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Asymptotic Properties of the Block-Type Statistics

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Abstract:
Extreme value theory is an issue extensively applied in many different fields. One of the central points of this theory is the estimation of a positive extreme value index. In this paper we introduce a new family of block type statistics related to this estimation. A weak consistency of the introduced statistics is proved. A bivariate central limit theorem for newly introduced statistics is derived. We provide the new family of semi-parametric estimators for the positive extreme value index. Asymptotic normality of the introduced estimators is proved. It is shown that new estimators have better asymptotic performance comparing with several block-type estimators over the whole range of parameters presented in the second order regular variation condition. An application to the estimation of the positive valued extreme value index for several real data sets is provided.

Keywords: asymptotic normality, extreme value index, block-type estimator

1. INTRODUCTION

Let $X_1, X_2, \ldots, X_n$ be a sample of independent identically distributed (i.i.d.) random variables (r.v.s) with an unknown distribution function (d.f.) F. In the present paper we will formulate our assumptions in terms of a quantile type function U associated to F, which is defined by

$$U(t) = \begin{cases} 0, & 0 < t \le 1 \\ \inf\{x : \ F(x) \ge 1 - (1/t)\}, & t > 1. \end{cases}$$

The main assumption is that U belongs to the class of regularly varying functions at infinity, with $\gamma > 0$ (shortly, $U \in \mathrm{RV}_\gamma$), i.e., for all $x > 0$,

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^\gamma. \tag{1.1}$$

The parameter γ in (1.1) is a positive extreme value index (EVI) that is the primary parameter of large extreme events. Models satisfying (1.1) are quite common in many application areas such as biostatistics, computer science, finance, insurance and social sciences, among others.

To exploit the block approach in the extreme value theory one has to divide the observations $X_1, \ldots, X_n$ into $1 \le s \le n$ non-overlapping blocks of size $m = [n/s]$:

$$B_i = \{X_{(i-1)m+1}, \ldots, X_{im}\}, \quad 1 \le i \le s,$$

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where $[\cdot]$ denotes the integer part. Let $X_{1,m}^{(i)} \leq X_{2,m}^{(i)} \leq \cdots \leq X_{m,m}^{(i)}$ denote the order statistics of m observations in the ith block.

The aim of the paper is to investigate asymptotic properties, including a weak consistency and an asymptotic normality, of the block-type statistics

$$Q_n(s,\ell,r) \;=\; \frac{1}{s}\sum_{i=1}^{s}\frac{1}{\ell}\sum_{j=1}^{\ell} h_r\left(\frac{X_{m-j+1,m}^{(i)}}{X_{m-\ell,m}^{(i)}}\right).$$

Here, $\ell \in \mathbb{N}$ and the family of functions h_r, $r \in \mathbb{R}$ is defined for $x > 0$ as follows:

$$h_r(x) = \begin{cases} (x^r - 1)/r, & r \neq 0, \\ \ln(x), & r = 0. \end{cases}$$

Note that the statistics $Q_n(s,\ell,r)$ are applicable particularly when only the largest observations $\{X_{m-j,m}^{(i)},\ 0 \leq j \leq \ell,\ 1 \leq i \leq s\}$ are available for the inference. Such type of data can be found in [4], where battle deaths in major power wars between 1495 and 1975 were analyzed. For more such data see [10] and references therein.

The statistics $Q_n(s,\ell,r)$ generalize two families of statistics. Firstly, by taking $r=0$ we turn back to the estimator for the positive EVI, proposed in [18]. Secondly, the statistics $Q_n(s,\ell,r)$ generalize the statistics $Q_n(s,1,r)$, introduced in [17], see also [21].

Note that the statistics $Q_n(s,\ell,r)$ are scale-free, that is, $Q_n(s,\ell,r)$ do not change when observations $X_1,\ldots,X_n$ are replaced by $cX_1,\ldots,cX_n$ with $c>0$.

The paper is organized as follows. In Section 2 our main results are presented. In the next section, we introduce a new family of estimators for positive EVIs and compare the latter estimators with several known estimators. In Section 4 the application to several real data sets is discussed. We finalize with conclusions. The last section contains proofs of the results.

2. MAIN RESULTS

Let $\xrightarrow{\mathrm{d}}$ and $\xrightarrow{\mathrm{p}}$ denote the convergence in distribution and in probability, respectively. The equality in distribution will be denoted by $\stackrel{\mathrm{d}}{=}$.

Theorem 2.1:
Suppose that $X_1,\ldots,X_n$ are i.i.d. r.v.s with the d.f. F, whose quantile type function U satisfies (1.1) with some $\gamma > 0$. Let $\ell \in \mathbb{N}$. Let $r \in \mathbb{R}$ be such that $\gamma r < 1$ holds. Let the block number $s = s_n$ and the block width $m = m_n$ tend to infinity as $n \to \infty$. Then

$$Q_n(s,\ell,r) \xrightarrow{\mathrm{p}} \Lambda_\gamma(\gamma r), \quad n \to \infty,$$

where $\Lambda_\gamma(t) := \gamma/(1-t)$, $t < 1$.

The second order parameter ρ rules the rate of convergence in the first order condition (1.1). Moreover, ρ is a non-positive parameter that appears in the limiting relation

$$\lim_{t\to\infty}\frac{1}{A(t)}\left(\frac{U(tx)}{U(t)} - x^\gamma\right) = x^\gamma h_\rho(x), \tag{2.2}$$

which we assume to be held for all $x>0$. Here, $A(t)$ is a measurable function with a constant sign near infinity and which is not identically zero, and $A(t) \to 0$ as $t \to \infty$. It is known that (2.2) implies that $|A(t)| \in RV_\rho$, see [11].

Let $Z = Z(t)$, $t < 1/2$ be a Gaussian random process with mean 0 and covariance function $\sigma^2(t_1,t_2) := \mathrm{E}\left(Z(t_1)Z(t_2)\right)$, where $\sigma^2(t_1,t_2) = (1-t_1-t_2)^{-1}$.

Theorem 2.2:
Suppose that $X_1, \ldots, X_n$ are i.i.d. r.v.s with the d.f. F, whose quantile type function U satisfies the second order condition (2.2) with some $\gamma > 0$ and $\rho \leq 0$. Let the block number $s = s_n$ and the block width $m = m_n$ satisfy the assumptions of Theorem 2.1 and, in addition,

$$\lim_{n\to\infty} \sqrt{s}A(m) = \mu \in (-\infty, +\infty) \tag{2.3}$$

hold. Let $\ell \in \mathbb{N}$ and $r_i \in \mathbb{R}$, $i = 1, 2$ be such that $\gamma r_i < 1/2$ holds. Then it holds

$$\begin{aligned}&\sqrt{s}\left(Q_n(s,\ell,r_1) - \Lambda_\gamma(\gamma r_1),\ Q_n(s,\ell,r_2) - \Lambda_\gamma(\gamma r_2)\right)\\ &\quad\xrightarrow{\mathrm{d}} \left(\mu\nu(\ell,\rho,r_1) + \frac{\Lambda_\gamma(\gamma r_1)}{\sqrt{\ell}}Z(\gamma r_1),\ \mu\nu(\ell,\rho,r_2) + \frac{\Lambda_\gamma(\gamma r_2)}{\sqrt{\ell}}Z(\gamma r_2)\right)\end{aligned} \tag{2.4}$$

as $n \to \infty$, where

$$\nu(\ell,\rho,r_i) = \frac{\Gamma(1+\ell-\rho)}{\Gamma(1+\ell)} \cdot \frac{1}{(1-\gamma r_i)(1-\gamma r_i - \rho)}, \quad i = 1, 2,$$

and $\Gamma(t) = \int_0^\infty x^{t-1}\exp\{-x\}\mathrm{d}x$, $t > 0$ is the gamma function.

Remark 1. If in conditions of Theorem 2.2 assumption (2.3) is replaced by

$$\lim_{n\to\infty} \sqrt{s}A(m) = \infty, \tag{2.5}$$

then it holds

$$\frac{1}{A(m)}\left(Q_n(s,\ell,r_1) - \Lambda_\gamma(\gamma r_1)\right) \xrightarrow{\mathrm{P}} \nu(\ell,\rho,r_1), \quad n \to \infty. \tag{2.6}$$

The extension of Theorem 2.2 to the multivariate case is quite trivial. Thus, Theorem 2.2 gives us a lot of possibilities to form asymptotically normal estimators of $\gamma > 0$ using statistics $Q_n(s,\ell,r)$ with different r. Considering the simplest case we propose the following family of the block-type estimators for $\gamma > 0$:

$$\hat{\gamma}_n^{(1)}(s,\ell,r) = \begin{cases} Q_n(s,\ell,0), & r = 0,\\ Q_n(s,\ell,r)\left(1 + rQ_n(s,\ell,r)\right)^{-1}, & r \neq 0.\end{cases}$$

It should be noted that the family of estimators $\hat{\gamma}_n^{(1)}(s,\ell,r)$ generalizes the family of estimators given in [18] (for $r = 0$) and the family of estimators proposed in [17] (for $\ell = 1$). We establish the asymptotic normality of $\hat{\gamma}_n^{(1)}(s,\ell,r)$ by applying the univariate form of Theorem 2.2.

