Adv Syst Sci Appl 2022; **03**:135–146
Published online at https://ijassa.ipu.ru.

Three-Currency Deposit Diversification: Savage's Principle Approach

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Abstract: The problem of optimal multi-currency deposit diversification with uncertain future exchange rates is studied as the problem of the minimization of the lost profit. It is assumed that only the ranges of these uncertain parameters are known. The Savage minimax regret conception is used to minimize the lost profit (risk by Savage) caused by uncertainty. The risk function and the function of the guaranteed risk are calculated in an explicit form. After that, the problem is reduced to finding the point of the minimum of a piecewise linear function under simple linear constraints. Explicit formulas for nine "representative" point-candidates for the optimal solution are found. The final choice is made by direct comparison of the values of the Savage criterion at these points.

Keywords: deposit diversification, uncertainty, risk by Savage, guaranteed on risk solution

INTRODUCTION

The paper deals with finding the guaranteed risk solution (the Savage minimax regret solution) in the deposit diversification problem with three types of currencies. An explicit form of the optimal solution is obtained, and simple and clear recommendations for Decision Makers are presented.

In general, Decision Makers (DM) can be classified into three categories: risk averse, risk takers, and neutral. Risk averse avoid any risk and try to maximize the guaranteed income. Risk takers consider the risk only and seek to minimize it. Neutral DMs try to consider simultaneously income and risk.

Very often, the main cause of risk is the incompleteness of information when making decisions. If stochastic characteristics of uncertain parameters are known, the optimization problem is usually reduced to a deterministic one with average values of optimized indexes. A more complex case of information incompleteness is that the DM knows neither the values of uncertain parameters nor or their stochastic characteristics, but only the ranges. This very case will be investigated below in the framework of a problem of optimality with respect to guaranteed risk deposit diversification. There exist many definitions of risk. In the current study, we understand risk in terms of Savage [4] (risk by Savage or regret). It may be interpreted as the loss of the income (regret) due to the lack of knowledge of the uncertain parameters.

We proceed from the risk taker's viewpoint and use the best guaranteed result approach to get an explicit form of the optimal solution (the Savage minimax regret solution) in a three-currency diversification problem.

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Other possible points of view (risk averse and neutral) were studied earlier in [9]. Particularly, the best guaranteed income solution for the problem under consideration with n types of currencies was obtained in an explicit form. A neutral DM, who takes into account both the outcomes and the risks, seeking to increase the value of the outcome and to reduce the value of the risk, has to take into account at the same time that any possible values of the uncertain parameters can occur. The correspondent bicriteria problem under the uncertainty was studied in [9], where the explicit form of the optimal solution in the diversification problem with two currencies was obtained and simple rules for DM were formulated. Various aspects of multiple-criteria optimization under uncertainty were investigated both in the static and dynamic cases in [1] – [3], [6] – [8] The Savage minimax regret solution in a two-currency problem was obtained in [9]. In this paper we investigate the same problem for three currency case.

1. FORMULATION OF THE PROBLEM

We assume that the optimal structure of the deposit diversification for any amount of money (in rubles) is completely determined by the optimal allocation of one ruble. At the beginning of the given time interval (here, a year) the DM distributes one ruble among three deposits, for definiteness, in rubles, dollars and euros. DM aims to obtain the highest possible value (in terms of rubles) at the end of the deposit period.

So, let K_d and K_e be the exchange rates of dollar and euro against ruble at the beginning of the year, and $1 - x_d - x_e$, x_d, x_e be the sizes of ruble, dollar and euro deposits respectively (in ruble terms). Interest rates of all types of the deposits r, d_d, d_e are assumed to be known. However, DM does not know the exact exchange rates of dollars y_d and euro y_e at the end of the deposit period and there are no available statistical characteristics concerning their possible values. Only the ranges of these uncertain parameters are known:

$$y_d \in [a_d, b_d], \;\; y_e \in [a_e, b_e] \,.$$

The consolidated result (income) at the end of the year after conversation to rubles depends both on a plan of diversification $x = (1 - x_d - x_e, x_d, x_e)$ and the exchange rates at the end of the period – uncertainties

$$y = (y_d, y_e) \in Y = [a_d, b_d] \times [a_e, b_e] \,. \tag{1.1}$$

This result is the sum of the future values of the different components of the deposits after back conversion into rubles. Therefore, it can be presented in the following form:

$$f(x, y) = (1 + r)\,(1 - x_d - x_e) + \xi_d x_d y_d + \xi_e x_e y_e, \tag{1.2}$$

where

$$\xi_d = \frac{1 + d_d}{K_d}, \quad \xi_e = \frac{1 + d_e}{K_e}. \tag{1.3}$$

At a meaningful level, the task of DM is the design of the optimal strategy

$$x = (x_d, x_e) \in X = \{x_d + x_e \leq 1, \; x_d \geq 0, \; x_e \geq 0\} \tag{1.4}$$

in order to achieve the greatest result $f\,(x, y)$. However, DM should take into account the possibility of realizing any values of uncertainty $y \in Y$.

