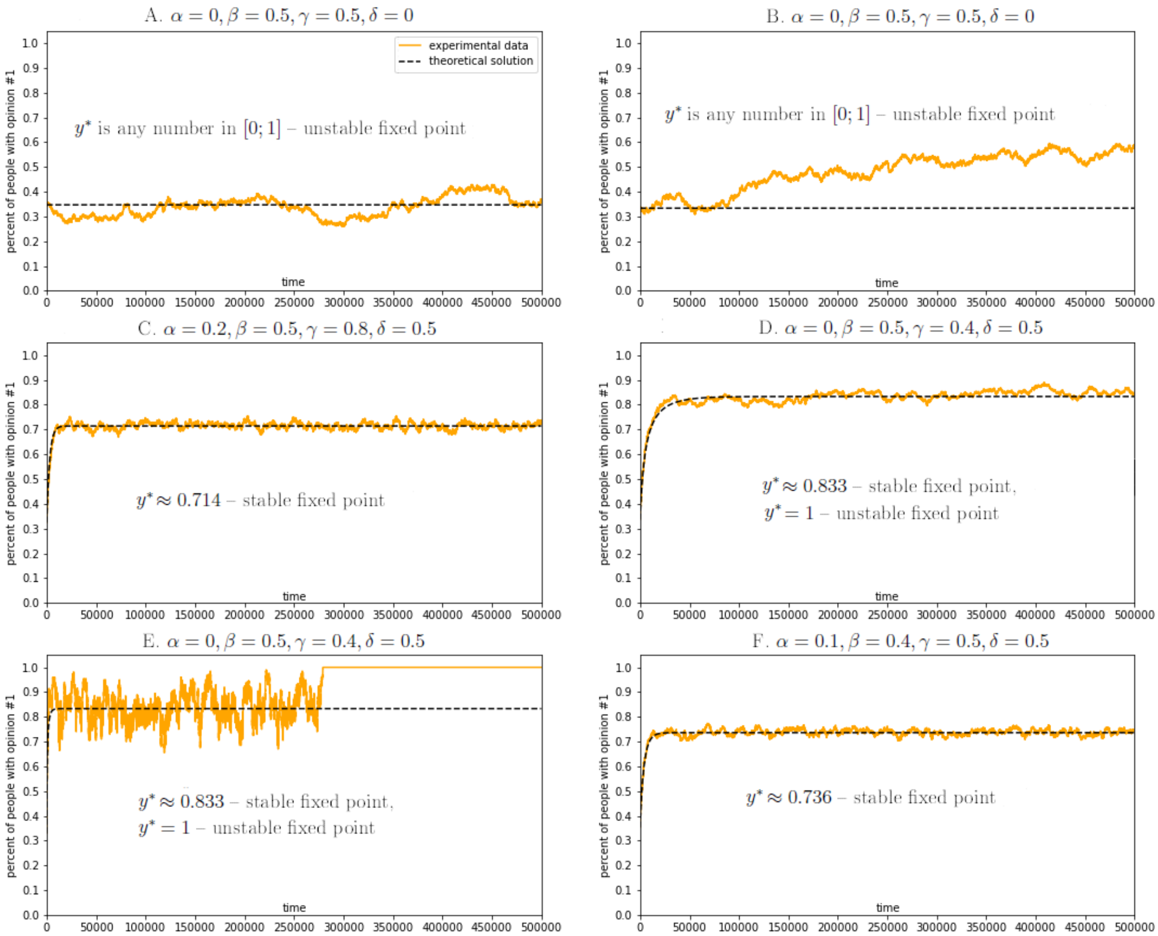

Fig. 4.1. Here, we demonstrate six simulation runs carried out on the fully-connected social graph for $N = 2000$ agents (excepting for the experiment on panel E, where there were only $N = 200$) for different transition tables (shown at the top of the Panels). The model's dynamics are presented with orange curves. Mean-field solutions are symbolized with dashed lines. On each plot, we depict what the fixed points look like for a given transition table.

5. CONCLUSION

This paper advances the recently published opinion formation model [13] by studying its behavior in the case of the binary opinion space—the most simple situation whereby only two opinions exist and compete in a society. In an attempt to get a comprehensive analytical description of the model, we focused on the mean-field approximation that describes the model's dynamics in terms of the populations of two competing opinion camps via the autonomous system of ordinary differential equations under the assumptions that the number of agents is huge and the agents communicate on a complete graph.

For this system, we found exact solutions, characterized fixed points, and investigated their stability properties across all possible configurations of the model parameters. Furthermore, we obtained a hyperplane in the space of the model parameters that define which opinion will win the competition.

We supported our theoretical results by computational experiments. These experiments revealed that for most points in the parameter space (excepting the set of measure zero, which represents absence of anticonformity effects in the model), the mean-field predictions manage to forecast the real model behavior. This finding is quite important because it means that for a wide range of transition tables (actually, this range does not include only specific transition table configurations, which are unlikely encountered in real life [1]), we have no need in searching for suitable network topologies and performing potentially expensive computational experiments (especially in the case of large systems)—the outcome of opinion dynamics (at the macroscopic level) does not depend on this factor and can be effectively predicted analytically, via the mean-field differential equations.

Further analysis may be devoted to incorporating ranking algorithms into the model [6] or to improving the mean-field predictions by accounting for local interactions [10].

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