

Numerical Soliton Solution for GRRLW Equation Using Fifth Order Runge-Kutta and Higher Order B-Spline Collocation Schemes

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Abstract: In this work, an efficient numerical method is proposed to obtain the approximate solution of the general Rosenau regularised long wave equation. The present method based on Butcher's fifth-order Runge-Kutta scheme, which consider a septic B-spline as the basis function, respectively. In this method, the approximate solutions are measured by neither transformation nor any kind of linearization technique to tackle the non linearity. Butcher's fifth-order Runge-Kutta approach without using a finite difference scheme at each time level is implemented for the numerical treatment of a system of first-order differential equations. The linear stability for the linearized septic B-spline scheme is investigated. The error norms are calculated on problem to enhance efficiency and accuracy of the present method and also compared to the existing results.

Keywords: BFRK approach, general Rosenau-RLW equation, linearization, septic B-spline scheme, stability

1. INTRODUCTION

In the present situation, when the analytical result of the nonlinear partial differential equation fails to exist, several numerical techniques are applied to obtain an approximate solution. Consider the general Rosenau-regularized long wave (GR-RLW) equation of the form

$$u_t - u_{xxt} + u_{xxxxt} = -(u_x + \omega(u^q)_x) \quad (1.1)$$

with the initial condition

$$u(x, 0) = u_0(x), \quad \gamma \leq x \leq \alpha \quad (1.2)$$

and the boundary conditions

$$u(\gamma, t) = u(\alpha, t) = u_x(\gamma, t) = u_x(\alpha, t) = u_{xx}(\gamma, t) = u_{xx}(\alpha, t) = 0, \quad 0 \leq t \leq T \quad (1.3)$$

where $q \leq 2$ is an integer and $u_0(x)$ is known as smooth function. Such types of equations frequently arise in various branches of applied sciences and physics. When $q=2$, (1.1) is referred to as the Rosenau-RLW equation. Numerous numerical techniques have been developed for solving (1.1). In particular, several finite difference schemes have been proposed in the literature [6, 12, 14, 15, 51–54, 58–60, 63, 65]. Chen et al. [7] developed a semi-explicit linearized Crank–Nicolson finite difference method for obtaining numerical

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solutions of (1.1). Wongsaijai and Poochinapan [61] and Janwised et al. [16] proposed three-level average implicit finite difference schemes to compute approximate solutions of the Rosenau–RLW and GR-RLW equations. In addition to finite difference techniques, several alternative numerical approaches have been developed. Atouani and Omrani [4] introduced a Galerkin finite element method, while Yagmurlu et al. [64] proposed a Galerkin cubic B-spline finite element approach. Foroutan and Ebadian [8] investigated a modified Chebyshev rational approximation technique, whereas Park [55] applied pointwise decay estimates. Mittal and Jain [45] considered a quintic B-spline collocation method combined with a finite element scheme. Moreover, Cai et al. [5] proposed a high-order structure-preserving scheme, and Wongsaijai et al. [62] developed a compact structure-preserving approach for obtaining approximate solutions of (1.1).

The primary objective of the present work is to obtain a more accurate approximate solution of the GR-RLW equation using the second method and to compare its performance with the first method as well as with existing results in the literature. In the proposed approach, the BFRK method is employed to solve the resulting system of first-order ordinary differential equations without using a finite difference scheme for the time-dependent variables at each time level. Additionally, several numerical techniques such as numerical quadrature schemes [17–19, 26–33, 35, 36, 39, 40, 43, 44, 49, 50, 66–80], block methods [10, 11, 20, 25, 34, 46–48], B-spline collocation techniques [13, 21–24, 37, 38, 56, 57], spline methods [1, 9, 42], and finite difference methods [2, 3, 41] are closely related to the development of the present scheme.

The remainder of the paper is organized as follows. Section 2 presents some basic characteristics of the proposed methods. Section 3 describes the septic B-spline scheme. Section 4 discusses the linear stability analysis. Numerical examples and discussions are provided in Section 5. Graphical and tabular results are presented in Section 6. Finally, the conclusions are given in the last section 7.

