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A Revisit to the Prevalent Producer Theory with Emphasis Placed on Firms' Individualism

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Abstract: As time elapses, an increasing number of theorists and practitioners expressed the urgent need for the community of economists to rebuild economic theories so that derived conclusions would be more readily applicable to real life. To answer this theoretically and practically important call, this paper attempts to reformulate some of the main conclusions of the producer theory so that firms are allowed to have their individually different criteria of optimality and methods of optimization, as the real-life business world dictates. To achieve this goal, on the basis of natural endowments of firms, this paper establishes a series of 7 generally true propositions, while it simultaneously examines how some of the presently well-known results hold true only conditionally. In the process of achieving this end, we generalize Hotelling's and Shepard's lemmas to much more relaxed scenarios than before. Because natural endowments are generally different from one firm to another, what is considered better is defined differently so that firms do not collectively produce a better society as a whole, even though each of them maximizes its self-interest. In other words, one of the main conclusions this paper derives formally is that the invisible hand, as proposed so convincingly by Adam Smith, is indeed not only invisible but also nonexistent in real life, unless a supernatural being is out there to tell what is better for everyone. In the conclusion, several open questions are listed for future research.

Keywords: decision making; invisible hand; modular function; natural endowments; optimization; preference order; rationality

1. INTRODUCTION

In studies of economic decision making, a commonly employed approach is to first introduce an objective function, such as a utility function, a production function, a profit function, etc., and then based on some kind cost-and-benefit analysis of the decision maker, this objective function is optimized (e.g., [15,16,18]). However, such an approach does not capture real-life scenarios [3,52] since not everybody is a maximizer or minimizer (e.g., [24, 26]), although it has been repeatedly confirmed with falsified empirical evidence, as so criticized by behavioral economists (e.g., [28,41]). Hence, the following question arises naturally at the most fundamental level underneath all investigations of economic decision making, if one focuses only on the micro-level of individual firms: Does a firm really go through such a general procedure when it decides on what to do in terms of making a production decision?

Theoretically speaking, the importance of this question is well witnessed by the vast amount of related literature in the name of rationality, where the aforementioned, commonly employed approach in studies of economic decision making is widely known as the assumption of rationality. Although such rationality has been criticized only in recent decades by behavioral economists, some degrees of an inherent uncertainty this assumption implicitly embodies has been broadly felt and explored by a good number of leading scholars [23], including, among numerous others, Gary Becker [4], Frank Lovett [36], Fritz Machlup [37],

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Ariel Rubinstein [46], Paul Samuelson [47], Herbert Simon [50], LL Thurstone [53], Max Weber [55] and Glen Weyl [57]. In summary, after using this approach for so many decades, scholars are still debating on what the assumption of rationality really means [23]. This end indirectly explains the reason why a compelling need for a meaningful reconstruction of economic theory has been called for by recent events, in particular, the 2008 financial crisis. For example, considering the inability for existing economic theories to describe, to predict and to explain in a timely manner the recent financial turmoil, Paul Krugman commented as follows in *New York Times* (2009-09-02),

> The economic profession went astray because economists, as a group, mistook beauty, clad in impressive-looking mathematics, for truth … As memories of the Depression faded, economists fell back in love with the old, idealized vision of an economy in which rational individuals interact in perfect markets … Unfortunately, this romanticized and sanitized vision of the economy led most economists to ignore … things that can go wrong. They turned a blind eye to the limitations of human rationality that often leads to bubbles and burst; to the problem of institutions that run amok; to the imperfection of markets … that can cause the economy … to undergo sudden, unpredictable crashes; and to the dangers created when regulators don't believe in regulation.

At the same time, Paul De Grauwe wrote the following in *Financial Times* (2009-07-21):

> Mainstream (economic) models take the view that economic agents are superbly inform and understand the deep complexities of the world ... they have "rational expectations" … they all understand the same "truth", they all act the same way. Thus modelling the behavior of just one agent (the "representative" consumer and the "representative" producer) is all one has to do to fully describe the intricacies of the world. Rarely has such a ludicrous idea been taken so seriously by so many academics.

Practically speaking, the significance of the previously posted question is witnessed by failures of predicting the occurrence of imminent economic crises, such as the Great Recession that started in 2008. In particular, economists surely failed to foretell that the crisis was brewing and failed to predict the surprisingly damaging aftermath when the noted home-price bubble burst [31]. In the contrast, many scholars provided inconsistent 20-20 hint-sights regarding the causes underneath the Recession (e.g., [25,29]). Additionally, such failures widely exist in applications of economic theories, such as what caused the Industrial Revolution [14,20,56] and whether or not economic policies actual work in real life [2,11].

That is, no matter whether one is talking about theoretical development or practical applications of economic theories and related business studies, there is a need to reestablish the relevant theories so that the derived conclusions are closer to real life than before. Aiming at addressing the aforementioned question of fundamental importance, this paper represents a first step towards this goal by basing our reasoning and analysis on the four natural endowments of a firm: self-awareness, imagination, conscience and free will. By doing so unconventionally, this research is able to pay a revisit to some of the most well-known results in the prevalent producer theory, one of the economic theories that desperately needs to be rewritten, and show how some known conclusions are only true conditionally, while others do not hold true at all.

The rest of this paper is organized as follows. Section 2 prepares for the smooth flow of analysis and discussion of the following sections. Section 3 develops all the main results of this work by generalizing the well-known Hotelling's lemma and Shepard's lemma and by examining how and why no societal-wide preference order can realistically exists. This presentation concludes in Section 4 with a few important open problems for future research.

2. MOTIVATION, FIRMS' DECISION MAKING AND OPERATION

This section prepares the presentation of the paper from two different angles. First, it quotes relevant conclusions about a firm's natural endowments and the role they play in the firm's decision making. And second, this section lays out the conventions and the necessary symbolic terms needed for later discussions.

2.1. Roles Natural Endowments Play in Firms' Decision Making

To help develop the desired methodology, we notice the fact that most part of modern science and mathematics is intrinsically based on the concept of numbers, although more abstract objects have been gradually introduced over time by using different names [30]. At the same time, the systemic structures behind numbers are ignored [34]. For instance, the concept of number 1 is abstracted from such observations as 1 apple, 1 table, 1 chair, etc. When number 1 is singled out from these situations, the systemic structures of the apple, table, chair, etc., are ignored. However, in terms of studies of business, the established theories are mainly about how a business entity evolves and how it interacts with others. That is, theories of business emphasize on systemic structures that are conventionally ignored by the well-adopted methods in economic studies [12]. To overcome this deficit in focus, in our effort to make the producer theory a better fit to real life, let us treat each firm as a living being so that its decisions are made based on its individually unique system of values and beliefs. And such system exists on top of and is determined by the natural endowments of the firm.

Specifically, parallel to the four natural endowments of an individual – self-awareness, imagination, conscience and free will [35], Forrest, Hafezalkotob et al. [10] develop a set of corresponding natural endowments for a firm. Specifically, by a firm's self-awareness, it means the firm's awareness that it exists as a business entity that is separate from other entities, such as people, firms and things, with its business secrets, such as adopted customer value propositions, operational strategies, protected product designs, etc. By a firm's imagination, it describes the firm's ability to learn and to acquire new knowledge, to innovatively imagine what might be the right offer, such as a newly designed product, or an improved product or new (or improved) service, to satisfy the deciphered information of market demand, and to develop the necessary process of materially introducing the imagined offer. By a firm's conscience, it represents the ability for the firm to evaluate which business effort among a group of alternatives is more beneficial than other efforts. By a firm's free will, it means the capability for the firm to keep, how to keep and to what degree to keep the promises written in its contracts with various business partners.

As is well-known from real-world experiences [39], although each firm naturally possesses these natural endowments, how well a firm can mobilize these endowments is dependent on the specific composition and constraints of the firm. That is, the degree of how the natural endowments can be employed is really different from one firm to another. That explains why some firms do well in certain aspects of business while not as well in other aspects. Real-life examples that can be used to confirm this end are plentiful. For instance, in any economic sector, there are a few companies that dominate the market while others do not seem prominent or matter at all.