Thus, the mathematical model of the problem of diversification is represented by the ordered triple

$$\Gamma = \langle X, Y, f(x, y) \rangle,$$

where $f(x, y)$ is the utility function of a depositor (DM) defined in (1.2), X is the set of DM's strategies defined in (1.4), and Y is the set of uncertainties defined in (1.1).

The problem Γ is a single-criterion decision-making problem under uncertainty: to maximize a linear function on x with uncertain coefficients on the polyhedron X taking into account the range Y of the uncertain factors. The presence of uncertainty leads to the concept of risk as the possibility of deviation of realistic results from the desired or expected values.

2. THE SAVAGE MINIMAX REGRET PRINCIPLE

Let $f(x, y)$ be the objective function (the income), X be the set of DM strategies, and Y be the set of uncertainties. Then $\max_{z\in X} f(z, y)$ is the best result (income), if uncertainty y occurs. Nevertheless, DM does not know which value of the uncertain parameter y will realize. The difference

$$\Phi(x, y) = \max_{z\in X} f(z, y) - f(x, y) \tag{2.5}$$

is called the Savage risk function (regret). It presents the loss due to non-acquaintance – the difference between the best result, obtained with known uncertainty, and the real result with any strategy x. This risk depends both on the strategy x and the uncertainty y. Trying to minimize the risk, DM may use the concept of the best guaranteed result (Wald's principle; see [5]). This leads to the following definition (Savage's principle of the minimax regret; [4]).

Definition 2.1:
A strategy $x^r \in X$ is called a guaranteed risk solution (GRS) of the problem Γ, if

$$\Phi^r = \max_{y\in Y} \Phi(x^r, y) = \min_{x\in X} \max_{y\in Y} \Phi(x, y), \tag{2.6}$$

where the Savage risk function $\Phi(x, y)$ is defined in (2.5).

Note some properties of the GRS:

- Due to (2.5), the risk function is nonnegative (the best risk is zero risk):

$$\Phi(x, y) \geq 0 \quad \forall x \in X,\ \forall y \in Y.$$

- If the function $f(x, y)$ is continuous and X, Y are compacts, then GRS exists.
- For every strategy $x \in X$, the inner maximum operation in (2.6) determines a guarantee on the risk

$$\Phi\left[x\right] = \max_{y\in Y} \Phi\left(x, y\right) \geq \Phi(x, y) \ \ \forall y \in Y.$$

  The outer minimum operation in (2.6)

$$\min_{x\in X} \Phi[x] = \Phi[x^r] = \Phi^r$$

  chooses the best (the least) guarantee, for

$$\Phi^r \leq \Phi[x],\ \forall x \in X, \ \text{ and } \ \Phi^r = \Phi[x^r] \geq \Phi(x, y),\ y \in Y.$$

Therefore, DM seeks to reduce his risk by choosing the strategy $x \in X$, assuming DM should take into account the possibility of realization every uncertainty $y \in Y$.

Remark 2.1:
The construction of GRS consists of four steps:

1. Construction of the function

$$f[y] = \max_{z\in X} f(z, y), \ \ \forall y \in Y.$$

2. Construction of the risk function $\Phi(x, y) = f[y] - f(x, y)$.

3. Computation of the inner maximum in (2.6) that determines the guarantee on risk:

$$\max_{y\in Y}\Phi(x,y)=\max_{y\in Y}(f[y]-f(x,y))=\Phi[x]\geq\Phi(x,y),\ \ \forall x\in X.$$

4. Calculation of the outer minimum in (2.6) yields the best guaranteed risk:

$$\Phi^r=\min_{x\in X}\Phi[x]=\Phi[x^r].$$

Finally, the obtained strategy x^r is the GRS of the problem Γ.

Remark 2.2:
Previous considerations are valid for arbitrary f, X, and Y. If the function f is defined in (1.2) and the sets X, Y are from (1.4), (1.1), then the diversification plan $(1-x_d^r-x_e^r, x_d^r, x_e^r)$ is the GRS in Γ.

3. EXPLICIT FORM OF GRS

Assume that the function f is defined in (1.2) and the sets X, Y are from (1.4), (1.1). In this section, we construct the GRS following the algorithm described above (Remark 2.1).

Step 1. The function $f(x,y)$ is linear with respect to the variable $x=(x_d,x_e)$. Therefore, for every fixed $y\in Y$ it attains its maximum on the polyhedron X at one of the vertices $(0,0)$, $(1,0)$ or $(0,1)$. Taking into account (1.2), we obtain:

$$f[y]=\max_{z\in X}f(z,y)=\max\{(1+r),\xi_d y_d,\xi_e y_e\}, \tag{3.7}$$

where ξ_d,ξ_e are defined in (1.3).