2. SOME CHARACTERISTICS

The usual Sobolev space of real-valued functions $Z^\beta(\Psi)$ on $\Psi = [\gamma, \alpha]$ with non negative integer β is illustrated in [16, 60].

$$Z_0^\beta(\Psi) = \{u(x) \in Z^\beta(\Psi) : \frac{\partial^d u}{\partial x^d} = 0, d = 0(1)\beta - 1\}, \quad (2.1)$$

$$Z_0^\beta(\Psi) = \{u(x) : \int_\Psi (u^{(i)})^2 dx < +\infty, i = 0(1)\beta\}.$$

where $u^{(i)}$ is the i -th order derivative.

The results of M_I (mass) and E_I (energy) for (1.1) - (1.3), are used to measure the conservative properties of the proposed scheme.

Lemma: (See Ref. [53]) Let $u_0(x) \in Z_0^2[\gamma, \alpha]$, then the solution of (1.1) satisfies the mass conservative property

$$M_I(t) = \int_\gamma^\alpha u(x, t) dx = \int_\gamma^\alpha u(x, 0) dx = \int_\gamma^\alpha u_0(x, 0) dx = M_I(0) \quad (2.2)$$

Theorem: Let $u_0(x) \in Z_0^2[\gamma, \alpha]$, then the solution of (1.1) satisfies the energy conservative property

$$E_I(t) = \int_{\gamma}^{\alpha} (u^2(x, t) + u_x^2(x, t) + u_{xx}^2(x, t)) dx = \int_{\gamma}^{\alpha} [u^2(x, 0) + u_x^2(x, 0) + u_{xx}^2(x, 0)] dx = E_I(0) \quad (2.3)$$

For any $t \in [0, T]$.

Proof: Rearranging (1.1) as

$$u_t - u_{xxt} + u_{xxxxt} = -u_x - q\omega u^{q-1}u_x \quad (2.4)$$

Multiplying (2.4) by $2u$ and integrating on the interval $[\gamma, \alpha]$, we obtain

$$\frac{d}{dt} \int_{\gamma}^{\alpha} u^2 dx - 2 \int_{\gamma}^{\alpha} uu_{xxt} dx + 2 \int_{\gamma}^{\alpha} uu_{xxxxt} dx = -2 \int_{\gamma}^{\alpha} uu_x dx - 2 \int_{\gamma}^{\alpha} q\omega u^q u_x dx \quad (2.5)$$

Applying the integration by parts and using the boundary conditions from (1.3), we obtain

$$\int_{\gamma}^{\alpha} uu_{xxt} dx = \frac{-d}{2dt} \int_{\gamma}^{\alpha} u_x^2 dx, \quad \int_{\gamma}^{\alpha} uu_{xxxxt} dx = \frac{d}{2dt} \int_{\gamma}^{\alpha} u_{xx}^2 dx$$

$$\int_{\gamma}^{\alpha} uu_x dx = \frac{(u^2(\alpha, t) - u^2(\gamma, t))}{2} = 0, \quad \int_{\gamma}^{\alpha} q\omega u^q u_x dx = \frac{q\omega(u^{q+1}(\alpha, t) - u^{q+1}(\gamma, t))}{q+1} = 0$$

Applying the integrated results in (2.5), we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\gamma}^{\alpha} (u^2 + u_x^2 + u_{xx}^2) dx &= 0, \\ E_I(t) &= \int_{\gamma}^{\alpha} (u^2(x, t) + u_x^2(x, t) + u_{xx}^2(x, t)) dx = \\ &= \int_{\gamma}^{\alpha} [u^2(x, 0) + u_x^2(x, 0) + u_{xx}^2(x, 0)] dx = E_I(0) \end{aligned}$$

As $E_I(t)$ depends on the initial data for any $t \in [0, T]$ is coined as a constant function.