By using systemic thinking and logical reasoning, Forrest, Shao et al. [13] establish the following results, when these scholars attempt to find the true meaning of the assumption of rationality (e.g., [15,16,23]): (i) The goal of each firm's effort is to materialize, at least partially or remotely, the firm's clearly stated mission. (ii) At the macro-firm level, the assumption of rationality stands for finding an optimal choice among all available alternatives with the criteria of optimality determined by the focal firm's management based on the firm's natural endowments. (iii) Each firm has its unique and dissimilar system of values and beliefs. (iv) When a firm faces a decision-making situation, it optimizes the potential subject to the given constraints by using its particular set of criteria of optimality, as formulated consistently with its system of values and beliefs.

Example 1:

To help confirm the importance of these systemic conclusions, let us revisit a decision-making situation from Hu [22] and Lin [34, p. 136]. Assume that the production routine of a firm can be abstracted into the directed and weighted network in Figure 1. The firm needs to minimize the path from node X, where the production starts, to node Z, where the production ends.

Case 1: The firm orders the real-number weights in the same way as how real numbers are conventionally ordered. In this case, $X \rightarrow X_1 \rightarrow Y \rightarrow Z_1 \rightarrow Z$ is the path the firm looks for. This path has the weight of 1. And other paths from node X to node Z respectively have weights 2, 3, and 4.

Case 2: The firm orders the real-number weights by referring to the mod4 function so that for any two real numbers x and y, $x < y$ if and only if $x(\text{mod}4) < y(\text{mod}4)$. Within such a system of decision-making, $X \rightarrow X_2 \rightarrow Y \rightarrow Z_2 \rightarrow Z$ is the path the firm looks for. The path's weight is equal to 3 + 0 + 0 + 1 = 4 (mod 4) = 0, where x (mod4) = the reminder of $x \div 4$. In comparison, the weights of other paths respectively have weights 1, 2, or 3.

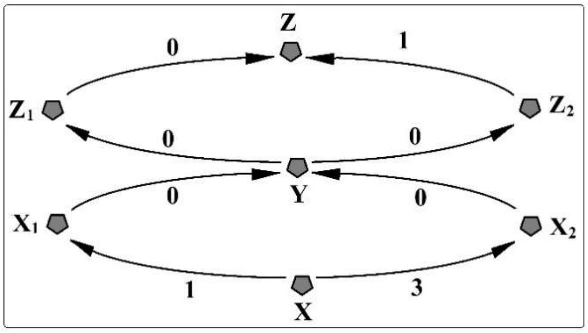

Figure 1. How minimization is defined differently in different value-belief systems

Before moving on, let us explain how the modular function, such as the mod4 function above, is actually applied in real life and how different systems of values and beliefs dictate different orderings of real numbers. Firstly, modular functions appear frequently in life, because in principle mod$r$ function represents periodicity r, for any positive real number r [10]. Commonly seen examples include 12-hour clocks, 7-day weeks, months of various numbers of days, where the first scenario represents a mod12 function, the second scenario the mod7 function, and the third scenario comprises a mixture of mod28, mod29, mod30 and mod31 functions. And, more generally, the projects a firm participates in also indirectly deal with modular operations. Specifically, every time when a new project starts, the involved firm begins its new round of counting of, for example, costs and profits, and measurement of, for example, how success a new business procedure will be.

Secondly, according to the conventional ordering of real numbers, one has \$30 K > \$3 million. That is the order commonly used in the literature of neoclassical economics. However, if it is revealed that \$30 K is the wage from a lawful employment, while \$3 million is the individual share of a group effort of robbing a bank, then a lot of people with certain kinds of values and beliefs will order \$30 K > \$3 million. In terms of business firms, one can readily construct scenarios where values and beliefs make a difference in the ordering of real numbers, for example, if such a concept as corporate social responsibilities, see, e.g., [9] for relevant details, is involved.

Now, we are ready to summarize what Example 1 tells us - different criteria of priority, such as different orderings of real numbers, empower different methods of optimization, leading to different optimal decisions. And all these differences stem from the varied systems of values and beliefs of individual firms. That is, although different firms might look at an identical objective function, the specifically employed criteria of priority or methods of optimization can be different from one firm to another. This end implies that Example 1 analytically confirms what Mises [40, p. 244] says – "the value judgements a man pronounces about another man's satisfaction do not assert anything about this other man's satisfaction. They only assert what condition of this other man better satisfies the man who pronounces the judgement." Specific to our current context, what is implied by Case 1 of the example above, which is the commonly studied situation, is that the economist asserts the focal firm's condition that better satisfies the economist, while Case 2 may actually be the state of affairs of the firm. That is, there is discrepancy between what the economist expects and what the firm desires to achieve [52]. This discrepancy surely represents one source of uncertainties and risks the economist experiences or takes when he draws conclusions and makes claims[†], if the firm goes after what its values and beliefs direct.

Based on the discussions above, assume that each firm has a particular way, defined by its system of values and beliefs, to order real numbers that fall within the domain D of its decision-making activities. Let the ordering relation be denoted by $\leq_F$ for less than or equal to, $\geq_F$ for greater than or equal to, and $=_F$ for equality.

2.2. Set-Theoretical Model of a Firm's Operations

This subsection and the rest of this paper consider a randomly chosen firm that will be referred to as the firm. When it purchases a quantity x of a commodity at unit price (or simply price) p_x, the firm creates for its account a debit in the amount of xp_x. In the contrary, when it sells a commodity in quantity y at unit price or price p_y, it creates in its account a credit of yp_y.

The quantities of all commodities the firm purchases from others, referred to as inputs to the firm, and offers to the market, referred to as outputs of the firm, can be written as a vector. To separate inputs from outputs in such a vector, the former quantities are represented by negative numbers because they create debits for the firm, while the latter quantities by positive numbers as they represent revenues of the firm. Assume that in the marketplace, all commodities are available for exchange and there are ℓ commodities in total that are ordered with such labels as 1, 2, …, ℓ, respectively. To make the following analysis possible, assume that the quantity of each commodity, either received as an input or offered to the market as an output, is a real number, as commonly done in economic analysis (e.g., [43])

If $p = (p_1, p_2, \dots, p_\ell)$ stands for the price system of the quantity vector $c = (c_1, c_2, \dots, c_\ell)$ of all commodities, then the firm's account will have the overall cash flow

$$p \cdot c = \sum_{h=1}^{\ell} p_h c_h, \tag{2.1}$$

where $\cdot$ stands for the dot product of vectors p and c. Without loss of generality, assume that each price system $p = (p_1, p_2, \dots, p_\ell)$ contains only positive components. That is, no commodity can be acquired without paying a price. And, for the commodity vector c, some or most of its components should be zero, representing that these commodities are neither inputs nor outputs of the firm.

As for commodities, considered include specific times when they are available for delivery, although they are exchanged at the present time, and locations where exchanges of ownership take place. As in real life, for such purposes, the time axis is divided into intervals of equal length, labeled by using natural numbers 1, 2, …, chronologically with the first interval starts at the origin 0, seen as the present moment. Assume that all time moments within an interval are not distinguishable. Similarly, the land is divided into finite many regions, in each of which

[†] Because this paper does not deal with the set of possible consumptions of a consumer, the concepts of risk and Knightian uncertainty defined for choice of possible consumptions [5] do not play any role here.

deliveries take place. So, when a commodity has different times of availability and/or different locations of delivery, it is seen as separate and different commodities with specified time and location. Correspondingly, the price of a commodity is the amount a consumer needs to pay now for each unit of that commodity while interests and discounts over time are omitted in order to simplify our analysis. Additionally, we ignore the value of money at different locations. That is, no issue with exchanges of money is involved.