Corollary 2.3:
Let the conditions of Theorem 2.2 be fulfilled for univariate case. Then it holds

$$\sqrt{s}\left(\hat{\gamma}_n^{(1)}(s,\ell,r) - \gamma\right) \xrightarrow{\mathrm{d}} \mathcal{N}\left(\mu\nu_1(\ell,\rho,\gamma r), \gamma^2\sigma_1^2(\ell,\gamma r)\right), \quad n \to \infty,$$

where $\mathcal{N}$ denotes a normal distribution and

$$\nu_1(\ell,\rho,R) = \frac{\Gamma(1+\ell-\rho)}{\Gamma(1+\ell)} \cdot \frac{1-R}{1-R-\rho}, \quad \sigma_1^2(\ell,R) = \frac{(1-R)^2}{\ell(1-2R)}, \quad R < 1/2.$$

Taking $r = 0$ in Corollary 2.3 we turn back to Theorem 1 in [18]. It should be noted that the asymptotic bias is written in a slightly different form in [18]. Another partial case of Corollary 2.3 can be found in [17] (see (2.12) therein). There, assuming more simple second order asymptotic condition than (2.3), the asymptotic normality of the estimators $1/\hat{\gamma}_n^{(1)}(s,1,r)$ is obtained.

3. A NEW FAMILY OF THE BLOCK-TYPE ESTIMATORS FOR POSITIVE EVI

Let $r \neq 0$ satisfy $\gamma r < 1$, while $s = s_n$ and $m = m_n$ satisfy the assumptions of Theorem 2.1. Then $Q_n(s, \ell, 0)/Q_n(s, \ell, r) \xrightarrow{\mathrm{P}} 1 - \gamma r$ holds as $n \to \infty$. So,

$$\hat{\gamma}_n^{(2)}(s, \ell, r) = \frac{Q_n(s, \ell, r) - Q_n(s, \ell, 0)}{r Q_n(s, \ell, r)}, \quad r \neq 0$$

presents a family of weakly consistent estimators for positive EVI. Let us define $\hat{\gamma}_n^{(2)}(s, \ell, 0) := \lim_{r \to 0} \hat{\gamma}_n^{(2)}(s, \ell, r)$. By using the L'Hospital's rule, it follows that

$$\hat{\gamma}_n^{(2)}(s, \ell, 0) = \frac{\tilde{Q}_n(s, \ell)}{2 Q_n(s, \ell, 0)},$$

where

$$\tilde{Q}_n(s, \ell) = \frac{1}{s} \sum_{i=1}^{s} \frac{1}{\ell} \sum_{j=1}^{\ell} \ln^2 \left(\frac{X_{m-j+1,m}^{(i)}}{X_{m-\ell,m}^{(i)}} \right).$$

An application of Theorem 2.2 yields the following result.

Theorem 3.1:
Let the conditions of Theorem 2.2 be fulfilled with $r_1 = 0$ and $r_2 = r$. Then it holds

$$\sqrt{s} \left(\hat{\gamma}_n^{(2)}(s, \ell, r) - \gamma \right) \xrightarrow{\mathrm{d}} \mathcal{N} \left(\mu \nu_2(\ell, \rho, \gamma r), \gamma^2 \sigma_2^2(\ell, \gamma r) \right), \quad n \to \infty, \tag{3.7}$$

where

$$\nu_2(\ell, \rho, R) = \frac{\Gamma(1 + \ell - \rho)}{\Gamma(1 + \ell)} \cdot \frac{1 - R}{(1 - R - \rho)(1 - \rho)}, \quad \sigma_2^2(\ell, R) = \frac{2(1 - R)}{\ell(1 - 2R)}, \quad R < 1/2.$$

Having the asymptotic normality of the introduced estimators $\hat{\gamma}_n^{(2)}(s, \ell, r)$ we can discuss an optimal choice of s, ℓ and r.

We assume further that $\rho < 0$ and $\mu \neq 0$. Keeping ℓ and r fixed, the limiting mean squared error (MSE) for $\hat{\gamma}_n^{(2)}(s, \ell, r)$ is, approximately,

$$\inf_s \left\{ \frac{\gamma^2 \sigma_2^2(\ell, \gamma r)}{s} + A^2 \left(\frac{n}{s} \right) \nu_2^2(\ell, \rho, \gamma r) \right\},$$

where the sequence $s = s_n$ satisfies the assumptions of Theorem 2.1.

It is known that there exists a positive decreasing function $b \in \mathrm{RV}_{2\rho - 1}$ such that $A^2(t) \sim \int_t^\infty b(x) \mathrm{d}x$, $t \to \infty$, see Here and below, we write an $a_n \sim c_n$ if $a_n / c_n \to 1$ as $n \to \infty$. Let $b^{\leftarrow}$ denote the inverse function of b. Following the lines in [12] we find

$$\mathrm{MSE} \left(\hat{\gamma}_n^{(2)}(s_2^*, \ell, r) \right) \sim \frac{1 - 2\rho}{-2\rho} \left(\nu_2^2(\ell, \rho, \gamma r) \left(\gamma^2 \sigma_2^2(\ell, \gamma r) \right)^{-2\rho} \right)^{1/(1-2\rho)} \frac{b^{\leftarrow}(1/n)}{n}, \tag{3.8}$$

as $n \to \infty$, where the optimal choice of the number of blocks s_2^* satisfies the relation

$$s_2^* \sim \left(\frac{\gamma^2 \sigma_2^2(\ell, \gamma r)}{\nu_2^2(\ell, \rho, \gamma r)} \right)^{1/(1-2\rho)} \frac{n}{b^{\leftarrow}(1/n)}, \quad n \to \infty.$$

Next, we minimize the right hand side of (3.8) with respect to ℓ and r. For this it is sufficient to minimize the product

$$\nu_2^2(\ell,\rho,\gamma r)\left(\gamma^2\sigma_2^2(\ell,\gamma r)\right)^{-2\rho}=\frac{2^{-2\rho}}{(1-\rho)^2}\left(\varphi(\ell,\rho)\right)^2\phi(\rho,\gamma r)$$

with respect to ℓ and r. Here, we have

$$\varphi(\ell,\rho)=\frac{\ell^\rho\Gamma(\ell+1-\rho)}{\Gamma(\ell+1)},\quad \phi(\rho,R)=\frac{(1-R)^{2-2\rho}}{(1-R-\rho)^2(1-2R)^{-2\rho}},\ R<1/2.$$

By using the standard technique of derivatives we obtain that the function $\phi(\rho,R)$ attains its minimum with $R_2^*=\rho$. Whence, the optimal choice of parameter r for $\hat{\gamma}_n^{(2)}(s_2^*,\ell,r)$ has the form $r_2^*=\rho/\gamma$.

Before we proceed, let us introduce the following lemma regarding the function $\varphi(\ell,\rho)$.

Lemma 3.2:
For any $\ell\in\mathbb{N}$ and $\rho<0$, $\varphi(\ell,\rho)>\varphi(\ell+1,\rho)$ holds.

By Lemma 3.2, for any $\ell\in\mathbb{N}$, the estimator $\hat{\gamma}_n^{(2)}(s_2^*,\ell,r_2^*)$ has a bigger asymptotic MSE comparing with $\hat{\gamma}_n^{(2)}(s_2^*,\ell+1,r_2^*)$. Thus, there is no an optimal estimator among estimators $\hat{\gamma}_n^{(2)}(s_2^*,\ell,r_2^*)$, $\ell\in\mathbb{N}$. The same conclusion holds for the estimators $\hat{\gamma}_n^{(1)}(s_1^*,\ell,r_1^*)$, $\ell\in\mathbb{N}$, where

$$s_1^*\sim\left(\frac{\gamma^2\sigma_1^2(\ell,\gamma r)}{\nu_1^2(\ell,\rho,\gamma r)}\right)^{1/(1-2\rho)}\frac{n}{b^{\leftarrow}(1/n)},\quad n\to\infty$$

and $r_1^*=(2\gamma)^{-1}\left(2-\rho-(2-4\rho+\rho^2)^{1/2}\right)$, see [17]. As for $\mathrm{MSE}\left(\hat{\gamma}_n^{(1)}(s_1^*,\ell,r)\right)$, it has the same asymptotic as in (3.8) with replacing $\nu_2(\ell,\rho,\gamma r)$ and $\sigma_2^2(\ell,\gamma r)$ by $\nu_1(\ell,\rho,\gamma r)$ and $\sigma_1^2(\ell,\gamma r)$, respectively.