3. SEPTIC B-SPLINE SCHEME

Let us identify a partition of the space interval $[\gamma, \alpha]$ into N equal space points x_i such that $\gamma = x_0 < x_1 < \dots < x_N = \alpha$ with the step length $h = \frac{(\alpha-\gamma)}{N} = x_{n+1} - x_n$. The septic B-splines $\phi_n(x)$, $n = -3(1)N + 3$ at the knots x_n are given as follows :

$$\phi_n(x) = \quad (3.1)$$

$$\frac{1}{h^7} \begin{cases} (x - x_{n-4})^7, & x \in [x_{n-4}, x_{n-3}) \\ (x - x_{n-4})^7 - \binom{8}{1}(x - x_{n-3})^7, & x \in [x_{n-3}, x_{n-2}) \\ (x - x_{n-4})^7 - \binom{8}{1}(x - x_{n-3})^7 + \binom{8}{2}(x - x_{n-2})^7, & x \in [x_{n-2}, x_{n-1}) \\ (x - x_{n-4})^7 - \binom{8}{1}(x - x_{n-3})^7 + \binom{8}{2}(x - x_{n-2})^7 - \binom{8}{3}(x - x_{n-1})^7, & x \in [x_{n-1}, x_n) \\ (x_{n+4} - x)^7 - \binom{8}{1}(x_{n+3} - x)^7 + \binom{8}{2}(x_{n+2} - x)^7 - \binom{8}{3}(x_{n+1} - x)^7, & x \in [x_n, x_{n+1}) \\ (x_{n+4} - x)^7 - \binom{8}{1}(x_{n+3} - x)^7 + \binom{8}{2}(x_{n+2} - x)^7, & x \in [x_{n+1}, x_{n+2}) \\ (x_{n+4} - x)^7 - \binom{8}{1}(x_{n+3} - x)^7, & x \in [x_{n+2}, x_{n+3}) \\ (x_{n+4} - x)^7, & x \in [x_{n+3}, x_{n+4}) \\ 0, & \text{otherwise} \end{cases} \quad (3.2)$$

Here $\phi_{-3}, \phi_{-2}, \dots, \phi_{N+3}$ is set of septic B-splines, which form a basis on the spatial domain $[\gamma, \alpha]$.

The approximate solution $U(x, t)$ to the analytic solution $u(x, t)$ of (1.1) is given as :

$$U(x, t) = \sum_{i=-3}^{N+3} \mu_i(t) \phi_i(x) \quad (3.3)$$

where, $\mu_i(t)$, $\phi_i(x)$ are the unknown time dependent parameters to be determined at each time level and known septic B-splines as per justification of (3.3) and (3.1), respectively. The nodal values of U_i and its higher order derivatives at the mesh points x_i with help of (3.1) and (3.3) are reported in (3.4).

$$\left. \begin{aligned} U_n &= \mu_{n-3} + 120\mu_{n-2} + 1191\mu_{n-1} + 2416\mu_n + 1191\mu_{n+1} + 120\mu_{n+2} + \mu_{n+3} \\ hU'_n &= 7(-\mu_{n-3} - 56\mu_{n-2} - 245\mu_{n-1} + 0\mu_n + 245\mu_{n+1} + 56\mu_{n+2} + \mu_{n+3}) \\ h^2U''_n &= 42(\mu_{n-3} + 24\mu_{n-2} + 15\mu_{n-1} - 80\mu_n + 15\mu_{n+1} + 24\mu_{n+2} + \mu_{n+3}) \\ h^3U'''_n &= 210(-\mu_{n-3} - 8\mu_{n-2} + 19\mu_{n-1} + 0\mu_n - 19\mu_{n+1} + 8\mu_{n+2} + \mu_{n+3}) \\ h^4U^{(iv)}_n &= 840(\mu_{n-3} + 0\mu_{n-2} - 9\mu_{n-1} + 16\mu_n - 9\mu_{n+1} + 0\mu_{n+2} + \mu_{n+3}) \\ h^5U^{(v)}_n &= 2520(-\mu_{n-3} + 4\mu_{n-2} - 5\mu_{n-1} + 0\mu_n + 5\mu_{n+1} - 4\mu_{n+2} + \mu_{n+3}) \\ h^5U^{(vi)}_n &= 5040(\mu_{n-3} - 6\mu_{n-2} + 15\mu_{n-1} - 20\mu_n + 15\mu_{n+1} - 6\mu_{n+2} + \mu_{n+3}) \end{aligned} \right\} \quad (3.4)$$