For the firm, its plan of action is to specify the quantity of each commodity it either consumes or offers. That is, its plan $y = (y_1, y_2, \dots, y_\ell)$ of action is to choose an element from the ℓth dimensional Euclidean space $\mathbb{R}^\ell$, representing the quantities of commodities it either consumes or offers, where $\mathbb{R}$ is the set of all real numbers and the subscripts 1, 2, …, ℓ denote the individual commodities that are available for exchanges. So, the price of action y is given by

$$p \cdot y = \sum_{h=1}^{\ell} p_h y_h, \tag{2.2}$$

where $p = (p_1, p_2, \dots, p_\ell)$ is the price system of all commodities and y_h stands for the quantity of commodity h (= 1, 2, …, ℓ). If the hth component of y is 0, it means that the firm neither consumers nor produces commodity h.

Within the boundary of its constraints, the firm chooses such a plan of action that best fits its specific system of values and beliefs, where maximizing profit, as conventionally studied in the literature, stands for only one such scenario that a particular system of values and beliefs might be demanding. Let Y be the set of all feasible production plans of the firm. That is, each $y \in Y$ is a production possibility of the firm that is technically materializable and meets the moral codes of the firm's system of values and beliefs.

As the last part of this preparation section, note that two specific binary relations, $\leq$ and $\leq_F$, exist. The first one is defined on Y such that for any $x, y \in Y$, $x \leq y$ if and only if $x_h \leq y_h$, for each $h = 1,2, \dots, \ell$. And the second is the firm-specific ordering $\leq_F$ of real numbers. As defined in Mas-Collel et al. [38], if $\leq$ satisfies the conditions of completeness, transitivity and reflexivity, then the firm is seen as rational. Evidently, this binary relation $\leq$ is not complete. On the other hand, because $\leq_F$ represents firm F's specific criteria of priority defined for the real numbers in the domain D of decision-making activities, when no confusion appears, assume that $\leq_F$ satisfies: (i) completeness (for any $x, y \in D$, $x \leq_F y$ or $y \leq_F z$ holds true); (ii) transitivity (for $x, y, z \in D$, if $x \leq_F y$ and $y \leq_F z$, then $x \leq_F z$); (iii) reflexivity (for any $x \in D$, $x \leq_F x$); and (iv) anti-symmetry (for different $x, y \in D$, $x \leq_F y$ and $y \leq_F x$ cannot hold true at the same time). In short, conditions (i) – (iv) are not equivalent to assuming that the firm considered in this paper is rational for the research economist who asserts conditions that lead to his expected optimal possibility, as so phrased in the language of Mises [40].

3. MAIN RESULTS

This section consists of three parts, where Subsection 3.1 looks at how the well-known Hotelling's lemma can be generalized; Subsection 3.2 similarly explores possible generalization of Shepard's lemma; and Subsection 3.3 examines how and why no societal-wide preference order can realistically exists.

3.1. Profits and Optimal Production Correspondence

For the firm, its profit function $\pi^F: \mathbb{R}_+^\ell \to \mathbb{R}$ is defined as follows

$$\pi^F(p) =_F \max\nolimits_{y \in Y}^F p \cdot y, \tag{3.1}$$

where $p \in \mathbb{R}^\ell$ is a price system of commodities, if the maximum value exists according to the firm's system of values and beliefs. That is, π^F is not necessarily defined for some $p \in \mathbb{R}^\ell$. In the rest of this paper, when the expression $\pi^F(p)$ appears, what is implicitly assumed is the existence of the maximum above unless stated otherwise.

In equation (3.1), Firm F's specific system of values and beliefs determines the meaning of maximum and how the maximization is carried out, if a potential result exists[‡]. This symbolism perfectly reflects the most common and most relevant concept of a rational action defined in psychology as an action that is in line with the values and beliefs of the individual concerned [27]. Related to this problem of maximization, neoclassical economics has embraced such a convention that the objective of each firm is to maximize its profit (e.g., [15,16,18,58]). However, this convention is not generally true in real life (e.g., [24,26]). For example, an organization of important chief executives from the United States, chaired by JP Morgan Chase CEO Jamie Dimon, recently ceased to support the doctrine that businesses must maximize profits for shareholders above all else (https://opportunity.businessroundtable.org/ourcommitment/, accessed on January 30, 2021). "Americans deserve an economy that allows each person to succeed through hard work and creativity and to lead to a life of meaning and dignity" and "we commit to deliver value to all of them, for the future success of our companies, our communities, and our country," (https://s3.amazonaws.com/brt.org/BRT-StatementonthePurposeofaCorporationOctober2020.pdf, accessed on January 30, 2021), as so declared by the organization. Once again, the reason why many firms don't put profit maximization as the number one priority can be explained by using their natural endowments. It is because the conscience of the managers directs them to contribute more to their respective causes. This end surely supports the notion that how a firm behaves is dictated by its system of values and beliefs.

Proposition 1:

The firm's profit function π^F is partially defined on $\mathbb{R}^\ell_+$; and if the firm's system of values and beliefs is consistent with the conventional ordering of real numbers, then π^F is homogeneous of degree one on $domain(\pi^F)$. And in general, π^F is not homogeneous of degree one.

Proof. The first conclusion follows from the fact that there is no guarantee that $\pi^F(p) =_F \max^F_{y \in Y} p \cdot y$ exists for each price system $p \in \mathbb{R}^\ell_+$. To see the second conclusion, for any scalar $\lambda > 0$ and any $p \in \text{domain}(\pi^F)$, $\pi^F(\lambda p) =_F \lambda \max^F_{y \in Y} p \cdot y = \lambda \pi^F(p)$.

As for the third conclusion, it suffices for us to construct a counterexample to confirm the statement. To achieve this end, let us modify Example 1 slightly. To do this, let us borrow the modular function (or mod function for short) that is generalized from the domain of integers to the set of all real numbers by Forrest, Hafezalkotob et al. [10]. In particular, let $a \in \mathbb{R}$ be a positive number. Define a linear order relation $<_{mod(a)}$ on $\mathbb{R}$ as follows: For any x and $y \in \mathbb{R}$,

$$x <_{mod(a)} y \text{ if and only if } x\, mod(a) < y\, mod(a), \tag{3.2}$$

where the ordering $<$ is the conventional one defined on $\mathbb{R}$, $x\, mod(a)$ is the remainder of $x \div a$ and $y\, mod(a)$ the remainder of $y \div a$, such that $0 \le x\, mod(a) < a$ and $0 \le y\, mod(a) < a$. When all the involved numbers a, x and y are integers, this order relation $<_{mod(a)}$ degenerate into the one widely studied in number theory [6]. In other words, for any $r \in \mathbb{R}$,

‡ One needs to pay attention to the fact that the concepts of optimality and optimization, employed here and throughout the rest of this paper, are firm-specific instead of those defined by the economist who looks at the issue in hands and believe the outcome is what the firm desires.

satisfying $0 \leq r < a$, the following set of real numbers are classified into one equivalence class, denoted by r without causing confusion:

$$r \equiv \{x \in \mathbb{R}: \exists q \in \mathbb{Z}(x = aq + r)\}, \tag{3.3}$$

where $\mathbb{Z}$ stands for the set of all integers, that is, $\mathbb{Z} = \{\dots, -3, -2, -1, 0, +1, +2, +3, \dots\}$.

Example 2:

Assume that a specific production of the firm involves one unit of each of the commodity inputs X, X_1, X_2, Y, Z_1, Z_2, Z, where X_1 and X_2 can substitute for each other and so do Z_1 and Z_2, Figure 2. The arrows stand for the sequence these commodities are fed into the production line one after another, while the weights the relevant profits created by the production sequence from one node to the next. Now, the manager of the production wants to maximize the total profit.