Let us find the limit of the ratio of $\mathrm{MSE}\left(\hat{\gamma}_n^{(i)}(s_i^*,\ell+1,r_i^*)\right)$ and $\mathrm{MSE}\left(\hat{\gamma}_n^{(i)}(s_i^*,\ell,r_i^*)\right)$:

$$\lim_{n\to\infty}\frac{\mathrm{MSE}\left(\hat{\gamma}_n^{(i)}(s_i^*,\ell+1,r_i^*)\right)}{\mathrm{MSE}\left(\hat{\gamma}_n^{(i)}(s_i^*,\ell,r_i^*)\right)}=\chi(\ell,\rho),\quad i=1,2,$$

where

$$\chi(\ell,\rho)=\left(\left(\frac{\ell}{\ell+1}\right)^\rho\frac{\ell+1-\rho}{\ell+1}\right)^{2/(1-2\rho)}.$$

For the well-known heavy-tailed models, like the Frechet and the Student's t, condition (2.2) holds with $\rho=-1$. We represent the numerical approximation of $\chi(\ell,-1)$ for several values of ℓ: $\chi(1,-1)\approx 0.825$, $\chi(2,-1)\approx 0.924$, $\chi(3,-1)\approx 0.958$, $\chi(4,-1)\approx 0.973$ and $\chi(5,-1)\approx 0.981$. Since the values of $\chi(\ell,-1)$ are close to 1 when $\ell\geq 3$, we recommend to take $3\leq\ell\leq 5$ dealing with real data sets.

At the end of this section, we compare the estimators $\hat{\gamma}_n^{(1)}(s_1^*,\ell,r_1^*)$ and $\hat{\gamma}_n^{(2)}(s_2^*,\ell,r_2^*)$. Denoting

$$\lim_{n\to\infty}\frac{\mathrm{MSE}\left(\hat{\gamma}_n^{(1)}(s_1^*,\ell,r_1^*)\right)}{\mathrm{MSE}\left(\hat{\gamma}_n^{(2)}(s_2^*,\ell,r_2^*)\right)}=\tilde{\chi}(\rho),$$

it is not difficult to get that

$$\tilde{\chi}(\rho) = \left(\frac{(1-2\rho)^{1-1/\rho}(1-R_1^*)^{2-1/\rho}}{2(1-\rho)(1-R_1^*-\rho)^{-1/\rho}(1-2R_1^*)} \right)^{-2\rho/(1-2\rho)},$$

where $R_1^* = \gamma r_1^*$. Let us note that

$$(\tilde{\chi}(\rho))^{(1-2\rho)/(-2\rho)} > 1 \tag{3.9}$$

for all $-\infty < \rho < 0$ (see Section 6 for the proof). Since (3.9) yields $\tilde{\chi}(\rho) > 1,\ \rho < 0$, we conclude that for any $\ell \in \mathbb{N}$ the new estimator $\hat{\gamma}_n^{(2)}(s_2^*, \ell, r_2^*)$ dominates the estimator $\hat{\gamma}_n^{(1)}(s_1^*, \ell, r_1^*)$ in the whole region of parameters $\{(\gamma, \rho) :\ \gamma > 0, \rho < 0\}$. Whence the conclusion related to the estimators introduced in [17] follows, namely, $\hat{\gamma}_n^{(2)}(s_2^*, 1, r_2^*)$ outperforms $\hat{\gamma}_n^{(1)}(s_1^*, 1, r_1^*)$ in the whole region of parameters $\{(\gamma, \rho) :\ \gamma > 0, \rho < 0\}$.

Comparing the estimators $\hat{\gamma}_n^{(1)}(s_1^*, \ell, 0)$ (proposed in [18]) and $\hat{\gamma}_n^{(2)}(s_2^*, \ell, 0)$ we have

$$\lim_{n\to\infty} \frac{\mathrm{MSE}\left(\hat{\gamma}_n^{(1)}(s_1^*, \ell, 0)\right)}{\mathrm{MSE}\left(\hat{\gamma}_n^{(2)}(s_2^*, \ell, 0)\right)} = \left((1-\rho)2^{\rho}\right)^{2/(1-2\rho)}.$$

Whence we get that the estimator $\hat{\gamma}_n^{(2)}(s_2^*, \ell, 0)$ has a smaller asymptotic MSE in the area $\{(\gamma, \rho) :\ \gamma > 0, -1 < \rho < 0\}$, while the estimator $\hat{\gamma}_n^{(1)}(s_1^*, \ell, 0)$ is better in the area $\{(\gamma, \rho) :\ \gamma > 0, \rho < -1\}$.

We end this section with comparison of $\hat{\gamma}_n^{(2)}(s_2^*, \ell, r_2^*)$ and the Hill's estimator ([14]) that has the form

$$\gamma_n^{(3)}(k) = \frac{1}{k} \sum_{i=1}^{k} \ln\left(\frac{X_{n-i+1,n}}{X_{n-k,n}} \right),$$

where $X_{1,n} \le \cdots \le X_{n,n}$ denote the ascending order statistics of the observations $X_1, \ldots, X_n$.

By [12] we know that the optimal choice of the sample fraction k satisfies the relation

$$k^* \sim \left(\gamma^2(1-\rho)^2\right)^{1/(1-2\rho)} \frac{n}{b^{\leftarrow}(1/n)}, \quad n \to \infty,$$

while

$$\mathrm{MSE}\left(\gamma_n^{(3)}(k^*)\right) \sim \frac{1-2\rho}{-2\rho} \left(\frac{\gamma^{-4\rho}}{(1-\rho)^2} \right)^{1/(1-2\rho)} \frac{b^{\leftarrow}(1/n)}{n}, \quad n \to \infty$$

holds. Thus, we get immediately

$$\lim_{n\to\infty} \frac{\mathrm{MSE}\left(\gamma_n^{(3)}(k^*)\right)}{\mathrm{MSE}\left(\hat{\gamma}_n^{(2)}(s_2^*, \ell, r_2^*)\right)} = \upsilon(\ell, \rho),$$

where

$$\upsilon(\ell, \rho) = \left(\left(\frac{\ell}{2}\right)^{-2\rho} \left(\frac{1-2\rho}{1-\rho}\right)^{2-2\rho} \left(\frac{\Gamma(1+\ell)}{\Gamma(1+\ell-\rho)} \right)^2 \right)^{1/(1-2\rho)}.$$

Functions $\upsilon(3, \rho)$ and $\upsilon(5, \rho)$ are shown in Fig.3.1. Whence one can deduce that for $\ell \ge 3$ there exists $\tilde{\rho} = \tilde{\rho}(\ell) < 0$ such that the estimator $\hat{\gamma}_n^{(2)}(s_2^*, \ell, r_2^*)$ outperforms the Hill's estimator $\gamma_n^{(3)}(k^*)$ within the area $(\gamma, \rho) \in (0, \infty) \times (\tilde{\rho}, 0)$.

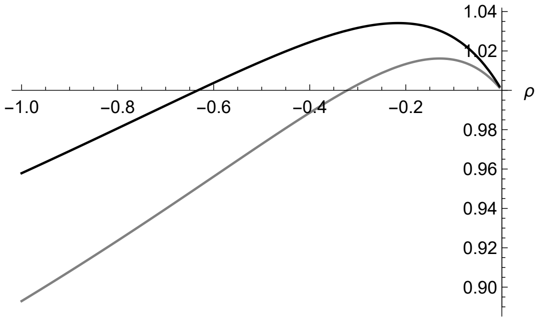

Fig. 3.1. Graph of of the functions $v(3,\rho)$ (grey) and $v(5,\rho)$ (black)

4. APPLICATIONS

We apply the estimators $\hat{\gamma}_n^{(i)}(s,\ell,0)$, $i=1,2$ with $\ell=1,3,5$ to real data sets. We analyze absolute log returns of two daily data sets: (i) natural gas prices (dollars USA per cubic feet) between 7 January 1997 and 1 September 2020; (ii) Europe Brent Spot Prices (dollars USA per barrel) between 20 May 1987 and 28 August 2020. The data sets contain $n=5389$ and $n=8261$ non-zero log returns, respectively. Let us denote the corresponding absolute log returns by $r_{1,t}$ and $r_{2,t}$. Fig. 1 displays graphs of $r_{1,t}$ (on the left) and $r_{2,t}$ (on the right).