3.1. Implementation of septic B-spline Scheme with BFRK scheme (SBSBFRKS)

The SBSBFRKS is implemented to obtain a better approximate solution of the governing (1.1). Employing (3.3) in (1.1) to obtain the approximate solution of the form as follows:

$$\begin{aligned} & \sum_{n=-3}^{N+3} (\phi_n(x) - \phi''_n(x) + \phi_n^{(iv)}(x)) \mu'_n(t) = \\ & - \left(1 + \omega q \left(\sum_{n=-3}^{N+3} \mu_n(t) \phi_n(x) \right)^{q-1} \right) \sum_{n=-3}^{N+3} \mu_n(t) \phi'_n(x) \end{aligned} \quad (3.5)$$

We apply (3.1) as well as (3.4) and its time derivatives in (3.8) to generate (3.6) which contains the first order differential equations as:

$$z_1\mu'_{n-3} + z_2\mu'_{n-2} + z_3\mu'_{n-1} + z_4\mu'_n + z_5\mu'_{n+1} + z_6\mu'_{n+2} + z_7\mu'_{n+3} = \xi_n, \quad 0 \leq n \leq N \quad (3.6)$$

$$\text{where, } z_1 = \left(\frac{840}{h^4} - \frac{42}{h^2} + 1\right), \quad z_2 = \left(120 - \frac{1008}{h^2}\right), \quad z_3 = \left(1191 - \frac{7560}{h^4} - \frac{630}{h^2}\right), \\ z_4 = \left(\frac{3360}{h^2} + \frac{13440}{h^4} + 2416\right), \quad z_5 = \left(1191 - \frac{7560}{h^4} - \frac{630}{h^2}\right), \quad z_6 = \left(120 - \frac{1008}{h^2}\right), \\ z_7 = \left(\frac{840}{h^4} - \frac{42}{h^2} + 1\right)$$

$$\xi_n = \\ \frac{-7}{h} \left[1 + \omega q(\mu_{n-3} + 120\mu_{n-2} + 1191\mu_{n-1} + 2416\mu_n + 1191\mu_{n+1} + 120\mu_{n+2} + \mu_{n+3})^{q-1} \right] \times \\ \left(-\mu_{n-3} - 56\mu_{n-2} - 245\mu_{n-1} + 0\mu_n + 245\mu_{n+1} + 56\mu_{n+2} + \mu_{n+3} \right)$$

A matrix representation of (3.7) or (3.8) is constructed with the elimination of the parameters $\mu'_{-3}, \mu'_{-2}, \mu'_{-1}, \mu'_{N+1}, \mu'_{N+2}, \mu'_{N+3}$ and $\mu_{-3}, \mu_{-2}, \mu_{-1}, \mu_{N+1}, \mu_{N+2}, \mu_{N+3}$ from (3.6) to obtain a unique solution by using the six constraints from the boundary conditions of (1.3).

$$D\mu'(t) = G(\mu(t)) \quad (3.7)$$

Or

$$\mu'(t) = Q(\mu(t)) \quad (3.8)$$

where, D is a square matrix of order $(N+1)$ and

$$Q(\mu(t)) = D^{-1}G(\mu(t)).$$

The highlight of this paper is to solve (3.8) with BFRK, without finite difference method at each time level (see Hossain et al. [13]).

$$\left. \begin{aligned} \bar{K}_1 &= kG(\bar{\mu}^n) \\ \bar{K}_2 &= kG\left(\bar{\mu}^n + \frac{1}{4}\bar{K}_1\right) \\ \bar{K}_3 &= kG\left(\bar{\mu}^n + \frac{1}{8}(\bar{K}_1 + \bar{K}_2)\right) \\ \bar{K}_4 &= kG\left(\bar{\mu}^n - \frac{1}{2}\bar{K}_2 + \bar{K}_3\right) \\ \bar{K}_5 &= kG\left(\bar{\mu}^n + \frac{3}{16}(\bar{K}_1 + 3\bar{K}_4)\right) \\ \bar{K}_6 &= kG\left(\bar{\mu}^n + \frac{1}{7}(-3\bar{K}_1 + 2\bar{K}_2 + 12\bar{K}_3 - 12\bar{K}_4 + 8\bar{K}_5)\right) \end{aligned} \right\} \quad (3.9)$$

$$\bar{\mu}^{n+1} = \bar{\mu}^n + \frac{1}{90}[7\bar{K}_1 + 0\bar{K}_2 + 32\bar{K}_3 + 12\bar{K}_4 + 32\bar{K}_5 + 7\bar{K}_6] \quad (3.10)$$