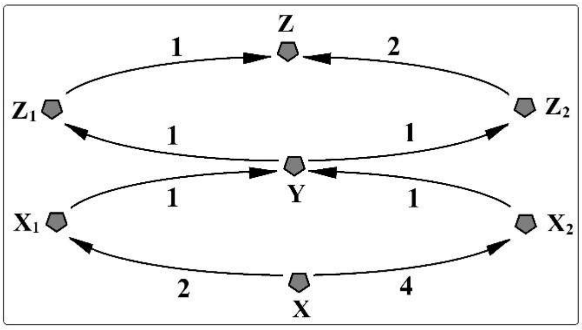

Figure 2. A specific production line with respective profits

Case 1: The firm orders the real-number weights in the same way as how real numbers are conventionally ordered. In this case, the four possible paths and their respective total weights are given as follows:

$$\begin{array}{ll} X \to X_1 \to Y \to Z_1 \to Z \text{ with weight 5;} & X \to X_1 \to Y \to Z_2 \to Z \text{ with weight 6;} \\ X \to X_2 \to Y \to Z_1 \to Z \text{ with weight 7;} & X \to X_2 \to Y \to Z_2 \to Z \text{ with weight 8.} \end{array} \tag{3.4}$$

Hence, the maximum total profit is equal to 8. To show that for any scalar $\lambda > 0$, $\pi^F(\lambda p) =_F \lambda \max_{y \in Y}^F p \cdot y = \lambda \pi^F(p)$, assume that all the commodities involved here are ordered as follows: $I = \{X, X_1, X_2, Y, Z_1, Z_2, Z\}$. Let the respective paths be

$$\begin{array}{ll} I_{1,1} = \{X, X_1, Y, Z_1, Z\}, & I_{1,2} = \{X, X_1, Y, Z_2, Z\}, \\ I_{2,1} = \{X, X_2, Y, Z_1, Z\}, & I_{2,2} = \{X, X_2, Y, Z_2, Z\}. \end{array} \tag{3.5}$$

Then the corresponding input and associated output vectors are given respectively by

$$^1y = (y_j)_{j \in I_{1,1}},\ ^2y = (y_j)_{j \in I_{1,2}},\ ^3y = (y_j)_{j \in I_{2,1}}, \text{ and } ^4y = (y_j)_{j \in I_{2,2}}, \tag{3.6}$$

and $^jZ_{AC}, j = 1, \dots, 4$. Now, each price system p can be written as follows:

$$p = \left((p_j)_{j \in I}, \left(^jZ_{AC}\right)_{j=1}^4\right). \tag{3.7}$$

And for any production y, there is k (= 1, …, 4) such that when all zero components are eliminated, we have

$$y^{in} = {}^ky \text{ and } y^{out} = {}^kZ_{AC}. \tag{3.8}$$

Therefore, for any price system p and any production y,

$$p \cdot y = p^{in} \cdot y^{in} + p^{out} \cdot y^{out}, \tag{3.9}$$

where p^{in} stands for the sub-vector of p of the prices of the sub-vector y^{in} of input commodities in y, and p^{out} the sub-vector of p of those of the sub-vector y^{out} of output commodities in y. Therefore, we have that for any scalar $\lambda > 0$,

$$\pi^F(\lambda p) =_F \max^F_{y\in Y}\left(\lambda p^{in}\cdot y^{in} + \lambda p^{out}\cdot y^{out}\right) = \lambda \max^F_{y\in Y} p\cdot y = \lambda\pi^F(p) = 8\lambda. \quad (3.10)$$

Case 2: The firm orders the real-number weights by referring to the mod4 function so that for any two real numbers x and y, $x < y$ if and only if $x(\text{mod}4) < y(\text{mod}4)$. In this case, the respective total profits of the four possible paths, as listed above, are given as follows: 5 (mod4) = 1, 6 (mod4) = 2, 7 (mod4) = 3 and 8 (mod4) = 0. Therefore, the maximum total profit is equal to 3. Now, let $\lambda = 3.2$ be the scalar that is multiplied to each of the individual local values. The corresponding total profits for the four paths are respective equal to 5 × 3.2 (mod4) = 0, 6 × 3.2 (mod4) = 3.2, 7 × 3.2 (mod4) = 2.4, and 8 × 3.2 (mod4) = 1.6. That is, we have (3.2 times 7 (mod4)) ≠ (3.2 × 6 (mod4)).

This end concludes the example and the proof of Proposition 1. QED

The optimal production correspondence of the firm [33] is the partial, set-valued function $\eta^F: \mathbb{R}^\ell_+ \to Y$ defined as follows: For any price system $p \in \mathbb{R}^\ell_+$,

$$\eta^F(p) = \{y \in Y: p\cdot y =_F \max^F_{y^q\in Y} p\cdot y^q\}, \quad (3.11)$$

assuming that there is a production $y \in Y$ such that $p\cdot y =_F \max^F_{y^q\in Y} p\cdot y^q$. Speaking differently, η^F maps each price system p to the subset $\eta^F(p) \subset Y$ of all profit-maximizing productions, if this subset is not empty.

In the rest of this section, assume that the order relation of real numbers, implied by the firm's system of values and beliefs, is the same as the conventional one.

Proposition 2 (Generalized Hotelling's lemma):

For a given price system $p = (p_1, p_2, \dots, p_\ell) \in \mathbb{R}^\ell$, if the firm's profit $\pi(p) = max_{y\in Y} p\cdot y$ exists, $\eta(p)$ is then a singleton in a neighborhood of p, if and only if $\pi(\cdot)$ is differentiable at p with respect to each p_h and

$$\frac{\partial\pi(p)}{\partial p_h} = y_h, for\ any\ y = (y_1, y_2, \dots, y_\ell) \in \eta(p), h = 1,2,\dots,\ell. \quad (3.12)$$

Proof. (⇒) Because $\eta(p)$ is a singleton in a neighborhood of p, the envelope theorem applies $\partial\pi(p)/\partial p_h = y_h$, for the element $y = (y_1, y_2, \dots, y_\ell) \in \eta(p), h = 1,2,\dots,\ell$.

(⇐) For any $y^1 = (y^1_1, y^1_2, \dots, y^1_\ell)$ and $y^2 = (y^2_1, y^2_2, \dots, y^2_\ell) \in \eta(p)$, we have $\pi(p) = p\cdot y^1 = p\cdot y^2$. So, equation (3.12) implies that

$$y^1_h = \frac{\partial\pi(p)}{\partial p_h} = y^2_h, h = 1,2,\dots,\ell. \quad (3.13)$$

Therefore, $y^1 = y^2$. That is, $\eta(p)$ is a singleton. QED

Define the following characteristic function $F: Y \to \mathbb{R}$:

$$F(y)\begin{cases} = 0, & \text{if } y \text{ is on the frontier of } Y \\ < 0, & \text{if } y \text{ is in the interior of } Y. \\ > 0, & \text{if } y \text{ is outside of } Y \end{cases} \quad (3.14)$$

With the help of this function, the following maximization problem $\max_y p\cdot y, \text{s.t.}\, y \in Y$ can be rewritten as $\max_y p\cdot y, \text{s.t.}\, F(y) \le 0$. The Lagrangian of this problem is

$$L = p\cdot y - \lambda F(y) \quad (3.15)$$

which implies the first-order conditions:

$$p_h = \lambda F_h(y^*), F(y^*) \leq 0, \text{for } y^* \in \eta(p), h = 1,2,\dots,\ell. \tag{3.16}$$

Proposition 3:

For any price system $p = (p_1, p_2, \dots, p_\ell) \in \mathbb{R}^\ell$ such that $p_h > 0$, for $h = 1,2, \dots \ell$, if $y(p) \in \eta(p)$ is continuously differentiable, then the matrix $D_p y(p) = D_p^2 \pi(p) = 0$.