Step 2. Consider the partition $Y=Y_1\cup Y_2\cup Y_3$ (some of the subsets Y_i can be empty) defined by the following conditions:

$$\begin{aligned}
Y_1&=\{y\in Y:\ 1+r\geq\xi_d y_d,\ 1+r\geq\xi_e y_e\},\\
Y_2&=\{y\in Y:\ \xi_d y_d\geq 1+r,\ \xi_d y_d\geq\xi_e y_e\},\\
Y_3&=\{y\in Y:\ \xi_e y_e\geq 1+r,\ \xi_e y_e\geq\xi_d y_d\}.
\end{aligned} \tag{3.8}$$

Then the risk function has the form

$$\Phi(x,y)=f[y]-f(x,y)=\begin{cases}\Phi_1(x,y), & y\in Y_1,\\ \Phi_2(x,y), & y\in Y_2,\\ \Phi_3(x,y), & y\in Y_3,\end{cases} \tag{3.9}$$

where

$$\begin{aligned}
\Phi_1(x,y)&=(1+r)-f(x,y)=(1+r-\xi_d y_d)x_d+(1+r-\xi_e y_e)x_e,\\
\Phi_2(x,y)&=\xi_d y_d-f(x,y)=(1-x_d)(\xi_d y_d-(1+r))+(1+r-\xi_e y_e)x_e,\\
\Phi_3(x,y)&=\xi_e y_e-f(x,y)=(1+r-\xi_d y_d)x_d+(\xi_e y_e-(1+r))(1-x_e).
\end{aligned} \tag{3.10}$$

The functions Φ_i are bilinear by the variables x and y. That is why the risk function $\Phi(x,y)$ with any fixed strategy x is a piece-wise linear function on the uncertainty y. The sets Y_i are the domains (polygons) of linearity of the risk function with respect to y. In other words, the risk function $\Phi(x,y)$ coincides with the linear (with respect to x) function $\Phi_i(x,y)$ for any fixed $y\in Y_i$. In this case, we shall say that the risk function $\Phi(x,y)$ is *specified* at the point (x,y) of the function $\Phi_i(x,y)$.

Step 3. For every strategy $x \in X$ the guaranteed risk $\Phi[x]$ is computed. Due to natural conditions $r > 0$, $d_i > 0$, $b_i > a_i > 0$, $K_i > 0$ for $i = d, e$, we obtain:

$$\Phi\,[x] = \max_{y \in Y} \Phi\,(x, y) = \max_{i=1,2,3} \Phi_i\,[x], \tag{3.11}$$

where

$$\begin{aligned} \Phi_1[x] &= \Phi_1(x, a_d, a_e) = (1 + r - \xi_d a_d)x_d + (1 + r - \xi_e a_e)x_e, \\ \Phi_2[x] &= \Phi_2(x, b_d, a_e) = (1 - x_d)(\xi_d b_d - (1 + r)) + (1 + r - \xi_e a_e)x_e, \\ \Phi_3[x] &= \Phi_3(x, a_d, b_e) = (1 + r - \xi_d a_d)x_d + (\xi_e b_e - (1 + r))(1 - x_e). \end{aligned} \tag{3.12}$$

For the sake of brevity, we introduce the following notations:

$$\begin{aligned} \alpha_d &= 1 + r - \xi_d a_d, \quad \alpha_e = 1 + r - \xi_e a_e, \\ \beta_d &= \xi_d b_d - (1 + r), \;\; \beta_e = \xi_e b_e - (1 + r). \end{aligned} \tag{3.13}$$

Then the formula (3.11) reads

$$\begin{aligned} \Phi[x] = \max\{\Phi_1[x],\, \Phi_2[x],\, \Phi_3[x]\} = \\ \max\{\alpha_d x_d + \alpha_e x_e,\, \beta_d(1 - x_d) + \alpha_e x_e,\, \alpha_d x_d + \beta_e(1 - x_e)\}. \end{aligned} \tag{3.14}$$

Step 4. Now we seek the best guaranteed risk

$$\Phi^r = \min_{x \in X} \Phi[x] = \Phi[x^r]$$

by considering the problem $\min_{x \in X} \Phi[x]$ separately in the interior and on the border of the set (triangle) X and choosing the final best result from these partial results. Namely,

$$\min_{x \in X} \Phi[x] = \min\left\{ \min_{x \in [O,A]} \Phi[x],\, \min_{x \in [O,B]} \Phi[x],\, \min_{x \in [A,B]} \Phi[x],\, \min_{x \in \operatorname{int} X} \Phi[x] \right\}. \tag{3.15}$$

Case 1. The cathetus OA.

The strategy $x \in [O, A] = \{x_d \in [0, 1],\; x_e = 0\}$, that is, $x = (x_d, 0)$, and we deal with the function

$$\begin{aligned} \varphi(x_d) = \Phi[x_d, 0] = \max\{\alpha_d x_d,\, \beta_d(1 - x_d),\, \alpha_d x_d + \beta_e\} = \\ \max\{\varphi_1(x_d),\, \varphi_2(x_d), \varphi_3(x_d)\}, \end{aligned}$$

where $\varphi_1(x_d) = \alpha_d x_d$, $\varphi_2(x_d) = \beta_d(1 - x_d)$, $\varphi_3(x_d) = \alpha_d x_d + \beta_e$. Note that $\varphi_3(x_d) - \varphi_1(x_d) = \beta_e - \text{const}$.