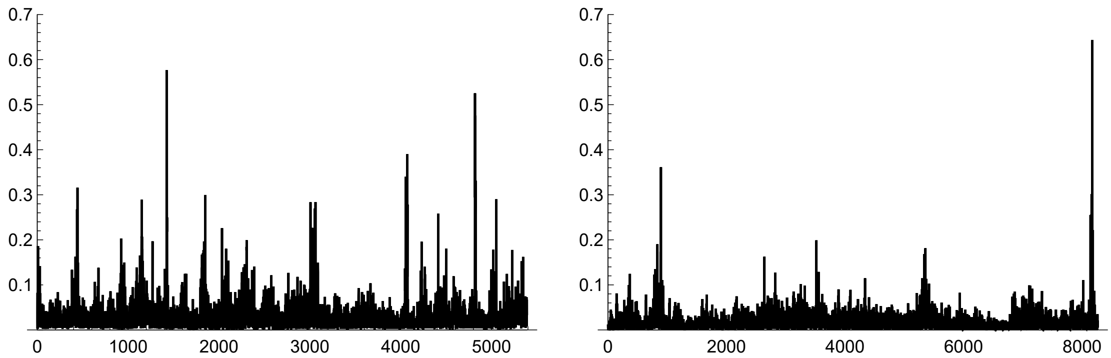

Fig. 4.2. Graphs of $r_{1,t}$, $1\le t\le 5389$ (on the left) and $r_{2,t}$, $1\le t\le 8261$ (on the right)

Let $r_i^{(1)}\ge r_i^{(2)}\ge\ldots$, $i=1,2$ be the order statistics in the decreasing order. We use QQ plots

$$\mathcal{T}^{(i)}(n,k)=\left\{\left(-\ln\left(\frac{j}{k}\right),\ln\left(\frac{r_i^{(j)}}{r_i^{(k)}}\right)\right),\quad 1\le j\le k\right\},\quad i=1,2$$

provided in [6] for the preliminary analyze. By Prop. 4.1 in [6] and assuming that the distribution of the non-zero $r_{i,t}$ belongs to the class $\mathrm{RV}_{-1/\gamma}$ and a sequence $k=k_n$ is such that $k\to\infty$, $n/k\to\infty$ as $n\to\infty$, the random sets $\mathcal{T}^{(i)}(n,s)$ converge in probability to the half-line $T^{(i)}=\left\{(x,\gamma^{(i)}x):\ 0\le x<\infty\right\}$. Fig. 2 shows the graph of $\mathcal{T}^{(1)}(n,k)$ (on the left) and $\mathcal{T}^{(2)}(n,k)$ (on the right) based on the upper 5% of the absolute log returns. The dotted least squares (LS) half-lines are presented in the corresponding graph. By Fig. 2, the absolute log returns for both data sets show a classic power law shape, with a slightly worse reflection at the largest values.

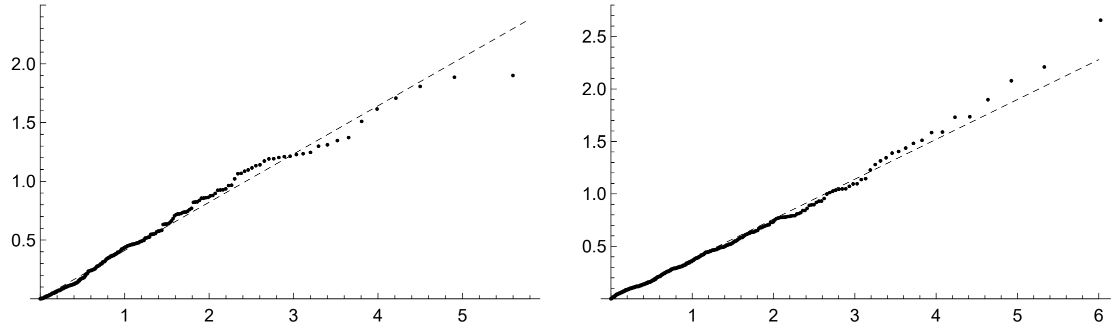

Fig. 4.3. Graphs of $\mathcal{T}^{(1)}\left(n^{(1)}, n^{(1)}/20\right)$ (on the left) and $\mathcal{T}^{(2)}\left(n^{(2)}, n^{(2)}/20\right)$ (on the right) with dotted LS half-lines

By considering the first data set we provide plots (suggested in [20])

$$\begin{aligned} \mathcal{P}_1(n,\ell) &= \left\{\left(\theta, \hat{\gamma}_n^{(1)}\left(\left[n^\theta\right], \ell, 0\right)\right),\ 0<\theta\leq 0.7\right\}, \\ \mathcal{P}_2(n,\ell) &= \left\{\left(\theta, \hat{\gamma}_n^{(2)}\left(\left[n^\theta\right], \ell, 0\right)\right),\ 0<\theta\leq 0.7\right\} \end{aligned}$$

for several ℓ values in Fig. 3-5. In practice, a stable region is visually detected in each plot and then an estimate of the parameter $\gamma > 0$ is taken as a mean over the stability region (shortly, the SR). The SR and the estimate of γ for each considered value of ℓ is given in Tab. 4.1. Moreover, the estimate of γ is presented in Fig. 3-5 as a dotted line.

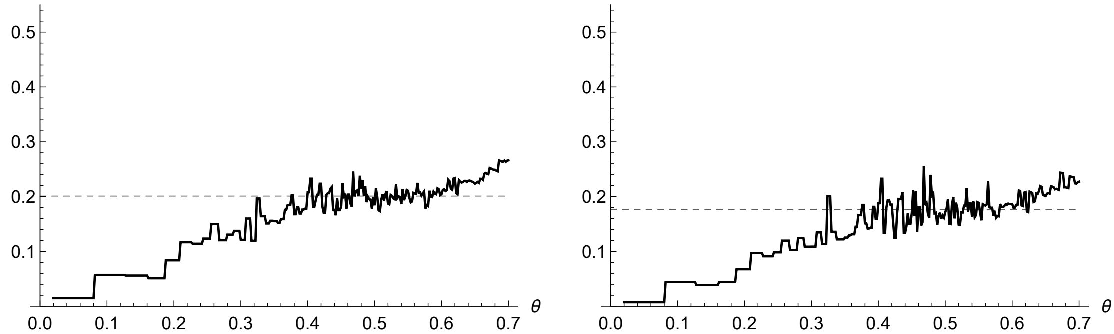

Fig. 4.4. Plots $\mathcal{P}_1(n,1)$ (on the left), $\mathcal{P}_2(n,1)$ (on the right) and the estimate of γ (dotted line)

The corresponding plots for the second data set are not represented due to their similarity to the plots in Fig. 3-5, but the corresponding results obtained by the SR and the estimate of γ are summarized in Tab. 4.2.

Table 4.1. Estimation results for the first data set with the size $n = 5389$

Plot	$\mathcal{P}_1(n,\ell)$			$\mathcal{P}_2(n,\ell)$		
ℓ	1	3	5	1	3	5
SR of θ	[0.48,0.6]	[0.34,0.56]	[0.36,0.54]	[0.48,0.6]	[0.34,0.56]	[0.36,0.54]
γ estimate	0.201	0.255	0.278	0.177	0.234	0.258

We end this section with several notings. Firstly, the estimates of $\gamma > 0$ with $\ell = 1$ are in close correspondence for both data sets, see Tab. 4.1 and 4.2. From the other hand, by

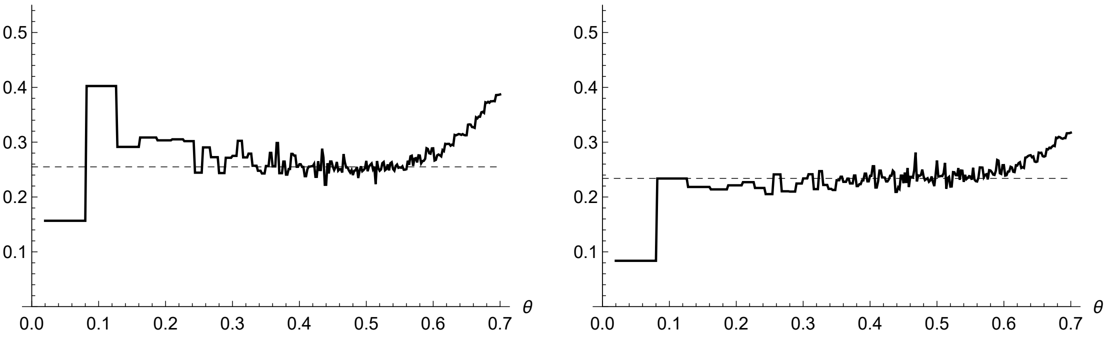

Fig. 4.5. Plots $\mathcal{P}_1\left(n,3\right)$ (on the left), $\mathcal{P}_2\left(n,3\right)$ (on the right) and estimate of γ (dotted line)

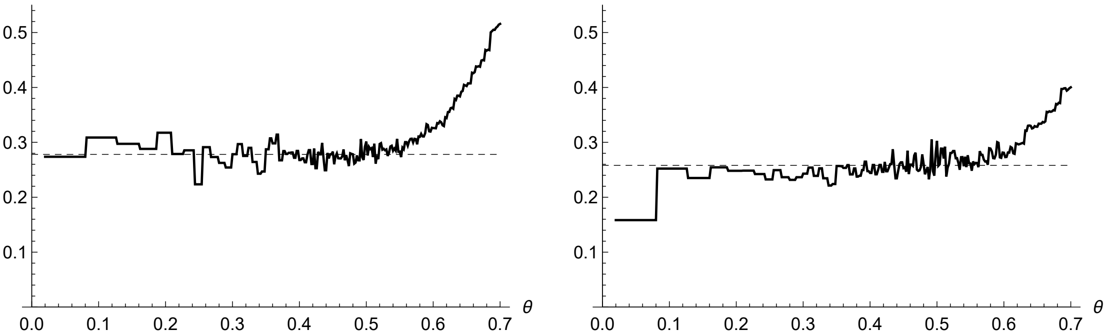

Fig. 4.6. Plots $\mathcal{P}_1\left(n,5\right)$ (on the left), $\mathcal{P}_2\left(n,5\right)$ (on the right) and estimate of γ (dotted line)