Proof. First, let us compute $D_p y(p)$ as follows:

$$\begin{aligned} D_p y(p) &= \frac{\partial y(p)}{\partial p} \\ &= \left[\frac{\partial y_1(p)}{\partial p}, \frac{\partial y_2(p)}{\partial p}, \dots, \frac{\partial y_\ell(p)}{\partial p}\right] \\ &= \begin{bmatrix} \frac{\partial y_1}{\partial p_1} & \frac{\partial y_2}{\partial p_1} & \cdots & \frac{\partial y_\ell}{\partial p_1} \\ \frac{\partial y_1}{\partial p_2} & \frac{\partial y_2}{\partial p_2} & \cdots & \frac{\partial y_\ell}{\partial p_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_1}{\partial p_\ell} & \frac{\partial y_2}{\partial p_\ell} & \cdots & \frac{\partial y_\ell}{\partial p_\ell} \end{bmatrix}_{\ell \times \ell} \\ &= \left[\frac{\partial^2 \pi(p)}{\partial p_i \partial p_j}\right]_{\ell \times \ell} \qquad \text{from Proposition 2} \\ &= D_p^2 \pi(p) \end{aligned} \tag{3.17}$$

The general (t, h) cell of $D_p y(p)$, as given in equation (3.17), is equal to

$$\begin{aligned} \frac{\partial y_h}{\partial p_t} &= \left(\frac{\partial y_h}{\partial p_t} p_h\right) \frac{1}{p_h} && \text{since } p_h > 0, \text{ for } h = 1,2, \dots \ell \\ &= \left(\frac{\partial y_h}{\partial p_t} F_h(y)\right) \frac{\lambda}{p_h} && \text{from equation (3.16)} \\ &= \frac{\partial F(y)}{\partial p_t} \frac{\lambda}{p_h} \\ &= 0 \cdot \frac{\lambda}{p_h} = 0, && \text{from envelope theorem} \end{aligned} \tag{3.18}$$

where a different way to explain $\partial F(y)/\partial p_t = 0$ is that $y \in \eta(p)$, no matter what value p is equal to, $F(y) = 0$. Therefore, the derivative of $F(y)$ with respect to any variable will be zero. Therefore, $D_p y(p) = D_p^2 \pi(p) = 0$. QED

Proposition 3 stands for a major improvement of a theorem in the producer theory [33,38], where the matrix $D_p y(p) = D_p^2 \pi(p)$ is shown to be symmetric, positive semi-definite under the assumptions, which are not imposed here, that the firm produces a single product and that $\eta(p)$ is a singleton.

3.2. The Cost Minimization Problem & Conditional Factor Demands

Let $q = \left(q_{h_1^{out}}, q_{h_2^{out}}, \dots, q_{h_s^{out}}\right)$ denote the required quantities of production outputs, satisfying that $q_{h_j^{out}} > 0$, for any $j = 1, 2, \dots, s$, and $h_1^{out} < h_2^{out} < \cdots < h_s^{out}$. Then, the firm's cost minimization problem can be written as follows, assuming that the firm is a price taker: For a given price system $p \in \mathbb{R}_+^\ell$,

$$\begin{cases} \min^F_{y \in Y} p^{in} \cdot y^{in} \\ \text{s.t. } f(y^{in}) \geq q' \end{cases} \tag{3.19}$$

where $f(y^{in}) = y^{out}$ satisfies that $\{h_1^{out}, h_2^{out}, \ldots, h_s^{out}\}$ is generally a subset of the set of all commodity subscripts that appear in the components of y^{out}. Without loss of generality, we assume that these two sets are the same, because producing additional products beyond what are listed in q requires at least an increased amount of labor input.

Let $z = z(p, q)$ be a solution of the minimization problem in equation (3.19). This solution is known as a conditional factor demand [33], because of its dependence on the required production outputs q.

Let $Z = \{z: \exists y \in Y(z = y^{in} \text{ and } f(z) \geq q)\}$ and the optimal value of the objective function in equation (3.19), be

$$c^F(p, q) =_F \min^F_{z \in Z} p^{in} \cdot z, \text{for } p \in \mathbb{R}^{\ell}_{+}, \tag{3.20}$$

which gives the minimum cost at which the required outputs q can be produced. Because the implementation of any non-zero production plan has to use certain amounts of some inputs, such as labor, work space, etc., and produce certain outputs, such as waste, if nothing useful, we have

$$Z \subseteq \{y^{in}: y \in Y\} \subseteq \bigcup_{t=1}^{\ell-1} \mathbb{R}^t_{-}. \tag{3.21}$$

And, for given $p \in \mathbb{R}^{\ell}_{+}$ and $q \in \mathbb{R}^s$, define the following set-valued function

$$\xi^F(p, q) = \{z \in Z: p^{in} \cdot z =_F \min^F_{z^q \in Z} p^{in} \cdot z^q\}, \tag{3.22}$$

known as the set of conditional factor demands (that is conditional on the desired level of outputs).

Proposition 4:

For any production $y \in Y$, if $y^{in} \in \mathbb{R}^t_{-}$, for some $t = 1,2, \ldots, \ell - 1$, and $y^{in} \notin Z \cap \mathbb{R}^t_{-}$ implies that there is a hyperplane L in $\mathbb{R}^t$ that separates y^{in} and $Z \cap \mathbb{R}^t_{-}$ in terms of the firm's order relation $\leq_F$ of real numbers so that $y^{in} \notin L$, then

$$Z = \bigcup_{t=1}^{\ell-1} \left\{z \in \mathbb{R}^t_{-}: \forall p \in \mathbb{R}^{\ell} \left(p^{in} \cdot z \geq_F c^F(p^{in}, q)\right)\right\}. \tag{3.23}$$

Proof. From equation (3.21), it follows that

$$Z = \bigcup_{t=1}^{\ell-1} Z \cap \mathbb{R}^t_{-}. \tag{3.24}$$

So, to show equation (3.23), it suffices to demonstrate that for each $t = 1,2, \ldots, \ell - 1$,

$$Z \cap \mathbb{R}^t_{-} = \left\{z \in \mathbb{R}^t_{-}: \forall p \in \mathbb{R}^{\ell} \left(p^{in} \cdot z \geq_F c^F(p^{in}, q)\right)\right\}, \tag{3.25}$$

due to the fact that terms in the union in equation (3.23) are pairwise disjoint.

Let the set defined by equation (3.25) be $\tilde{Z}$. It suffices to demonstrates that $Z \cap \mathbb{R}^t_{-} \subseteq \tilde{Z}$ and $\tilde{Z} \subseteq Z \cap \mathbb{R}^t_{-}$. The former follows directly from the definition of $c^F(p, q)$ in equation (3.20). Next, let us examine that $\tilde{Z} \subseteq Z \cap \mathbb{R}^t_{-}$.

To this end, let us pick an arbitrary $y \in Y$ such that $y^{in} \notin Z \cap \mathbb{R}^t_{-}$. Then, the if-condition guarantees the existence of a hyperplane L in $\mathbb{R}^t$ that separates y^{in} and $Z \cap \mathbb{R}^t_{-}$. Assume that the equation of the plane is $p^{in} \cdot x = \beta$, for $x \in \mathbb{R}^t$, some non-zero $p \in \mathbb{R}^{\ell}$ and a scalar $\beta \in$

$\mathbb{R}$, satisfying that for any $z' \in Z \cap \mathbb{R}_-^t$, $p^{in} \cdot z' \geq_F \beta$ and $\beta >_F p^{in} \cdot y^{in}$. Hence, taking minimum or infimum produces

$$p^{in} \cdot y^{in} <_F \beta \leq_F \min_{z \in Z \cap \mathbb{R}_-^t}^F (p^{in} \cdot z) \text{ or } \inf_{z \in Z \cap \mathbb{R}_-^t}^F (p^{in} \cdot z). \tag{3.26}$$

Once again, due to differences in dimensionality between $\mathbb{R}_-^{t_1}$ and $\mathbb{R}_-^{t_2}$, for $t_1, t_2 = 1,2, \ldots, \ell - 1$, $t_1 \neq t_2$, it can be seen that $\min_{z \in Z \cap \mathbb{R}_-^t}^F (p^{in} \cdot z) = c^F(p^{in}, q)$, because $p^{in} \in \mathbb{R}_+^t$ the corresponding z in the operation $p^{in} \cdot z$ has to be from $\mathbb{R}_-^t$. Therefore, equation (3.26) is the same as

$$p^{in} \cdot y^{in} <_F \beta \leq_F \min_{z \in Z}^F (p^{in} \cdot z) \text{ or } \inf_{z \in Z}^F (p^{in} \cdot z). \tag{3.27}$$

That is, what is shown is that $y^{in} \notin Z \cap \mathbb{R}_-^t \to y^{in} \notin \tilde{Z}$, which means $\tilde{Z} \subseteq Z \cap \mathbb{R}_-^t$. Therefore, equation (3.25), and then equation (3.23) follows from equation (3.24). QED

Proposition 5 (Generalized Shepard's lemma):

For a given price system $p \in \mathbb{R}_+^\ell$, assume that the firm's minimization of production cost is based on the conventional Lagrangian approach. Then, $\xi^F(p,q) = \{z(p,q)\}$ is a singleton in a neighborhood of p^{in}, if and only if $c(\cdot, q)$ is differentiable at p^{in} with respect to each $p_{h_j^{in}}$ such that

$$\frac{\partial c(p,q)}{\partial p_{h_j^{in}}} = z_{h_j^{in}}(p,q), \text{for } j = 1,2, \ldots, t, \tag{3.28}$$

assuming that $p^{in} = \left(p_{h_1^{in}}, p_{h_2^{in}}, \ldots, p_{h_t^{in}}\right) \in \mathbb{R}_+^t$, for some $t = 1,2, \ldots, \ell - 1$.