where, k is time step and

$$\bar{\mu}^{n+1} = [\mu_0^{n+1} \mu_1^{n+1} \dots \mu_N^{n+1}]^T.$$

The vector $\bar{\mu}^n$ is applied for the time evaluation of $U(x, t)$ on SBSBFRKS. It is obtained by repeated application of the recurrence relations (3.9) and (3.10), after finding the initial

$$\left. \begin{aligned} U_x(\gamma, 0) &= 0 \\ U_{xx}(\gamma, 0) &= 0 \\ U_{xxx}(\gamma, 0) &= 0 \\ U(x, 0) &= u_0(x_n), \quad 0 \leq n \leq N \\ U_x(\alpha, 0) &= 0 \\ U_{xx}(\alpha, 0) &= 0 \\ U_{xxx}(\alpha, 0) &= 0 \end{aligned} \right\} \quad (3.11)$$
$$A_{\bar{\mu}}{}^0 = B \quad (3.12)$$

[illegible]

$b_4 = \frac{6709}{54}$, $b_5 = 1$, $e_1 = \frac{3200}{27}$, $e_2 = \frac{10733}{9}$, $e_3 = \frac{21752}{9}$, $e_4 = \frac{32158}{27}$, $e_5 = 120$, $e_6 = 1$. Finally, the initial vector $\bar{\mu}^0$ is obtained from (3.12).

$$u_t - u_{xxt} + u_{xxxxt} = -(u_x + \omega q \theta u_x) \quad (4.1)$$

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in (4.1), respectively.

$$\begin{aligned}
 (c_1 + c_2)\mu_{n-3}^{m+1} + (c_3 + c_4)\mu_{n-2}^{m+1} + (c_5 + c_6)\mu_{n-1}^{m+1} + c_7\mu_n^{m+1} + (c_5 - c_6)\mu_{n+1}^{m+1} \\
 + (c_3 - c_4)\mu_{n+2}^{m+1} + (c_1 - c_2)\mu_{n+3}^{m+1} = (c_1 - c_2)\mu_{n-3}^m + (c_3 - c_4)\mu_{n-2}^m \\
 + (c_5 - c_6)\mu_{n-1}^m + (c_5 + c_6)\mu_n^m + (c_3 + c_4)\mu_{n+1}^m + (c_1 + c_2)\mu_{n+3}^m
 \end{aligned} \quad (4.2)$$

where, k is a time step

$$\begin{aligned}
 c_1 &= \frac{840}{h^4} - \frac{42}{h^2} + 1, \\
 c_2 &= -\frac{7k}{2h} - \frac{7\omega q k \theta}{2h}, \\
 c_3 &= 120 - \frac{1008}{h^2}, \\
 c_4 &= -\frac{196\omega q k \theta}{h} - \frac{196k}{h}, \\
 c_5 &= -\frac{630}{h^2} - \frac{7560}{h^4} + 1191, \\
 c_6 &= -\frac{1715\omega q k \theta}{2h} - \frac{1715k}{2h}, \\
 c_7 &= \frac{3360}{h^2} + \frac{13440}{h^4} + 2416.
 \end{aligned}$$

Applying the Fourier mode $\mu_n^m = H^m e^{inwh}$, $i = \sqrt{-1}$, where, w is a mode number and h is the element size, in (4.2) and implementation of Euler's formula ($e^{iwh} = \cos(wh) + i\sin(wh)$) and further simplifying the growth factor H becomes

$$H = \frac{B + iC}{B - iC} \quad (4.3)$$

where, $B = c_7 + 2c_5\cos(wh) + 2c_1\cos(3wh) + 2c_3\cos(2wh)$

$$C = -2c_6\sin(wh) - 2c_2\sin(3wh) - 2c_4\sin(2wh)$$

$$|H|^2 = \frac{B^2 + C^2}{B^2 + C^2} = 1 \quad (4.4)$$

Then modulus of $|H| = 1$, this indicates that the linearized septic B-spline collocation approach is unconditionally stable.