Table 4.2. Estimation results for the second data set with the size $n = 8261$

Plot	$\mathcal{P}_1\left(n,\ell\right)$			$\mathcal{P}_2\left(n,\ell\right)$		
ℓ	1	3	5	1	3	5
SR of θ	[0.32,0.64]	[0.32,0.56]	[0.34,0.52]	[0.32,0.64]	[0.32,0.55]	[0.34,0.52]
γ estimate	0.204	0.228	0.244	0.180	0.212	0.220

comparing the graphs in Fig. 1 one can observe that the left graph contains more intermediate size peaks than the right graph. This should be reflected on estimates of $\gamma > 0$. Such insensitivity of estimators $\hat{\gamma}_n^{(i)}(s,1,0)$, $i = 1,2$ in the practical applications is one more reason to use $\hat{\gamma}_n^{(i)}(s,\ell,0)$ with $\ell > 1$. Secondly, summarizing the results on stability region of parameter θ, we have $\theta \in [1/3, 1/2]$ for both data sets. This gives that the stability region for block number s is $\left[[n^{1/3}], [n^{1/2}]\right]$. Consequently, $\left[[n^{1/2}], [n^{2/3}]\right]$ is stability region for the block width m. Keeping in mind that $[5389^{1/2}] = 73$ and $[8261^{1/2}] = 90$, we believe that the block width is large enough in its stability region to take $\ell = 5$ for both data sets. Finally, we estimate the second order parameter ρ by the estimator proposed in [8] and get $\hat{\rho} = -0.69$ and $\hat{\rho} = -0.73$ for absolute log returns of the first and second data set, respectively. Since the estimator $\hat{\gamma}_n^{(2)}(s,5,0)$ is better than the estimator $\hat{\gamma}_n^{(1)}(s,5,0)$ when $-1 < \rho < 0$, we conclude that 0.258 and 0.220 are the estimates of γ for absolute log returns of the data sets under consideration.

5. CONCLUSIONS

The new family of block-type statistics $Q_n(s,\ell,r)$ is introduced. Under the classical assumptions on quantile type function U, block number $s=s_n$, block width $m=m_n$ and additionally on the parameter r, the weak consistency and the asymptotic normality (the bivariate central limit theorem) of the statistics $Q_n(s,\ell,r)$ are proved.

The statistics $Q_n(s,\ell,r)$ can be used to construct new block-type estimators for positive EVI. We list practical reasons for using the block type estimators. Firstly, as it is mentioned in Introduction, the block type estimators are applicable when only a few largest values are observed within blocks. Secondly, due to the recursiveness of the block-type estimators (see, pg. 17 in [15]) they are well suited for the on-line estimation of the positive EVI.

We proposed the new family of semiparametric estimators $\hat{\gamma}_n^{(2)}(s,\ell,r)$ for positive EVI. This family is based on statistics $Q_n(s,\ell,0)$ and $Q_n(s,\ell,r)$, $r\in\mathbb{R}$. Hence, the asymptotic normality of $\hat{\gamma}_n^{(2)}(s,\ell,r)$ is quite simple application of our bivariate central limit theorem for the statistics $Q_n(s,\ell,r)$, $r\in\mathbb{R}$. The optimal choices s_2^* and r_2^* for $s=s_n$ and r are proposed. Also, it is proved that for any $\ell\in\mathbb{N}$, the estimator $\hat{\gamma}_n^{(2)}(s_2^*,\ell,r_2^*)$ has a bigger asymptotic MSE comparing with $\hat{\gamma}_n^{(2)}(s_2^*,\ell+1,r_2^*)$. Despite that there is no an optimal estimator among estimators $\hat{\gamma}_n^{(2)}(s_2^*,\ell,r_2^*)$, $\ell\in\mathbb{N}$, we gave some heuristic argument that it is enough to take $3\le\ell\le 5$ considering a real data set.

Moreover, a new family of semiparametric estimators $\hat{\gamma}_n^{(1)}(s,\ell,r)$ is proposed. It includes several known families of estimators for positive EVI. New estimators $\hat{\gamma}_n^{(2)}$ are shown to be better than $\hat{\gamma}_n^{(1)}$ in the sense of the asymptotic MSE.

The performance of the estimators $\hat{\gamma}_n^{(1)}(s,\ell,0)$ and $\hat{\gamma}_n^{(2)}(s,\ell,r)$ is demonstrated by considering real data sets. We left numerous open questions related to the behavior of new estimators for middle size samples. We admit that quite extensive Monte-Carlo simulations are needed and we hope to fill this gap in the nearest future.

6. PROOFS

Proof of Lemma 3.2.

The inequality $\varphi(\ell+1,\rho)<\varphi(\ell,\rho)$ is equivalent to

$$\frac{-\rho}{\ell+1}<\left(1+\frac{1}{\ell}\right)^{-\rho}-1. \tag{6.10}$$

If $\rho=-1$, then (6.10) holds obviously. Let $0<-\rho<1$. By using a symmetric form of the Bernoulli's inequality (see, pg.5 in [3]) we obtain that the right hand side of (6.10) is not less than $(-\rho/\ell)(1+1/\ell)^{-1-\rho}$. It remains to check that for any $\ell\in\mathbb{N}$ the last quantity exceeds $-\rho/(\ell+1)$.

Let us consider the case $-\rho>1$. By applying the symmetric form of the Bernoulli's inequality one more time we get

$$\left(1+\frac{1}{\ell}\right)^{-\rho}-1>\frac{-\rho}{\ell}>\frac{-\rho}{\ell+1},\quad \ell\in\mathbb{N}.$$

This completes the proof of Lemma 3.2. □

Let $Y_1,Y_2,\dots,Y_m$ be i.i.d. Pareto r.v.s with $\mathrm{P}(Y_1\ge x)=1/x$, $x\ge 1$. Let $Y_{\ell,m}$, $1\le\ell\le m$ denote, as usual, the ℓ-th ascending order statistic. If $\ell\in\mathbb{N}$, then by combining Example

8.3.3 and Theorem 8.4.1 in [1], we get that $m^{-1}Y_{m-\ell,m}$ converges in distribution to some non-degenerate distribution as $m \to \infty$. The next lemma establishes the asymptotic behavior of the moments of $m^{-1}Y_{m-\ell,m}$.

Lemma 6.1:
Let $\zeta \in \mathbb{R}$ satisfy $\zeta < \ell + 1$, where $\ell \in \mathbb{N}$. Then it holds

$$\mathrm{E}\left(\frac{Y_{m-\ell,m}}{m}\right)^{\zeta} \to \frac{\Gamma(\ell-\zeta+1)}{\Gamma(\ell+1)}, \quad m \to \infty. \tag{6.11}$$

Proof
The trivial case $\zeta = 0$ does not require a proof. Let $\zeta > 0$. By using (2.2.2) in [1] we derive that the density function of $(m^{-1}Y_{m-\ell,m})^{\zeta}$ is

$$p(x) = \frac{m^{-(\ell+1)}m!}{\zeta\ell!(m-\ell-1)!}x^{-(\ell+1)/\zeta-1}\left(1-\frac{1}{mx^{1/\zeta}}\right)^{m-\ell-1}, \quad m^{-\zeta} \le x < \infty.$$

From the above density function we obtain the expectation of $(m^{-1}Y_{m-\ell,m})^{\zeta}$:

$$\begin{aligned}\mathrm{E}\left(\frac{Y_{m-\ell,m}}{m}\right)^{\zeta} &= \frac{m^{-(\ell+1)}m!}{\zeta\ell!(m-\ell-1)!}\int_{m^{-\zeta}}^{\infty} x^{-(\ell+1)/\zeta}\left(1-\frac{1}{mx^{1/\zeta}}\right)^{m-\ell-1}\mathrm{d}x \\ &= \frac{m^{-\zeta}m!}{\ell!(m-\ell-1)!}B(\ell-\zeta+1, m-\ell),\end{aligned}$$

where $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x$, $a > 0$, $b > 0$ is the beta function. The gamma and beta functions are related as $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$. Thus, we have

$$\mathrm{E}\left(\frac{Y_{m-\ell,m}}{m}\right)^{\zeta} = \left\{\frac{m^{-\zeta}\Gamma(m+1)}{\Gamma(m-\zeta+1)}\right\}\frac{\Gamma(\ell-\zeta+1)}{\Gamma(\ell+1)}.$$