Proof. ($\Rightarrow$) Because $\xi^F(p,q) = \{z(p,q)\}$ is a singleton in a neighborhood of p, the assumption on how the firm minimizes its cost means that the envelope theorem applies so that we have

$$\frac{\partial c(p,q)}{\partial p_{h_j^{in}}} = \frac{\partial}{\partial p_{h_j^{in}}}\left[p^{in} \cdot z(p,q)\right] = \frac{\partial p}{\partial p_{h_j^{in}}} \cdot z(p,q) = z_{h_j^{in}}(p,q), \tag{3.29}$$

for $j = 1,2, \ldots, t$.

($\Leftarrow$) For any $z^1, z^2 \in \xi^F(p,q)$, by filling unmatching components with zeros if needed, we can assume that $z^1 = \left(z_{h_1^{in}}^1, z_{h_2^{in}}^1, \ldots, z_{h_t^{in}}^1\right)$ and $z^2 = \left(z_{h_1^{in}}^2, z_{h_2^{in}}^2, \ldots, z_{h_t^{in}}^2\right)$. Hence, we have $c(p,q) = p \cdot z^1 = p \cdot z^2$. So, equation (3.28) implies that

$$z_{h_j^{in}}^1 = \frac{\partial c(p,q)}{\partial p_{h_j^{in}}} = z_{h_j^{in}}^2, j = 1,2, \ldots, t. \tag{3.30}$$

Therefore, $z^1 = z^2$. That is, $\xi^F(p,q)$ is a singleton. QED

For the rest of this subsection, assume that the firm's order relation $\leq_F$ of real numbers is the same as the conventional one $\leq$.

Proposition 6:

For a given price system $p \in \mathbb{R}_+^\ell$, if $\xi^F(p,q) = \{z(p,q)\}$ is a singleton in a neighborhood of p and $z(p,q)$ is continuously differentiable with respect to $p^{in} = \left(p_{h_1^{in}}, p_{h_2^{in}}, \ldots, p_{h_t^{in}}\right) \in \mathbb{R}_+^t$, then the matrix $D_p z(p,q) = D_p^2 c(p,q) = 0_{t \times t}$.

Proof. First, let us compute $D_p z(p,q)$ as follows:

$$\begin{aligned} D_p z(p,q) &= \frac{\partial z(p,q)}{\partial p} \\ &= \left[\frac{\partial z_{h_1^{in}}(p)}{\partial p}, \frac{\partial z_{h_2^{in}}(p)}{\partial p}, \dots, \frac{\partial z_{h_t^{in}}(p)}{\partial p}\right] \\ &= \left[\frac{\partial^2 c(p,q)}{\partial p \partial p_{h_1^{in}}}, \frac{\partial^2 c(p,q)}{\partial p \partial p_{h_2^{in}}}, \dots, \frac{\partial^2 c(p,q)}{\partial p \partial p_{h_t^{in}}}\right] \quad \text{from Proposition 5} \\ &= \left[\frac{\partial^2 c(p,q)}{\partial p_{h_i^{in}} \partial p_{h_j^{in}}}\right]_{t\times t} \\ &= D_p^2 c(p,q) \end{aligned} \tag{3.31}$$

That is $D_p z(p,q) = D_p^2 c(p,q)$. To show $D_p z(p,q) = 0$, define

$$F(z)\begin{cases} = 0, & \text{if } z \text{ is on the frontier of } Z \\ < 0, & \text{if } z \text{ is in the interior of } Z \,. \\ > 0, & \text{if } z \text{ is outside of } Z \end{cases} \tag{3.32}$$

Then, the following minimization problem $\min_z p \cdot z, \text{s.t.}\, z \in Z$ can be rewritten as $\min_z p \cdot z, \text{s.t.}\, F(z) \leq 0$. The Lagrangian of this problem is

$$L = p \cdot z - \lambda F(z) \tag{3.33}$$

which implies the first-order conditions:

$$p_{h_j^{in}} = \lambda F_{h_j^{in}}(z^*), F(x^*) \leq 0, \text{for } j = 1,2,\dots,t, \tag{3.34}$$

where $z^* \in \xi^F(p,q)$. From equation (3.31), it follows that for $z(p,q) = \left(z_{h_1^{in}}, z_{h_2^{in}}, \dots, z_{h_t^{in}}\right) \in \xi^F(p,q)$,

$$\begin{aligned} D_p z(p,q) &= \left[\frac{\partial z_{h_j^{in}}}{\partial p_{h_i^{in}}}\right]_{t\times t} = \left[\frac{\partial z_{h_j^{in}}}{\partial p_{h_i^{in}}} \cdot \frac{p_{h_j^{in}}}{p_{h_j^{in}}}\right]_{t\times t} \\ &= \left[\frac{\lambda}{p_{h_j^{in}}} \cdot \frac{\partial z_{h_j^{in}}}{\partial p_{h_i^{in}}} \cdot F_{h_j^{in}}(z)\right]_{t\times t} \quad \text{from equation (3.34)} \\ &= \left[\frac{\lambda}{p_{h_j^{in}}} \frac{\partial F(z)}{\partial p_{h_i^{in}}}\right]_{t\times t} \\ &= \left[\frac{\lambda}{p_{h_j^{in}}} \cdot 0\right]_{t\times t} = 0_{t\times t} \quad \text{from equation (3.33) or the envelope theorem} \end{aligned} \tag{3.35}$$

where $\partial F(z)/\partial p_{h_i^{in}} = 0$ also comes from the fact that $z \in \xi^F(p,q)$ so that $F(z) = 0$ no matter how p changes. Therefore, the derivative of $F(z)$ with respect to $p_{h_i^{in}}$ is zero. Therefore, $D_p z(p,q) = 0_{t\times t}$. QED

This result carries a well-known conclusion in the producer theory much forward. In particular, the known result claims [33,38] that the matrix $D_p z(p,q) = D_p^2 c(p,q)$ is symmetric, negative semidefinite and $D_p z(p,q) p^{in} = 0$.

3.3. The Non-Existence of Invisible Hand in Any Large-Scale Economy

Let $p = (p_1, p_2, \dots, p_\ell) \in \mathbb{R}_+^\ell$ be a price system. Then for any production $y_j \in Y_j$, the profit π_j of producer j is $p \cdot y_j$ and the total profit π of all producers is $\pi = \sum_{j=1}^n \pi_j = \sum_{j=1}^n p \cdot y_j = p \cdot \sum_{j=1}^n y_j = p \cdot y$, where assumed is $y = \sum_{j=1}^n y_j$, which is known as a total production. In this setup, each producer is treated as a price taker, while trying to maximize the realization of its mission by choosing an optimal production y_j, where both the definition of maximum and method of maximization are defined by the producer j's system of values and beliefs. Considering that each price and every commodity are time and location specific, this setup requires each producer to choose its production so that its inputs and outputs are optimally distributed over both time and space. Such an optimal production is referred to as an equilibrium production of producer j with respect to the price system p [7].