Subcase 1 a. If $\beta_e \geq 0$, then $\varphi_3(x_d) \geq \varphi_1(x_d)$ for every x_d, and the function φ_1 can be excluded from consideration. The graph of φ on the segment $[0, 1]$ consists of one or two line segments. The function φ attains its minimum at the point $A = (1, 0)$ or $O = (0, 0)$ or at the intersection x_d^{OA} of the graphs φ_2 and φ_3 if $x_d^{OA} \in [0, 1]$. The point x_d^{OA} can be determined from the equation $\beta_d(1 - x_d) = \alpha_d x_d + \beta_e$, which yields

$$x_d^{OA} = \frac{\beta_d - \beta_e}{\alpha_d + \beta_d}. \tag{3.16}$$

The condition $x_d^{OA} \in [0, 1]$ is equivalent to the following relations between the parameters:

$$a_d \leq \frac{(1 + d_e)K_d}{(1 + d_d)K_e} \leq b_d.$$

Subcase 1 b. If $\beta_e \leq 0$, then $\varphi_3(x_d) \leq \varphi_1(x_d)$ for every x_d, and the function φ_3 can be excluded from consideration. The graph of φ on $[0,1]$ consists of one or two line segments. The function φ attains its minimum at the point $A = (1,0)$ or $O = (0,0)$ or at the intersection x_d^{OA} of the graphs φ_1 and φ_2 if $x_d^{OA} \in [0,1]$. The point x_d^{OA} can be determined from the equation $\alpha_d x_d = -\beta_d x_d + \beta_d$, which yields

$$x_d^{OA} = \frac{\beta_d}{\alpha_d + \beta_d}. \tag{3.17}$$

The condition $x_d^{OA} \in [0,1]$ is equivalent to the following relations between the parameters:

$$a_d \leq \frac{1+r}{1+d_d} K_d \leq b_d.$$

The both formulas (3.16) and (3.17) may be combined in the single expression

$$x_d^{OA} = \frac{\beta_d - \beta_e(\operatorname{sign}\beta_e + 1)/2}{\alpha_d + \beta_d}. \tag{3.18}$$

Further investigations of maximum of the function $\Phi[x]$ on the interval $[OA]$ is based on the monotonicity of the functions φ_1, φ_2, φ_3. Another approach is to calculate and compare the values of the function $\Phi[x]$. This comparison may be done after the calculation of all candidates for GRS of the problem Γ.

Thus we have proved the following assertion.

Proposition 3.1:
The guaranteed risk function $\Phi[x]$ attains its minimum on the interval $[OA]$ at a point of the set

$$[0,1] \cap \{O = (0,0),\ A = (1,0),\ x^{OA} = (x_d^{OA}, 0)\}, \tag{3.19}$$

where x_d^{OA} is defined in (3.18). *In other words, if the GRS $x^r = (x_d, x_e)$ lies in the part $[OA]$ of the boundary of the set X ($x_e = 0$), then the set* (3.19) *contains the GRS.*

Case 2. The cathetus OB.

The strategy $x \in [0,B] = \{x_d = 0,\ x_e \in [0,1]\}$, that is, $x = (0, x_e)$, and we deal with the function

$$\psi(x_e) = \Phi[0, x_e] = \max\{\alpha_e x_e,\ \alpha_e x_e + \beta_d,\ \beta_e(1 - x_e)\}.$$

In this case, all reasonings are similar to the previous one. We present the corresponding results:

Subcase 2 a. If $\beta_d \geq 0$, then the function ψ attains its minimum at the point $B = (0,1)$ or $O = (0,0)$ or the point

$$x_e^{OB} = \frac{\beta_e - \beta_d}{\alpha_e + \beta_e}, \tag{3.20}$$

if the condition $x_e^{OB} \in [0,1]$ holds true. The letter is equivalent to the following relations between the parameters:

$$a_e \leq \frac{(1+d_d)K_e}{(1+d_e)K_d} \leq b_e.$$

Subcase 2 b. If $\beta_d \leq 0$, then ψ attains its minimum at the point $B = (0,1)$ or $O = (0,0)$ or

$$x_e^{OB} = \frac{\beta_e}{\alpha_e + \beta_e}, \tag{3.21}$$

if the condition $x_e^{OB} \in [0,1]$ holds true. The letter is equivalent to the following relations between the parameters:

$$a_e \leq \frac{1+r}{1+d_e} K_e \leq b_e.$$

The both formulas (3.20) and (3.21) may be combined in the single expression

$$x_e^{OB} = \frac{\beta_e - \beta_d(\operatorname{sign}\beta_d + 1)/2}{\alpha_e + \beta_e}. \tag{3.22}$$

Recall that the final comparison will be done after the analysis of the remaining calculation for GRS. Thus we have proved the following assertion.