By using the Stirling's formula for the gamma function one can verify that the quantity in the curly brackets tends to 1 as $m \to \infty$. For the rest case $\zeta < 0$ the proof of (6.11) is similar and thus, it is omitted. Lemma 6.1 is proved. □

Proof of the Theorem 2.1.
We use the Potter's bounds (see, e.g., Prop. B.1.9 in [13]). Namely, for any $\varepsilon > 0$ and $\delta > 0$ there exists $t_0 = t_0(\varepsilon, \delta)$, such that for $t \ge t_0$ and $x \ge 1$,

$$(1-\varepsilon)\,x^{\gamma-\delta} \le \frac{U(tx)}{U(t)} \le (1+\varepsilon)\,x^{\gamma+\delta}$$

holds. Without loss of generality let us assume that $0 < \varepsilon < 1$. Since the function h_r is strictly increasing at $(0, \infty)$ for any $r \in \mathbb{R}$, we get

$$h_r\left((1-\varepsilon)\,x^{\gamma-\delta}\right) \le h_r\left(\frac{U(tx)}{U(t)}\right) \le h_r\left((1+\varepsilon)\,x^{\gamma+\delta}\right),$$

or equivalently,

$$x^{\gamma r}h_r\left((1-\varepsilon)\,x^{-\delta}\right) \le h_r\left(\frac{U(tx)}{U(t)}\right) - h_r\left(x^{\gamma}\right) \le x^{\gamma r}h_r\left((1+\varepsilon)\,x^{\delta}\right). \tag{6.12}$$

Keeping in mind

$$U(Y_j^{(i)}) \stackrel{\mathrm{d}}{=} X_j^{(i)}, \quad 1 \le j \le \ell,\ 1 \le i \le s, \tag{6.13}$$

where $Y_1^{(1)}, \ldots, Y_\ell^{(1)}, \ldots, Y_1^{(s)}, \ldots, Y_\ell^{(s)}$ are i.i.d. Pareto r.v.s with $\mathrm{P}\left(Y_1^{(1)} > x\right) = 1/x$, $x \ge 1$, we have

$$Q_n(s,\ell,r) \stackrel{\mathrm{d}}{=} \frac{1}{s}\sum_{i=1}^{s}\frac{1}{\ell}\sum_{j=1}^{\ell} h_r\left(\frac{U\left(Y_{m-j+1,m}^{(i)}\right)}{U\left(Y_{m-\ell,m}^{(i)}\right)}\right).$$

The right endpoint of the Pareto distribution is equal to infinity. Thus, for any $1 \le i \le s$, $Y_{m-\ell,m}^{(i)} \xrightarrow{\mathrm{p}} +\infty$ as $m \to \infty$. Substituting

$$t = Y_{m-\ell,m}^{(i)}, \quad x = \frac{Y_{m-j+1,m}^{(i)}}{Y_{m-\ell,m}^{(i)}}, \tag{6.14}$$

into (6.12) we get

$$h_r\left(\frac{U\left(Y_{m-j+1,m}^{(i)}\right)}{U\left(Y_{m-\ell,m}^{(i)}\right)}\right) - h_r\left(\left(\frac{Y_{m-j+1,m}^{(i)}}{Y_{m-\ell,m}^{(i)}}\right)^{\gamma}\right)$$
$$\ge \left(\frac{Y_{m-j+1,m}^{(i)}}{Y_{m-\ell,m}^{(i)}}\right)^{\gamma r} h_r\left((1-\varepsilon)\left(\frac{Y_{m-j+1,m}^{(i)}}{Y_{m-\ell,m}^{(i)}}\right)^{-\delta}\right), \tag{6.15}$$

$$h_r\left(\frac{U\left(Y_{m-j+1,m}^{(i)}\right)}{U\left(Y_{m-\ell,m}^{(i)}\right)}\right) - h_r\left(\left(\frac{Y_{m-j+1,m}^{(i)}}{Y_{m-j,m}^{(i)}}\right)^{\gamma}\right)$$
$$\le \left(\frac{Y_{m-j+1,m}^{(i)}}{Y_{m-\ell,m}^{(i)}}\right)^{\gamma r} h_r\left((1+\varepsilon)\left(\frac{Y_{m-j+1,m}^{(i)}}{Y_{m-\ell,m}^{(i)}}\right)^{\delta}\right). \tag{6.16}$$

By summing inequalities (6.15), (6.16) over $1 \le j \le \ell$ and then over $1 \le i \le s$ we obtain

$$\tilde{Q}_n(s,\ell,r) - \Delta_n^{(-)}(s,\ell,r) \le Q_n(s,\ell,r) \le \tilde{Q}_n(s,\ell,r) + \Delta_n^{(+)}(s,\ell,r),$$

where

$$\begin{aligned}\tilde{Q}_n(s,\ell,r) &= \frac{1}{s}\sum_{i=1}^{s}\frac{1}{\ell}\sum_{j=1}^{\ell} h_r\left(\left(\frac{Y_{m-j+1,m}^{(i)}}{Y_{m-\ell,m}^{(i)}}\right)^{\gamma}\right),\\ \Delta_n^{(\pm)}(s,\ell,r) &= \frac{1}{s}\sum_{i=1}^{s}\frac{1}{\ell}\sum_{j=1}^{\ell}\left(\frac{Y_{m-j+1,m}^{(i)}}{Y_{m-\ell,m}^{(i)}}\right)^{\gamma r} h_r\left((1\pm\varepsilon)\left(\frac{Y_{m-j+1,m}^{(i)}}{Y_{m-\ell,m}^{(i)}}\right)^{\pm\delta}\right).\end{aligned}$$

Now it suffices to prove that

$$\tilde{Q}_n(s,\ell,r) \xrightarrow{\mathrm{p}} \Lambda_\gamma(\gamma r), \tag{6.17}$$
$$\Delta_n^{(\pm)}(s,\ell,r) \xrightarrow{\mathrm{p}} 0 \tag{6.18}$$

as $n \to \infty$. The Rényi's representation ([19]) enables us to write

$$\frac{1}{s}\sum_{i=1}^{s}\frac{1}{\ell}\sum_{j=1}^{\ell} g\left(\frac{Y^{(i)}_{m-j+1,m}}{Y^{(i)}_{m-\ell,m}}\right) \stackrel{\mathrm{d}}{=} \frac{1}{s}\sum_{i=1}^{s}\frac{1}{\ell}\sum_{j=1}^{\ell} g\left(Y^{(i)}_j\right), \tag{6.19}$$

where g is any measurable function, see (4.14) in [9]. By applying (6.19) we get

$$\tilde{Q}_n(s,\ell,r) \stackrel{\mathrm{d}}{=} \frac{1}{s}\sum_{i=1}^{s}\eta_i, \quad \eta_i := \frac{1}{\ell}\sum_{j=1}^{\ell} h_r\left(\left(Y^{(i)}_j\right)^{\gamma}\right).$$

Note that η_i, $1 \le i \le s$ is a sequence of i.i.d. r.v.s with $\mathrm{E}\eta_1 = \Lambda_\gamma(\gamma r)$. Thus, by applying the Khintchine weak law of large numbers the claim (6.17) follows.

Let $\delta > 0$ be small enough that such $\gamma(r+\delta) < 1$ holds. Similarly as in (6.17) we find

$$\Delta^{(\pm)}_n(s,\ell,r) \xrightarrow{\mathrm{P}} \begin{cases} h_0(1\pm\varepsilon) \pm \delta, & r = 0, \\ \frac{h_r(1\pm\varepsilon)(1-\gamma r)\pm\gamma\delta/r}{(1-\gamma(r\pm\delta))(1-\gamma r)}, & r \neq 0. \end{cases} \tag{6.20}$$

Let us recall that for any $r \in \mathbb{R}$, h_r is a continuous function on $(0,\infty)$. This, together with $h_r(1) = 0$, gives $h_r(1\pm\varepsilon) \to 0$ as $\varepsilon \downarrow 0$. So, for any $r \in \mathbb{R}$, the right hand side of (6.20) tends to 0 as $\varepsilon \downarrow 0$ and $\delta \downarrow 0$. Thus, the claim (6.18) follows. Theorem 2.1 is proved. □

Proof of the Theorem 2.2.
By using (6.13) one more time we get that for any $(\theta_1,\theta_2) \in \mathbb{R}^2$,

$$\theta_1\left(Q_n(s,\ell,r_1) - \Lambda_\gamma(\gamma r_1)\right) + \theta_2\left(Q_n(s,\ell,r_2) - \Lambda_\gamma(\gamma r_2)\right) \stackrel{\mathrm{d}}{=} T_n$$

holds, where

$$\begin{aligned} T_n \;:=\; & \frac{1}{s}\sum_{i=1}^{s}\frac{1}{\ell}\sum_{j=1}^{\ell}\theta_1\left(h_{r_1}\left(\frac{U(Y^{(i)}_{m-j+1,m})}{U(Y^{(i)}_{m-\ell,m})}\right) - \Lambda_\gamma(\gamma r_1)\right) \\ & + \theta_2\left(h_{r_2}\left(\frac{U(Y^{(i)}_{m-j+1,m})}{U(Y^{(i)}_{m-\ell,m})}\right) - \Lambda_\gamma(\gamma r_2)\right). \end{aligned}$$

By the Cramér-Wold Theorem (see, e.g., pg. 16 in [22]), the claim (2.4) will be proved if we show that

$$\sqrt{s_n}T_n \xrightarrow{\mathrm{d}} \theta_1\left(\mu\nu(\ell,\rho,r_1) + \frac{\Lambda_\gamma(\gamma r_1)}{\sqrt{\ell}}Z(\gamma r_1)\right) + \theta_2\left(\mu\nu(\ell,\rho,r_2) + \frac{\Lambda_\gamma(\gamma r_2)}{\sqrt{\ell}}Z(\gamma r_2)\right) \tag{6.21}$$

as $n \to \infty$.