Let $\leq_j$ be the order relation of real numbers producer j employs in its determination of optimal decisions and $\leq$ the conventional order relation between real numbers. For the economy of n producers, the collective order relation $\leq_E$ of real numbers is defined as follows, assuming that the society is democratic: For any a and $b \in \mathbb{R}$,

$$a \leq_E b \text{ if and only if } a \leq_j b, \text{for } j = 1,2, \dots, n. \tag{3.36}$$

Proposition 7:

Assume that all involved order relations of real numbers are consistent with the conventional one. Then, for a given price system $p = (p_1, p_2, \dots, p_\ell) \in \mathbb{R}_+^\ell$, *and a total production* $y = y_1 + y_2 + \dots + y_n \in Y_1 + Y_2 + \dots + Y_n$, *where* $y_j \in Y_j$, *for j = 1, 2, ..., n,* $p \cdot y =_E max_{y^q \in Y}^E p \cdot y^q$ *if and only if* $p \cdot y_j =_j max_{y_j^q \in Y_j}^j p \cdot y_j^q$ *, for* $j = 1, 2, \dots, n$.

Proof. (⇒) Assume that $p \cdot y = p \cdot y_1 + p \cdot y_2 + \dots + p \cdot y_n =_E \max_{y^q \in Y}^E p \cdot y^q$, where $y_j \in Y_j$, for j = 1, 2, …, n, but there is a producer j, $1 \leq j \leq n$, such that $p \cdot y_j <_j \max_{y_j^q \in Y_j}^j p \cdot y_j^q$. Hence, we have

$$\begin{aligned} \max_{y^q \in Y}^E p \cdot y^q &=_E p \cdot y_1 + p \cdot y_2 + \dots + p \cdot y_n \\ &<_E p \cdot y_1 + \dots + p \cdot y_{j-1} + \max_{y_j^q \in Y_j}^j p \cdot y_j^q + p \cdot y_{j+1} + \dots + p \cdot y_n. \end{aligned} \tag{3.37}$$

That contradicts the meaning of $p \cdot y =_E \max_{y^q \in Y}^E p \cdot y^q$.

(⇐) Assume that $p \cdot y_j =_j \max_{y_j^q \in Y_j}^j p \cdot y_j^q$, for j = 1, 2, …, n, but $p \cdot y =_E p \cdot y_1 + p \cdot y_2 + \dots + p \cdot y_n <_E \max_{y^q \in Y}^E p \cdot y^q$. Hence, there is $y^* = y_1^* + y_2^* + \dots + y_n^* \in Y$, where $y_j^* \in Y_j$, for j = 1, 2, …, n, such that $p \cdot y_1 + p \cdot y_2 + \dots + p \cdot y_n <_E p \cdot y_1^* + p \cdot y_2^* + \dots + p \cdot y_n^*$. So, there is at least one such j, $1 \leq j \leq n$, that

$$p \cdot y_j =_j \max_{y_j^q \in Y_j}^j p \cdot y_j^q <_j p \cdot y_j^*. \tag{3.38}$$

That is a contradiction. Therefore, the assumption $p \cdot y <_E \max_{y^q \in Y}^E p \cdot y^q$ is false. QED

What this proposition says is that if all involved order relations of real numbers are consistent with the conventional one, then one has

$$\max_{y^q \in Y}^E p \cdot y^q =_E \sum_{j=1}^n \max_{y_j^q \in Y_j}^j p \cdot y_j^q. \tag{3.39}$$

This equation (3.39) provides a condition under which Adam Smith's invisible hand exists, for more details, see the discussion right after Example 3 below. However, the following example shows that this equation is not generally true when some of the individual order relations of real numbers are allowed to be different from the conventional one.

Example 3:

Assume that an economy has two producers, named 1 and 2. Both of them order real numbers with the order relation $\leq_{mod(4)}$. Without loss of generality, assume that one unit of each involved commodity is either needed for production or produced as market offer.

The particular production of producer 1 is shown in Figure 3; it involves one unit of each of the commodity inputs A, $A_?$, B, $C_?$, C, where $A_?$ can be either A_1 and A_2, but not both, and similarly, $C_?$ can be either C_1 and C_2, but not both. That is, commodities A_1 and A_2 can substitute for each other and the same holds true for commodities C_1 and C_2. In Figure 3, the arrows stand for the sequence the corresponding commodities are fed into the production line one after another, while the weights the relevant profits created by the production sequence from one node to the next.

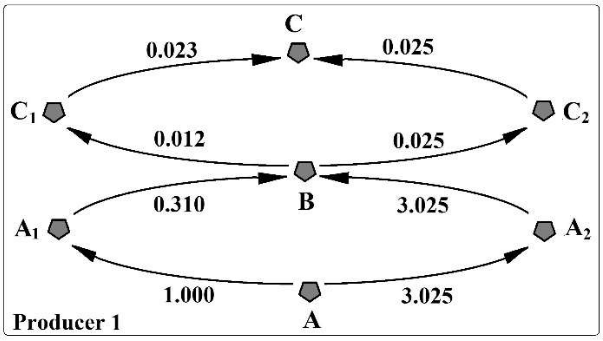

Figure 3. Productions of producer 1

Let us label the individual productions of producer 1 as follows:

$$I_{11}: A \to A_1 \to B \to C_1 \to C; \qquad I_{12}: A \to A_1 \to B \to C_2 \to C;$$
$$I_{21}: A \to A_2 \to B \to C_1 \to C; \qquad I_{22}: A \to A_2 \to B \to C_2 \to C. \tag{3.40}$$

So, the set of production possibilities of producer 1 is $Y_1 = \{I_{11}, I_{12}, I_{21}, I_{22}\}$ and corresponding profits are 1.345, 1.36, 6.085 mod(4) = 2.085, and 6.1 mod(4) = 2.1, respectively. That is, we have

$$\max^1_{y \in Y_1} p \cdot y =_1 2.1. \tag{3.41}$$

For producer 2, its production is shown in Figure 4; it involves one unit of each of the commodity inputs U, $U_?$, V, $W_?$, W, where $U_?$ can be either U_1 and U_2, but not both, and similarly, $W_?$ can be either W_1 and W_2, but not both. That is, commodities U_1 and U_2 can substitute for each other and the same holds true for commodities W_1 and W_2. In Figure 4, the arrows stand for the sequence the corresponding commodities are fed into the production line one after another, while the weights the relevant profits created by the production sequence from one node to the next.

Let us label the individual productions of producer 2 as follows:

$$J_{11}: U \to U_1 \to V \to W_1 \to W; \qquad J_{21}: U \to U_2 \to V \to W_1 \to W;$$
$$J_{12}: U \to U_1 \to V \to W_2 \to W; \qquad J_{22}: U \to U_2 \to V \to W_2 \to W. \tag{3.42}$$

So, the set of production possibilities of producer 2 is $Y_1 = \{J_{11}, J_{12}, J_{21}, J_{22}\}$ and corresponding profits are 6.1 mod(4) = 2.1, 4.135 mod(4) = 0.135, 3.31, and 1.315, respectively. That is, we have

$$\max^2_{y \in Y_2} p \cdot y =_2 3.31. \tag{3.43}$$

Therefore, from equations (3.41) and (3.43), we have

$$\max^1_{y \in Y_1} p \cdot y + \max^2_{y \in Y_2} p \cdot y =_{mod(4)} 2.1 + 3.31\, mod(4) =_{mod(4)} 1.41. \tag{3.44}$$

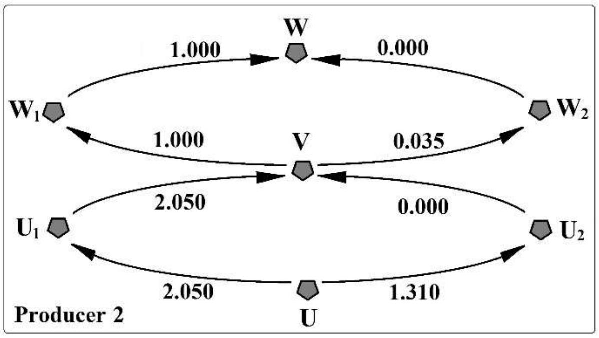

Figure 4. Productions of producer 2

To compute $\max^E_{y \in Y} p \cdot y$, we first have $Y = \{y_1 + y_2 : y_1 \in Y_1, y_2 \in Y_2\}$, where the order relation of real numbers of the two-producer economy is equal to $\leq_E = \leq_{mod(4)}$. That is, elements in Y are given in the form of $I_{ij} + J_{kl}$, for $i, j, k, l = 1,2$. The computational results of $p \cdot y = p \cdot y_1 + p \cdot y_2$ mod(4) are shown in Table 3.1.