5. NUMERICAL EXAMPLES AND DISCUSSIONS

One test problem is considered to validate the efficiency and accuracy of the present work. In this section for the sake of comparison of the present work with others, the proposed schemes are applied for selected values of $q = 2, 4$. The exactness of the methods are incorporated by computing the error norms L_∞ and L_2 as follows:

$$L_\infty = \max_{0 \leq n \leq N} |u_n - U_n|$$

$$L_2 = \sqrt{h \sum_{n=0}^N |(u_n - U_n)^2|}$$

where u_n and U_n represents exact and numerical solutions, respectively.

The numerical solution of the two conservative laws related to conservation of mass and energy for the GR-RLW equations are given in [62], follows:

$$M_I = \int_a^b u(x, t) dx = h \sum_{n=1}^{N-1} U_n^m$$

$$E_I = \int_a^b (u^2(x, t) + u_x^2(x, t) + u_{xx}^2(x, t)) dx = h \sum_{n=1}^{N-1} ((U_n^m)^2 + [(U_n^m)_x]^2 + [(U_n^m)_{xx}]^2)$$

Example 1. Let us consider the following generalized Rosenau-RLW equation with $\omega = 1$ as

$$u_t - u_{xxt} + u_{xxxxt} + u_x + qu^{q-1}u_x = 0$$

subject to the initial condition $u(x, 0) = D \operatorname{sech}^{\frac{4}{q-1}}(Ax)$, $x \in [\gamma, \alpha]$ and boundary conditions

$$u(\gamma, t) = u(\alpha, t) = u_x(\gamma, t) = u_x(\alpha, t) = u_{xx}(\gamma, t) = u_{xx}(\alpha, t) = 0, \quad t \in [0, T]$$

The analytical solution of this problem is

$$u(x, 0) = D \operatorname{sech}^{\frac{4}{q-1}}(A(x - Bt))$$

$$\text{where, } A = \frac{q-1}{\sqrt{4q^2+8q+20}}, \quad B = \frac{q^4+4q^3+14q^2+20q+25}{(q^4+4q^3+10q^2+12q+21)},$$

$$E = \ln\left(\frac{(q+3)(3q+1)(q+1)}{2(q^2+3)(q^2+4q+7)}\right), \quad D = e^{\left(\frac{E}{q-1}\right)}$$

Case I. In this case, the computed results for the error norms L_2 and L_∞ at $k = h = 0.1$, $q = 2$, $x \in [-30, 120]$ with different time values $t \leq 10$ using SBSBFRKS and the solid comparison to others [45,53] are reported in Table 6.1. In similar manner the results using the present approaches and comparison with others [45,53] for the two error norms at different values of h and with the same measures for the above parameters k, q, x for $t = 10$ are reported in Table 6.2. It is found that the SBSBFRKS is a better approach as compared to others in literature. We can also profile the numerical and exact solutions at selected time values $t = 4, 6, 8, 10$ and $p = 2$, $h = k = 0.4$, $x \in [-30, 120]$ in Fig. 6.1. Fig. 6.1 illustrates the solitary waves are moved to the right with almost unchanged amplitude as the time simulation increases.

Case II. The results of error norms L_2 and L_∞ for different values of h with $k = 0.1$, $q = 4$, $x \in [-30, 120]$ and $t = 60$ using the present method is reported in Table 6.3. It is investigated that the SBSBFRKS is much better than the results of [45]. The conservative values M_I and E_I and L_2 and L_∞ for different time levels $t \leq 60$ with $h = k = 0.1$, $q = 2$, $x \in [-30, 120]$ are computed by the present methods and compared with [45] are reported in Table 6.4. It is clearly viewed that the approximate solution using SBSBFRKS of the two invariants are closed to their exact solution $M_I = 1.897658228777$, $E_I = 0.533177532044$ and the error norms L_2 and L_∞ are improved much better than [45].

Case III. In this numerical simulation, the error norms L_2 and L_∞ for different values of $k = h$ with $t = 24$, $q = 4$, $x \in [-50, 150]$ are obtained using proposed method and compared

with others [52, 59, 62, 63] are reported in Table 7.5. From Table 7.5, we have observed that the results obtained using SBSBFRKS is better other existing methods in literature. The numerical solutions of the two invariants M_I and E_I are examined by present schemes and compared with [62] at different values of time t for $h = k = 0.4, x \in [-50, 150]$ are displayed in Table 6.6. The remarkable points are numerical solutions of M_I and E_I with help of SBSBFRKS are close to the analytical solutions of $M_I = 6.2658061620$ and $E_I = 2.8676945570$ to other existing schemes in literature.