Proposition 3.2:
The guaranteed risk function $\Phi[x]$ attains its minimum on the interval $[OB]$ at a point of the set

$$[0,1] \cap \{O = (0,0),\ B = (0,1),\ x^{OB} = (0, x_e^{OB})\}, \tag{3.23}$$

where x_e^{OB} is defined in (3.22). *In other words, if the GRS $x^r = (x_d, x_e)$ lies in the part $[OB]$ of the boundary of the set X ($x_e = 0$), then the set* (3.23) *contains the GRS.*

Case 3. The hypotenuse of the triangle AOB: $AB = \{x_d + x_e = 1,\ x_d \geq 0,\ x_e \geq 0\}$, the strategy $x = (1 - x_e, x_e)$ and the guaranteed risk function (3.14) takes the form

$$\chi(x_e) = \Phi[1 - x_e, x_e] = \max\{\chi_1(x_e), \chi_2(x_e), \chi_3(x_e)\}, \quad x_e \in [0,1], \tag{3.24}$$

where

$$\begin{aligned} \chi_1(x_e) &= \Phi_1[1 - x_e, x_e] = (\alpha_e - \alpha_d)x_e + \alpha_d, \\ \chi_2(x_e) &= \Phi_2[1 - x_e, x_e] = (\beta_d + \alpha_e)x_e, \\ \chi_3(x_e) &= \Phi_3[1 - x_e, x_e] = (\alpha_d + \beta_e)(1 - x_e). \end{aligned} \tag{3.25}$$

Let l_i be the graph of the linear function χ_i, $i = 1,2,3$. The function $\chi(x_e)$ is a convex piecewise linear function on the variable $x_e \in [0,1]$. The graph of $\chi(x_e)$ is the lower envelope of the triple l_1, l_2, l_3 and it consists of one, two or three segments.

A function of such kind attains its minimum at one of the ends of the segment $[0,1]$ or at the point of interception of some pair of l_1, l_2, l_3. Therefore, three new candidates for possible points of maximum of the function $\chi(x_e)$ can be found from the following equations:

$$\chi_1(x_e) = \chi_2(x_e), \quad \chi_1(x_e) = \chi_3(x_e), \quad \chi_2(x_e) = \chi_3(x_e).$$

Solutions of these equations have the following form:

$$x_e^{12} = \frac{\alpha_d}{\alpha_d + \beta_d}, \quad x_e^{13} = \frac{\beta_e}{\alpha_e + \beta_e}, \quad x_e^{23} = \frac{\alpha_d + \beta_e}{\alpha_e + \alpha_d + \beta_e + \beta_d}. \tag{3.26}$$

The solutions outside the open interval $(0,1)$ should be eliminated from consideration. The remaining points from the set (3.26) together with two inner points – candidates from the Case 1 and the Case 2 – and three vertices of the triangle X constitute the full set of possible points of the minimum of the function $\Phi[x]$ on the boundary of the set X.

Thus we have proved the following assertion.

Proposition 3.3:
The guaranteed risk function $\Phi[x]$ attains its minimum on the interval (A,B) at a point of the set

$$(0,1) \cap \left\{x^{12} = (x_e^{12}, 1 - x_e^{12}),\ x^{13} = (x_e^{13}, 1 - x_e^{13}),\ x^{23} = (x_e^{23}, 1 - x_e^{23})\right\}. \tag{3.27}$$

In other words, if the GRS $x^r = (x_d, x_e)$ lies in the part (A,B) of the boundary of the set X ($x_e = 0$), then the set (3.27) *contains the GRS.*

Case 4. The interior of the triangle AOB: $\operatorname{int} X = \{x_d + x_e < 1,\ x_d > 0,\ x_e > 0\}$. The strategy $x = (x_d, x_e)$ and the guaranteed risk function (3.14) takes the form

$$\Phi[x] = \max\{\Phi_1[x],\ \Phi_2[x],\ \Phi_3[x]\} = \\ \max\{\alpha_d x_d + \alpha_e x_e,\ \beta_d(1 - x_d) + \alpha_e x_e,\ \beta_e(1 - x_e) + \alpha_d x_d\}, \quad x \in \operatorname{int} X. \quad (3.28)$$

Now we introduce the following

Condition TM (Total Mixing).

The guaranteed risk function $\Phi\,[x]$ attains its minimal value Φ^* on the set X at an inner point of X, and $\Phi[x] > \Phi^*$ for all points $x \in \operatorname{int} X$.

The condition TM means that the optimal diversification plan necessarily uses all three currencies. Later on, in this section, we restrict our consideration by the condition TM, since other possible situations are covered by the Propositions 3.1 – 3.3.

The guaranteed risk $\Phi\,[x]$ is a convex continuous function, it may attain its minimum at inner point of the convex set X. This minimum is the global minimum on the set X.