The relation (2.2) is equivalent to

$$\lim_{t\to\infty}\frac{h_r\left((tx)^{-\gamma}U(tx)\right) - h_r\left(t^{-\gamma}U(t)\right)}{\left(t^{-\gamma}U(t)\right)^r A(t)} = h_\rho(x), \tag{6.22}$$

see [16] for details. By applying the Drees inequality ([7], see also Prop. 2.1 in [5]), the relation (6.22) implies that there exists the function $A_0(t)$, such that $A_0(t) \sim \left(t^{-\gamma}U(t)\right)^r A(t)$, $t \to \infty$ and for all $\varepsilon > 0$ and $\delta > 0$ there is $t_0 = t_0(\delta,\epsilon)$ such that for $t \ge t_0$ and $x \ge 1$,

$$\left|\frac{h_r\left((tx)^{-\gamma}U(tx)\right) - h_r\left(t^{-\gamma}U(t)\right)}{A_0(t)} - h_\rho(x)\right| \le \varepsilon x^{\rho+\delta}$$

holds, and consequently, it follows

$$h_r\left(\frac{U(tx)}{U(t)}\right) \geq h_r\left(x^\gamma\right) + x^{\gamma r} h_\rho(x) A_1(t) - \varepsilon x^{\gamma r+\rho+\delta} \left|A_1(t)\right|, \tag{6.23}$$

$$h_r\left(\frac{U(tx)}{U(t)}\right) \leq h_r\left(x^\gamma\right) + x^{\gamma r} h_\rho(x) A_1(t) + \varepsilon x^{\gamma r+\rho+\delta} \left|A_1(t)\right|, \tag{6.24}$$

where $A_1(t) = \left(t^\gamma U(t)\right)^{-r} A_0(t)$. We note that $A_1(t) \sim A(t)$ as $t \to \infty$. This, together with $|A(t)| \in \mathrm{RV}_\rho$, implies $|A_1(t)| \in \mathrm{RV}_\rho$.

Next, we apply (6.23), (6.24) with substitutions (6.14). Then we get, as in the proof of Theorem 2.1,

$$T_n^{(1)} + \theta_1 T_n^{(2)}(r_1) + \theta_2 T_n^{(2)}(r_2) - \varepsilon|\theta_1| T_n^{(3)}(r_1) - \varepsilon|\theta_2| T_n^{(3)}(r_2) \leq T_n, \tag{6.25}$$

$$T_n^{(1)} + \theta_1 T_n^{(2)}(r_1) + \theta_2 T_n^{(2)}(r_2) + \varepsilon|\theta_1| T_n^{(3)}(r_1) + \varepsilon|\theta_2| T_n^{(3)}(r_2) \geq T_n, \tag{6.26}$$

where

$$\begin{aligned}
T_n^{(1)} &= \frac{1}{s}\sum_{i=1}^{s}\frac{1}{\ell}\sum_{j=1}^{\ell}\theta_1\left(\gamma h_{\gamma r_1}\left(\frac{Y_{m-j+1,m}^{(i)}}{Y_{m-\ell,m}^{(i)}}\right) - \Lambda_\gamma\left(\gamma r_1\right)\right) \\
&\qquad + \theta_2\left(\gamma h_{\gamma r_2}\left(\frac{Y_{m-j+1,m}^{(i)}}{Y_{m-\ell,m}^{(i)}}\right) - \Lambda_\gamma\left(\gamma r_2\right)\right), \\
T_n^{(2)}(r) &= \frac{1}{s}\sum_{i=1}^{s} A_1\left(Y_{m-\ell,m}^{(i)}\right)\Upsilon_i(r), \\
\Upsilon_i(r) &= \frac{1}{\ell}\sum_{j=1}^{\ell}\left(\frac{Y_{m-j+1,m}^{(i)}}{Y_{m-\ell,m}^{(i)}}\right)^{\gamma r} h_\rho\left(\frac{Y_{m-j+1,m}^{(i)}}{Y_{m-\ell,m}^{(i)}}\right), \\
T_n^{(3)}(r) &= \frac{1}{s}\sum_{i=1}^{s}\left|A_1\left(Y_{m-\ell,m}^{(i)}\right)\right|\frac{1}{\ell}\sum_{j=1}^{\ell}\left(\frac{Y_{m-j+1,m}^{(i)}}{Y_{m-\ell,m}^{(i)}}\right)^{\gamma r+\rho+\delta}.
\end{aligned}$$

Let us prove the relation

$$\sqrt{s_n} T_n^{(1)} \xrightarrow{\mathrm{d}} \theta_1 \frac{\Lambda_\gamma(\gamma r_1)}{\sqrt{\ell}} Z(\gamma r_1) + \theta_2 \frac{\Lambda_\gamma(\gamma r_2)}{\sqrt{\ell}} Z(\gamma r_2), \quad n \to \infty. \tag{6.27}$$

By applying (6.19) we get

$$T_n^{(1)} \stackrel{\mathrm{d}}{=} \frac{1}{s}\sum_{i=1}^{s}\frac{1}{\ell}\sum_{j=1}^{\ell}\theta_1\left(\gamma h_{\gamma r_1}\left(Y_j^{(i)}\right) - \Lambda_\gamma\left(\gamma r_1\right)\right) + \theta_2\left(\gamma h_{\gamma r_2}\left(Y_j^{(i)}\right) - \Lambda_\gamma\left(\gamma r_2\right)\right). \tag{6.28}$$

The summands over i in the right hand side of (6.28) present a sequence of the i.i.d. zero mean r.v.s. Moreover, under assumptions $\gamma r_1 < 1/2$ and $\gamma r_2 < 1/2$ we have

$$\begin{aligned}
&\mathrm{E}\left(\frac{1}{\ell}\sum_{j=1}^{\ell}\theta_1\left(\gamma h_{\gamma r_1}\left(Y_j^{(1)}\right) - \Lambda_\gamma\left(\gamma r_1\right)\right) + \theta_2\left(\gamma h_{\gamma r_2}\left(Y_j^{(1)}\right) - \Lambda_\gamma\left(\gamma r_2\right)\right)\right)^2 \\
&\quad = \frac{1}{\ell}\big(\theta_1^2\Lambda_\gamma^2(\gamma r_1)\sigma^2(\gamma r_1,\gamma r_1) + 2\theta_1\theta_2\Lambda_\gamma(\gamma r_1)\Lambda_\gamma(\gamma r_2)\sigma^2(\gamma r_1,\gamma r_2) \\
&\qquad + \theta_2^2\Lambda_\gamma^2(\gamma r_2)\sigma^2(\gamma r_2,\gamma r_2)\big).
\end{aligned}$$

Thus, the relation (6.27) follows by applying the Lindeberg-Lévy central limit theorem (see, e.g., pg. 16 in [22]).

We claim that under assumption $\gamma r < 1/2$,

$$\sqrt{s_n} T_n^{(2)}(r) \xrightarrow{\mathrm{P}} \mu \cdot \nu(\ell, \rho, r), n \to \infty, \tag{6.29}$$

and, assuming additionally, $0 < \delta < 1 - \gamma r - \rho$,

$$\sqrt{s_n} T_n^{(3)}(r) \xrightarrow{\mathrm{P}} \frac{|\mu|}{1 - \gamma r - \rho - \delta}, \quad n \to \infty. \tag{6.30}$$

The relations (6.29), (6.30), together with (6.27) prove (6.21).