6. FIGURES AND TABLES

This section provides figures and tables.

6.1. Figures

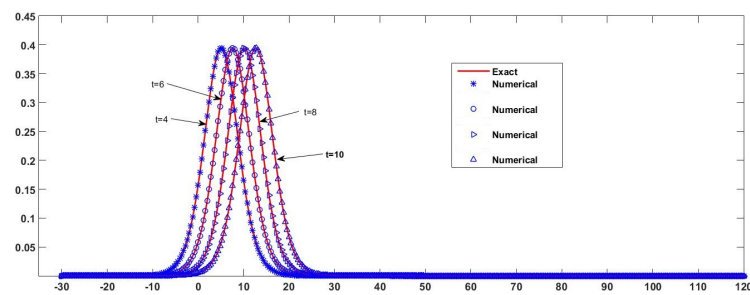


Fig. 6.1. Numerical and exact solutions at the selected time values $t = 4, 6, 8, 10$ of Example 1(Case I) with $h = k = 0.4, p = 2, x \in [-30, 120]$

6.2. Tables

Table 6.1. The results of error norms for Example 1 (Case I) at different value t and comparison with others

<i>Methods</i>	SBSBFRKS		[45]		[53]
t	L_∞	L_2	L_∞	L_2	L_∞
2	$2.4285e^{-7}$	$2.7684e^{-7}$	$2.8908e^{-6}$	$5.8217e^{-6}$	$1.7282e^{-5}$
4	$3.1281e^{-7}$	$3.9337e^{-7}$	$5.1694e^{-6}$	$1.0632e^{-5}$	$3.3671e^{-5}$
6	$3.3624e^{-7}$	$5.1873e^{-7}$	$6.4804e^{-6}$	$1.4223e^{-5}$	$4.8067e^{-5}$
8	$3.4304e^{-7}$	$6.5073e^{-7}$	$7.1646e^{-6}$	$1.6523e^{-5}$	$1.0355e^{-4}$
10	$3.4306e^{-7}$	$7.7492e^{-6}$	$7.6292e^{-6}$	$1.8123e^{-5}$	$4.1701e^{-4}$

Table 6.2. The comparison of L_∞ and L_2 for Example 1 (Case I) at different value of h with others

<i>Methods</i>	SBSBFRKS		[45]		[53]	
h	L_∞	L_2	L_∞	L_2	L_∞	L_2
0.5	$3.7107e^{-7}$	$1.3374e^{-6}$	$1.9117e^{-4}$	$4.5438e^{-4}$	...	...
0.25	$4.3678e^{-7}$	$1.1462e^{-6}$	$4.7606e^{-5}$	$1.1308e^{-4}$	$1.9743e^{-4}$	...
0.125	$4.6190e^{-7}$	$1.2114e^{-6}$	$1.1908e^{-5}$	$2.8287e^{-5}$	$5.0101e^{-5}$	...
0.0625	$4.7041e^{-7}$	$1.2353e^{-6}$	$2.9950e^{-6}$	$7.1601e^{-6}$	$1.2489e^{-5}$	...

Table 6.3. Comparison of error norms for Example 1 (Case II) at $t = 60$ with [45]

<i>Methods</i>	SBSBFRKS		[45]	
h	L_∞	L_2	L_∞	L_2
1	$1.2337e^{-4}$	$3.1100e^{-4}$	$1.1541e^{-2}$	$3.0217e^{-4}$
0.5	$2.2417e^{-5}$	$1.1658e^{-4}$	$2.2164e^{-3}$	$5.5853e^{-4}$
0.25	$2.8796e^{-5}$	$1.6559e^{-4}$	$5.1848e^{-4}$	$1.3010e^{-4}$
0.125	$3.1282e^{-5}$	$1.9000e^{-4}$	$1.2789e^{-4}$	$3.5071e^{-4}$
0.0625	$3.2092e^{-6}$	$7.9856e^{-5}$	$3.1864e^{-5}$	$1.8243e^{-4}$