Let P_1, P_2, P_3 be two-dimensional planes defined by the equations $z = F_i\,(x)$, $x \in \mathbb{R}^2$, $i = 1, 2, 3$. The graph of the guaranteed risk function $\Phi\,[x]$ is the lower envelope of a family of the planes P_1, P_2, P_3, or, more precisely, its part located above the triangle X. The epigraph of the function $\Phi\,[x]$, $x \in X$, is an inverted obelisk-shaped three-dimensional body. Calculation of the minimum of the function $\Phi\,[x]$ on X is equivalent to finding the lowest point of that body. Such a geometric interpretation allows us to simplify the minimization of the non-smooth function $\Phi\,[x]$ with linear restrictions.

Since the minimum of $\Phi[x]$ on the boundary of X is already investigated, it remains to consider the minimum of $\Phi[x]$ on $\operatorname{int} X$.

Lemma 3.1:

Suppose that $\Phi_i[x]$ is not identically constant for at least one $i \in \{1, 2, 3\}$ and there exists a point $x^0 \in \operatorname{int} X$ such that

$$\Phi_i[x^0] > \Phi_j[x^0],\ \ \Phi_i[x^0] > \Phi_k[x^0], \quad \forall\, j \neq i,\ k \neq i. \quad (3.29)$$

Then x^0 is not a point of minimum of $\Phi[x]$ on the set $\operatorname{int} X$.

Proof

Due to the equality

$$\Phi[x] = \max\{\Phi_1[x],\ \Phi_2[x],\ \Phi_3[x]\},$$

the condition of the lemma means that the value of $\Phi\,[x]$ is determined at the point x^0 only by the function $\Phi_i\,[x]$.

Since all functions $\Phi_i[x]$ are continuous, the inequalities (3.29) hold true in a neighborhood of the point x^0. In this neighborhood the risk function coincides with the linear function $\Phi\,[x] = \Phi_i\,(x)$, which is not constant. Therefore, it cannot reach the extremum (even local) at a point $x^0 \in \operatorname{int} X$. □

Lemma 3.1 does not cover degenerate cases that all functions $\Phi_i[x]$ are constant. However, it is obvious that in these cases the function $\Phi[x]$ is also constant, whence all admissible solutions $x \in X$ are optimal.

Let the function $\Phi_1[x] = \alpha_d x_d + \alpha_e x_e$ is identically constant, that is, $\alpha_d = \alpha_e = 0$. Then the risk function is defined by the formula

$$\Phi[x] = \max\{0,\ \Phi_2[x],\ \Phi_3[x]\} = \max\{0,\ \beta_d(1 - x_d),\ \beta_e(1 - x_e)\},$$

where $0 < (1 - x_d) < 1$, $0 < (1 - x_e) < 1$ for any point $x = (x_d, x_e) \in \operatorname{int} X$. Let us show that then the condition TM is not satisfied, and the minimum is attained on the boundary of the set X.

Indeed, consider all possible combination of the signs of β_d, β_e:

1. Let $\beta_d \le 0$ and $\beta_e \le 0$, then $\Phi_2[x] \le 0$ and $\Phi_3[x] \le 0$. Therefore, $\Phi[x] = \Phi^* = 0$ for every $x \in X$. All admissible diversification plans are optimal. The condition TM is not satisfied.
2. Let $\beta_d \le 0$ and $\beta_e > 0$, then $\Phi_2[x] \le 0$ and $\Phi_3[x] > 0$. Therefore, the function
$$\Phi[x] = \Phi_3[x] = \beta_e(1 - x_e)$$
for every $x \in \operatorname{int} X$. The condition TM is not satisfied, since the linear function $\Phi_3[x]$ does not reach minimum on the open set $\operatorname{int} X$.
3. Let $\beta_d > 0$ and $\beta_e \le 0$, then $\Phi_2[x] > 0$ and $\Phi_3[x] \le 0$. Therefore, the function
$$\Phi[x] = \Phi_2[x] = \beta_d(1 - x_d)$$
for every $x \in \operatorname{int} X$. As before, the condition TM is not satisfied.
4. Let $\beta_d > 0$ and $\beta_e > 0$, then $\Phi_2[x] > 0$ and $\Phi_3[x] > 0$. Therefore, the function
$$\Phi[x] = \max\{\beta_d(1 - x_d),\ \beta_e(1 - x_e)\}$$
for every $x \in \operatorname{int} X$. Note that the functions $\Phi_2\,[x]$, $\Phi_3\,[x]$ are not constant. Then, as well as in Lemma 3.1, the function $\Phi[x]$ can attain its minimum in $\operatorname{int} X$ only on the line $\beta_d(1 - x_d) = \beta_e(1 - x_e)$. However, the linear function
$$\Phi[x] = \beta_d(1 - x_d) = \beta_e(1 - x_e)$$
does not attains its minimum on an open interval. The condition TM is not satisfied.