Table 3.1. Computing $\max^E_{y \in Y} p \cdot y$

P2 / P1	2.1	0.135	3.31	1.345
1.345	3.445	1.48	0.655	2.69
1.36	3.46	1.495	0.67	2.705
2.085	0.185	2.22	1.395	3.43
2.1	0.2	2.235	1.41	3.445

Note: P1 = producer 1; p2 = producer 2

From Table 3.1, we have $\max^E_{y \in Y} p \cdot y = 3.46$. Combining this end with equation (3.44) leads to the conclusion that

$$\max^E_{y \in Y} p \cdot y >_E \sum_{j=1}^{n} \max^j_{y_j^q \in Y_j} p \cdot y_j^q. \tag{3.45}$$

where the order relation $\leq_E$ is given by $\leq_{mod(4)}$.

In summary, it has been shown that equation (3.39) does not hold true in general in terms of systems of values and beliefs. QED

A more general situation for equation (3.36) and in turn equation (3.39) not to hold is that in general the economy does not have any order relation $\leq_E$ of real numbers. For instance, assume that a small economy contains only two producers, named 1 and 2, such that producer 1's order relation of real numbers is $\leq_{mod(3)}$, while that of producer 2 is $\leq_{mod(4)}$, then 2 and 3.1 cannot be ordered in the economy, because these producers have inconsistent order relations: $3.1 \leq_{mod(3)} 2$ *and* $2 \leq_{mod(4)} 3.1$, so that equation (3.36) cannot be applied to define any $\leq_E$ that is consistent with each of the order relations of the individual producers.

This end is very important in terms of the "invisible hand," as introduced by Adam Smith in 1759 in his work *The Theory of Moral Sentiments* in Part IV and Chapter 1. In this work, the concept describes the unintended greater social benefits and public good brought about by individuals acting in their own self-interests [48]. Here, the key is the word "greater". What does it mean? According to the discussion above, the order relation $\leq_E$ of the economy might not exist in general, as long as there are producers whose order relations of real numbers are inconsistent with each other. That is, in general, within any economy in real life there is not a universally accepted way to tell what is meant by "greater social benefits and public good", because real numbers in such a large-scale system are not ordered consistently in the individual systems of values and beliefs of the incumbent producers. In other words, what is obtained above analytically confirms what Joseph E. Stiglitz [1] believes - the invisible hand is often not there, as he claimed based on Greenwald and Stiglitz [17].

4. SOME FINAL WORDS

There have been very loud and meaningful calls from both theorists and practitioners to reconstruct the prevalent business and economic theories. The reason underlying the calls is that applications of these theories don't generally provide timely guidance for practical purposes, for related discussions, see, for example, [12,28,41], and the comments by Paul Krugman and Paul De Grauwe, given in the start of this presentation. To positively answer such calls and to address the question posted in the introduction section, this paper represents one step towards this end by examining how the widely accepted producer theory [33,38] can be made more realistic by basing logical reasoning on the four natural endowments of a firm [10,13]: self-awareness, imagination, conscience and free will.

Other than continuing the tradition, the reason why this work chooses set theory as the tool of analysis and reasoning is because this tool can readily bring systems methodology [34] into rigorous studies of economic theories. Such a convenience is very significant, because studies of business and economies tend to involve organizations, their interactions and evolutions, while the focus of systems science is on the creation of tools for such studies. For relevant studies that confirm this point, see, for example, [12,45].

As for why we choose natural endowments as the basis of our reasoning and analysis is because, systemically speaking, they represent the most fundamental elements underlying human intellectual activities in general and decision-making in particular [35]. And we like to reconstruct relevant economic and business theories in a similar fashion as theories of mathematics, where every conclusion is derived from a small set of basic postulates of clear validity. One example of such a mathematical theory is the Euclidean plane geometry, which is constructed on the basis of five axioms. By doing so, we expect that many endless debates that occurred in the history of economic studies can be possibly ended fruitfully with more or less definite verdicts.

Because of the novelty of choosing natural endowments as our starting points, this paper is able to examine a series of 7 most fundamental propositions of the producer theory. At the same time, it convincingly demonstrates the existence of firm-specific order relations of real numbers, reflecting differences in the decision criteria of priority from one firm to another, and consequently the existence of firm-specific methods of optimization. For related discussions, see, for example, Hammerton [19], Van Fleet [54] and Yang and Andersson [59]. This end represents a major contribution this work makes to the literature beyond checking which known results of the producer theory hold true generally or only under specific conditions.

In terms of the literature, several schools of economic thoughts also attempted similarly to either reconstruct or revisit the prevalent economic theory in general or traditional production theory in particular from different angles or starting points. For example, Richard Nelson and Sidney G. Winter contributed to the development of the whole evolutionary school by reviving relevant discussions in the 1980s. They highlighted that the endogenous and permanent change in the environment, "largely driven by innovation, is a central characteristic of modern capital economies" [42, p. 3]. Based on Piero Sraffa's [51] interpretation of David Ricardo, the neo-Recardian school emerged. Scholars in this school argue, among other topic fronts, that natural growth is primarily demand-driven because growth in the labor force as well as in labor productivity both respond to the pressure of demand, both domestic and foreign [8,32]. The post-Keynesian economics attempts to rebuild economic theory in the light of Keynes' ideas and insights with its theoretical foundation placed on the principle of effective demand, that demand matters in the long as well as the short run so that a competitive market economy has no natural or automatic tendency towards full employment [21]. Recently, Shaikh [49] demonstrated by using statistics and mathematical modeling that he could derive most of the key propositions of economics without employing such basic assumptions as rationality, optimization, perfect competition, perfect information, representative agents or so-called rational expectations.

In comparison, it can be seen that no matter which school one is looking at, its highlighted starting points should be theoretically confirmed instead of being claimed or believed to be. For instance, what the whole evolutionary school believes to be central to modern capital economies should be established by using a confirmative means. To this end, the approach adopted in this paper can fulfil this need. In particular, by using natural endowments, it has been shown that the never-ending need for innovation in modern capital economy is relentlessly driven by the forever changing consumer needs for better lives; that in turn increasingly intensifies competitions among the suppliers in the marketplace [12, Chapter 3, 13, Proposition 9]. In other words, whatever conclusions derived in the so-called evolutionary economics can be seen as rooted in the concept of natural endowments. As for Shaikh's [49] work, similar scenarios appear throughout, because many of his data-based claims are subject to different interpretations. For instance, in his critical review, Patomäki [44] was able to draw a conclusion inconsistent to the one Shaikh claimed. In other words, the approach adopted in this research can be expected to provide a new foundation for economic and business studies, because it problematizes the commonly used methodology, while reconstructs the ontology of related theories at a deeper level.

As for topics for possible future research, many aspects of this paper call for follow-ups. For example, this paper explores two different order relations of real numbers – the conventional one and the one defined by a modular function. However, managers and entrepreneurs in real life surely employ other means to prioritize the alternatives available for their decision making. So, to make this work more compatible for real-life applications, it will be desirable for scholars to explore how the results developed in this paper remain reliable for different criteria of priority or different orderings of real numbers. Specifically, Propositions 2, 3, 5 and 6, most likely do not hold true for systems of values and beliefs that define order relations of real numbers through the modular function. So, with such an order relation, how will these results look like? Or, in what forms will these results appear?

Other two important open questions for future studies are about (i) how generally one can specify the ordering of real numbers or criteria of priority when a particular system of values and beliefs is given; and (ii) how to systemically and strategically investigate the interactions of heterogeneous firms with their individually different systems of values and beliefs.

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