Table 6.4. Comparative study of the error norms and invariants for Example 1 (Case II) at $t \leq 60$ with other

<i>Methods</i>	SBSBFRS				[45]			
t	$L_\infty e^7$	$L_2 e^7$	M_I	E_I	$L_\infty e^5$	$L_2 e^5$	M_I	E_I
10	3.341	7.749	1.8976590	0.533177532	0.7629	1.813	1.89765990	0.53317753
20	3.340	11.91	1.8976612	0.533177532	0.9095	2.251	1.89766149	0.53317753
30	3.417	14.63	1.8976626	0.533177532	1.027	2.546	1.89766306	0.53317753
40	3.444	16.73	1.8976638	0.533177532	1.138	2.814	1.89766459	0.53317753
50	3.470	18.47	1.8976651	0.533177532	1.245	3.075	1.89766608	0.53317753
60	3.485	19.97	1.8976663	0.533177532	1.350	3.338	1.89766756	0.53317753

Table 6.5. The comparative study of error norms for Example 1 (Case III) at $t = 24$ with others

Methods		SBSBFRKS	[62]	[52]	[63]	[59]
h=0.8	L_∞	$1.54154e^{-5}$	$1.73987e^{-4}$	$1.64873e^{-2}$	$9.21264e^{-2}$	$2.03188e^{-2}$
	L_2	$3.83604e^{-5}$	$4.33437e^{-4}$	$4.33356e^{-2}$	$2.29013e^{-1}$	$5.47448e^{-2}$
h=0.4	L_∞	$1.23509e^{-6}$	$1.03213e^{-5}$	$4.25828e^{-3}$	$2.01925e^{-2}$	$4.78559e^{-3}$
	L_2	$2.95514e^{-6}$	$2.59246e^{-5}$	$1.11909e^{-2}$	$5.00535e^{-2}$	$1.29577e^{-2}$
h=0.2	L_∞	$8.06728e^{-8}$	$6.37832e^{-7}$	$1.07810e^{-3}$	$4.89501e^{-3}$	$1.17720e^{-3}$
	L_2	$2.00480e^{-7}$	$1.60587e^{-6}$	$2.82645e^{-3}$	$1.21867e^{-2}$	$3.20256e^{-3}$
h=0.1	L_∞	$1.79380e^{-8}$	$3.97932e^{-8}$	$7.09140e^{-4}$	$1.21680e^{-3}$	$2.93288e^{-4}$
	L_2	$7.27691e^{-8}$	$1.00259e^{-7}$	$2.70375e^{-4}$	$3.03019e^{-3}$	$7.99013e^{-4}$

Table 6.6. The numerical solution of L_2 and L_∞ and comparison of M_I and E_I for Example 1 (Case III) at different value t

Methods		SBSBFRKS		[62]		
t	$L_\infty e^7$	$L_2 e^6$	M_I	E_I	M_I	E_I
exact	...	...	6.265806162	2.867694557	...	...
$t = 4$	8.176	1.644	6.265806163	2.86769446	6.265806174	2.8667709
$t = 8$	7.983	1.856	6.265806168	2.86769436	6.265806174	2.8667711
$t = 12$	8.858	2.084	6.265806168	2.86769426	6.265806173	2.8667711
$t = 16$	9.899	2.343	6.265806168	2.86769417	6.265806173	2.8667710
$t = 20$	11.02	2.634	6.265806169	2.86769407	6.265806174	2.8667710
$t = 24$	12.35	2.955	6.265806169	2.86769397	6.265806174	2.8667710

7. CONCLUSIONS

In this study, a septic B-spline method combined with a direct BFRK technique is proposed for solving the GR-RLW equation. In the proposed approach, septic B-spline basis functions are employed to transform the governing equation into a system of first-order ordinary differential equations, which is subsequently solved using the BFRK scheme. The proposed SBSBFRKS method computes approximate solutions of the GRRLW equation without employing any linearization or transformation techniques. In addition, the invariants and error norms L_2 and L_∞ are evaluated to assess the accuracy of the method. The numerical results demonstrate that the proposed method provides solutions that are in excellent agreement with the analytical solutions. Furthermore, the computed invariants and error norms obtained using the SBSBFRKS method are more accurate than those reported by existing numerical methods in the literature.

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