From now, we may assume that the function $\Phi_1[x]$ is not constant. Thus, the minimum of the function $\Phi[x]$ on the set $\operatorname{int} X$ (if it exists) can be reached only at the intersection of some pair of the planes P_i, $i = 1, 2, 3$. Therefore, in what follows we consider only those interior points of the set X where the risk function is determined by more than one function Φ_i: $\Phi_1[x] = \Phi_2[x]$ or $\Phi_1[x] = \Phi_3[x]$ or $\Phi_2[x] = \Phi_3[x]$. We denote the corresponding sets by L_1, L_2, L_3, respectively. Due to the linearity of Φ_i, all L_i are open intervals (possibly, empty).

Consider the function $\Phi[x]$ on the interval L_1, where $\Phi_1[x] = \Phi_2[x]$. Then only two situations are possible.

Subcase A. $\Phi_3[x] < \Phi_1[x] = \Phi_2[x]$, $x \in L_1$ (the function Φ_3 is omissible). If the linear function $\Phi[x] = \Phi_1[x]$ is constant on an open interval L_1, then it attains its global minimum at the border of the set X as well as on X. This minimum will be found when examining the boundary of the set X. If $\Phi[x] = \Phi_1[x]$ is not constant on L_1, then it has no points of minimum on the open interval L_1.

Subcase B. $\Phi_1[x^*] = \Phi_2[x^*] = \Phi_3[x^*]$ at some point $x^* \in L_1$. This point corresponds to an interception of three plains P_1, P_2, P_3, where P_i is the graph of the function $z = \Phi_i[x]$, $i = 1, 2, 3$. The uniqueness of such points under natural assumptions about the parameters is shown below.

Therefore, the internal point of the minimum of the guaranteed risk function satisfies the following system of linear equations and inequalities:

$$\Phi_1[x] = \Phi_2[x],\ \ \Phi_3[x] = \Phi_2[x],\ \ x_d + x_e < 1,\ \ x_d > 0,\ \ x_e > 0. \tag{3.30}$$

Taking into account the above expressions for the functions $\Phi_i[x]$, this yields the system of linear equations

$$\begin{aligned} \alpha_d x_d + \alpha_e x_e &= \beta_d(1 - x_d) + \alpha_e x_e, \\ \alpha_d x_d + \beta_e(1 - x_e) &= \beta_d(1 - x_d) + \alpha_e x_e, \end{aligned} \tag{3.31}$$

whose solution

$$x^* = (x_d^*, x_e^*) = \Big(\frac{\beta_d}{\alpha_d + \beta_d},\ \frac{\beta_e}{\alpha_e + \beta_e}\Big) \tag{3.32}$$

is a candidate to be the GRS, if the inequalities in (3.30) hold true. We would like to emphasize that x^* is the point of inner minimum, if the function $\Phi\,[x]$ attains its minimum on $\operatorname{int} X$.

Remark 3.1:
Due to the natural assumptions $b_d > a_d$, $b_e > a_e$ and the notation (3.13) we have $\alpha_d + \beta_d > 0$, $\alpha_e + \beta_e > 0$. Therefore, the solution of the linear system (3.31) exists, it is unique and defined by the formula (3.32).

Thus we have proved the following assertion.

Proposition 3.4:
Suppose that function $\Phi\,[x]$ *attains its minimum on the open set* $\operatorname{int} X$. *Then the point* x^* *given by the formula* (3.32) *is the point of minimum, if* $x_d^*, x_e^* \in (0,1)$ *and* $x_d + x_e < 1$.

Remark 3.2:
Recall that any point of local minimum of a convex function on a convex set is at the same time the point of global minimum on the given set. Hence, if the point of inner minimum (3.32) exists and it belongs to $\operatorname{int} X$, it gives the GR solution. Whether it exists or not depends on the interrelations between the parameters of the problem. The GR-solution very often belongs to the boundary of the set of diversification plans. It means that the optimal plan of diversification contains only two (or even one) of three currencies.

Let us rewrite the inequalities for (x_d^*, x_e^*) from (3.30) in terms of the initial parameters of the problem under consideration. Natural conditions $b_i > a_i$, $i = d, e$, imply the following inequalities (see (3.13)):

$$\alpha_i + \beta_i = \xi_i(b_i - a_i) > 0, \;\; i = d, e,$$

where ξ_i are defined in (1.3). Hence the condition $(x_d^* > 0) \wedge (x_e^* > 0)$ is equivalent to

$$(\beta_d > 0) \wedge (\beta_e > 0), \tag{3.33}$$

which, in turn, is equivalent to

$$\xi_i b_i > 1 + r, \;\; i = d, e. \tag{3.34}$$

Assume that the condition (3.33) holds true. Then the condition $(x_d^* < 1) \wedge (x_e^* < 1)$ is equivalent to $(\beta_d < 1) \wedge (\beta_e < 1)$, which, in turn, is equivalent to

$$\xi_i a_i < 1 + r, \;\; i = d, e. \tag{3.35}$$

Finally, we remark that the inequality $x_d^* + x_e^* < 1$ is equivalent to

$$\beta_d/(\alpha_d + \beta_d) + \beta_e/(\alpha_e + \beta_e) < 1,$$

that is, $\beta_d\beta_e < \alpha_d\alpha_e$, which is equivalent to

$$(\xi_d b_d - (1+r))(\xi_e b_e - (1+r)) < (\xi_d a_d - (1+r))(\xi_e a_e - (1+r)). \tag{3.36}$$

Thus, the union of the inequalities (3.34), (3.35) and (3.36) is a necessary and sufficient condition for $x^* \in \operatorname{int} X$.

Remark 3.3:
We emphasize that x^* is only the candidate, but not necessarily a point of minimum.

4. FINAL ALGORITHM TO CONSTRUCT THE GUARANTEED RISK SOLUTION

Combining the above results (Propositions 3.1 – 3.4), we obtain the following algorithm to calculate the guaranteed risk diversification strategy and the minimal guaranteed risk:

1. Write down the numerical values of the interest rates r, d_d, d_e and the current exchange rates K_d, K_e. Set the possible ranges (per year) for dollar and euro: $[a_d, b_d]$ and $[a_e, b_e]$, respectively.
2. Calculate the secondary parameters: α_d, α_e, β_d, β_e using the formula (3.13).
3. Calculate nine points – the candidates to be the optimal solution. It is convenient to represent these points in the form presents in the table below.
4. Remove the points that do not belong to the set X of the admissible plans.
5. Calculate the values of the guaranteed risk at the remaining points using the formula (3.15). Choose the solution with the best (minimal) guaranteed risk.

	Candidate	Formula	Condition
1	The vertex O	$x_d = 0, x_e = 0$	–
2	The vertex A	$x_d = 1, x_e = 0$	–
3	The vertex B	$x_d = 0, x_e = 1$	–
4	The inner point $(x_d^{OA}, 0)$ of the cathetus (OA)	$x_d^{OA} = \frac{\beta_d - \beta_e(\operatorname{sign}\beta_e + 1)/2}{\alpha_d + \beta_d}$	$0 < x_d^{OA} < 1$
5	The inner point $(0, x_e^{OB})$ of the cathetus (OB)	$x_e^{OB} = \frac{\beta_e - \beta_d(\operatorname{sign}\beta_d + 1)/2}{\alpha_e + \beta_e}$	$0 < x_e^{OB} < 1$
6	The inner point $(1 - x_e^{12}, x_e^{12})$ of the hypotenuse (AB)	$x_e^{12} = \frac{\alpha_d}{\alpha_d + \beta_d}$	$0 < x_e^{12} < 1$
7	The inner point $(1 - x_e^{13}, x_e^{13})$ of the hypotenuse (AB)	$x_e^{13} = \frac{\beta_e}{\alpha_e + \beta_e}$	$0 < x_e^{13} < 1$
8	The inner point $(1 - x_e^{23}, x_e^{23})$ of the hypotenuse (AB)	$x_e^{23} = \frac{\alpha_d + \beta_e}{\alpha_e + \alpha_d + \beta_e + \beta_d}$	$0 < x_e^{23} < 1$
9	The inner point (x_d^*, x_e^*) of the set X	$x_d^* = \frac{\beta_d}{\alpha_d + \beta_d}$ $x_e^* = \frac{\beta_e}{\alpha_e + \beta_e}$	$0 < x_d^*, x_e^* < 1$ $x_d^* + x_e^* < 1$

5. CONCLUSION

We considered the problem of optimal multi-currency deposit diversification with uncertain future exchange rates as a problem of minimization the lost profits. It was assumed that only the ranges of the uncertain parameters are known, and there are no statistical characteristics. The Savage minimax regret conception is used to minimize the lost profits due to uncertainty.

It should be noted that the Savage criterion is more sophisticated than the more often used Wald criterion. Therefore, the corresponding mathematical constructions are more complicated. Nevertheless, in the problem under consideration, it is possible to find the risk function and the function of the guaranteed risk in an explicit form. After that, the problem is reduced to finding the point of the minimum of a piecewise linear function under simple linear constraints. To solve it, the problem is decomposed into two stages: finding the minimum on the boundary and on the interior of the given set. Explicit formulas for nine representative points being candidates for the optimal solution are found. The final choice is made by the direct comparison of the values of the Savage criterion at these points.

Under certain ratios of the parameters, the optimal solution is not unique. Without dwelling in more detail, we only note that in the case of non-uniqueness, it is possible to consider an additional criterion (for example, the guaranteed income). This leads to multi-criteria optimization problems with uncertainty. Some problems of this class were considered earlier in [9]. It seems promising to apply the approaches developed in these works to the analysis of financial management problems with incomplete information.

ACKNOWLEDGEMENTS

This article is an output of a research project implemented as part of the Basic Research Program at the National Research University Higher School of Economics (HSE University). The work was partially supported by the International Centre of Decision Choice and Analysis of the National Research University Higher School of Economics.

The author is grateful to the reviewers for their thorough and in-depth study of the paper, constructive comments, and valuable suggestions, which gave possibility to improve significantly the presentation of the paper.

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