Let us prove that (6.29) holds. By the definition of the function A, there exists t_1, such that the function $A_1(t)$ has a constant sign when $t \geq t_1$. Let us recall that $|A_1| \in \mathrm{RV}_\rho$ holds. By applying the Drees inequalities for regularly varying functions (see, e.g., Prop. B.1.10 in [13]) we get that for any $\tilde{\varepsilon} > 0$ and $\tilde{\delta} > 0$ there exists $t_2 = t_2(\tilde{\epsilon}, \tilde{\delta})$, such that for $t \geq t_1 \vee t_2$ and $tx \geq t_1 \vee t_2$, it holds

$$-\tilde{\varepsilon} \max\left\{x^{\rho+\tilde{\delta}}, x^{\rho-\tilde{\delta}}\right\} \leq \frac{A_1(tx)}{A_1(t)} - x^\rho \leq \tilde{\varepsilon} \max\left\{x^{\rho+\tilde{\delta}}, x^{\rho-\tilde{\delta}}\right\}.$$

We apply the latter inequality with t replaced by m and x replaced by $Y_{m-\ell,m}^{(i)}/m$. Put

$$T_n^{(4)}(\rho, r) \quad = \quad \frac{1}{s} \sum_{i=1}^{s} \left(\frac{Y_{m-\ell,m}^{(i)}}{m} \right)^\rho \Upsilon_i(r).$$

As in the proof of Theorem 2.1, we get

$$-\tilde{\varepsilon} A_1(m) \max\left\{T_n^{(4)}(\rho - \tilde{\delta}, r), T_n^{(4)}(\rho + \tilde{\delta}, r)\right\} + A_1(m) T_n^{(4)}(\rho, r) \quad \leq \quad T_n^{(2)}(r), \tag{6.31}$$

$$\tilde{\varepsilon} A_1(m) \max\left\{T_n^{(4)}(\rho - \tilde{\delta}, r), T_n^{(4)}(\rho + \tilde{\delta}, r)\right\} + A_1(m) T_n^{(4)}(\rho, r) \quad \geq \quad T_n^{(2)}(r), \tag{6.32}$$

when $A_1(m) > 0$ for $m \geq t_1 \vee t_2$ and the reverse inequalities to (6.31), (6.32) hold in the case $A_1(m) < 0$ for $m \geq t_1 \vee t_2$.

Since assumption (2.3) implies $\sqrt{s} A_1(m) \to \mu$ as $n \to \infty$, the relation (6.29) will be proved if we show that for any $\tilde{\rho} \in \mathbb{R}$ satisfying $2\tilde{\rho} < \ell + 1$,

$$T_n^{(4)}(\tilde{\rho}, r) \xrightarrow{\mathrm{P}} \nu(\ell, \tilde{\rho}, r), \quad n \to \infty. \tag{6.33}$$

As for relation (6.33), it follows from

$$T_n^{(4)}(\tilde{\rho}, r) - \mathrm{E}\left(\left(\frac{Y_{m-\ell,m}^{(1)}}{m} \right)^{\tilde{\rho}} \Upsilon_1(r) \right) \quad \xrightarrow{\mathrm{P}} \quad 0, \tag{6.34}$$

$$\mathrm{E}\left(\left(\frac{Y_{m-\ell,m}^{(1)}}{m} \right)^{\tilde{\rho}} \Upsilon_1(r) \right) \quad \to \quad \nu(\ell, \tilde{\rho}, r), \quad n \to \infty. \tag{6.35}$$

We will prove

$$\mathrm{E}\left(T_n^{(4)}(\tilde{\rho}, r) - \mathrm{E}\left(\left(\frac{Y_{m-\ell,m}^{(1)}}{m} \right)^{\tilde{\rho}} \Upsilon_1(r) \right) \right)^2 \to 0, \quad n \to \infty, \tag{6.36}$$

which, in turn, will complete the proof of (6.34), see Section 8.3 in [2]. By using the Renyi's representation one can get that for any $1 \le i \le s$, $Y^{(i)}_{m-\ell,m}$ and $\Upsilon_i(r)$ are independent r.v.s. Thus, the left hand side of (6.36) is equal to

$$\begin{aligned}&\frac{1}{s}\left(\mathrm{E}\left(\frac{Y^{(1)}_{m-\ell,m}}{m}\right)^{2\tilde\rho}\mathrm{E}\left(\Upsilon_1(r)\right)^2-\mathrm{E}^2\left(\frac{Y^{(1)}_{m-\ell,m}}{m}\right)^{\tilde\rho}\mathrm{E}^2\left(\Upsilon_1(r)\right)\right)\\&\le\frac{2}{s}\mathrm{E}\left(\frac{Y^{(1)}_{m-\ell,m}}{m}\right)^{2\tilde\rho}\mathrm{E}\left(\Upsilon_1(r)\right)^2.\end{aligned}$$

By our assumptions $s = s_n \to \infty$ holds as $n \to \infty$. This, together with $\mathrm{E}\left(Y^{(1)}_{m-\ell,m}/m\right)^{2\tilde\rho} \to \Gamma(\ell-2\tilde\rho+1)/\Gamma(\ell+1)$ as $n\to\infty$ (see Lemma 6.1) and $\mathrm{E}\left(\Upsilon_1(r)\right)^2<\infty$ yields (6.36). To prove that Υ_1 is square integrable r.v. it is enough to use (6.19) and then to apply a direct integration.

It rests to prove (6.35). Since $Y^{(1)}_{m-\ell,m}$ and $\Upsilon_1(r)$ are independent r.v.s we get that the left hand side of (6.35) is equal to $\mathrm{E}\left(Y^{(1)}_{m-\ell,m}/m\right)^{\tilde\rho}\mathrm{E}\left(\Upsilon_1(r)\right)$. Now (6.35) follows by applying Lemma 6.1 and (6.19) one more time.

The proof of (6.30) is similar to the proof of (6.29) and thus, it is omitted. This completes the proof of Theorem 2.2. □

Proof of Remark 1.
With the notation of the previous proof, we have $T^{(1)}_n/A(m) = \sqrt{s}T^{(1)}_n/\left(\sqrt{s}A(m)\right)$, where we take $\theta_1 = 1$ and $\theta_2 = 0$ in the definition of $T^{(1)}_n$. By combining (2.5) and (6.27) we get $T^{(1)}_n/A(m) \xrightarrow{\mathrm{P}} 0$ as $n\to\infty$. From the proof of Theorem 2.2 it follows that $T^{(2)}_n(r_1) \xrightarrow{\mathrm{P}} \nu(\ell,\rho,r_1)$, $n\to\infty$ and the sequence $T^{(3)}_n(r_1)$, $n=1,2,\ldots$ is bounded in probability. Thus, keeping in mind inequalities (6.25), (6.26), the relation (2.6) follows. This completes the proof. □

Proof of Theorem 3.1.
Let $r \neq 0$. We have

$$\hat\gamma^{(2)}_n(s,\ell,r)-\gamma = \frac{(1-\gamma r)\left(Q_n(s,\ell,r)-\Lambda_\gamma(\gamma r)\right)-\left(Q_n(s,\ell,0)-\Lambda_\gamma(0)\right)}{rQ_n(s,\ell,r)}. \qquad (6.37)$$

By Theorem 2.2, the numerator of the right hand side of (6.37), multiplied by $\sqrt{s_n}$, converges in distribution to

$$\mathcal{N}\left(\frac{\mu\gamma r}{(1-\rho)(1-\gamma r-\rho)}\cdot\frac{\Gamma(1+\ell-\rho)}{\Gamma(1+\ell)},\frac{2\gamma^4r^2}{\ell(1-\gamma r)(1-2\gamma r}\right).$$

As the denominator, it tends in probability to $r\Lambda_\gamma(\gamma r)$ as $n\to\infty$. Now (3.7) follows by applying the Slutsky's lemma (see, e.g., pg. 11 in [22]).

Let $r = 0$. The similar argument used in proving (2.4) shows that

$$\sqrt{s}\left(Q_n(s,\ell,0)-\gamma,\tilde Q(s,\ell)-2\gamma^2\right)\xrightarrow{\mathrm{d}}\frac{\mu\Gamma(1+\ell-\rho)}{(1-\rho)\Gamma(1+\ell)}\left(1,\frac{2\gamma(2-\rho)}{1-\rho}\right)+(\Pi_1,\Pi_2)$$

holds as $n\to\infty$. Here, μ is the same as in (2.3), while (Π_1,Π_2) is a Gaussian random vector with zero means and $\mathrm{E}\Pi_1^2=\gamma^2/\ell$, $\mathrm{E}\Pi_2^2=20\gamma^4/\ell$, $\mathrm{E}\left(\Pi_1\Pi_2\right)=4\gamma^3/\ell$. Now, by using the identity

$$\hat{\gamma}_n^{(2)} - \gamma = \frac{\left(\tilde{Q}(s,\ell) - 2\gamma^2\right) - 2\gamma\left(Q_n(s,\ell,0) - \gamma\right)}{2Q_n(s,\ell,0)},$$

it is not difficult to verify that (3.7) holds with $r = 0$. This ends the proof. □

Proof of Corollary 2.3.
This follows by a similar argument as used in proving Theorem 3.1. □

Proof of the inequality (3.9).
Note that $(\tilde{\chi}(\rho))^{(1-2\rho)/(-2\rho)}$ can be rewritten as follows

$$(\tilde{\chi}(\rho))^{(1-2\rho)/(-2\rho)} = \left(\frac{(\chi(\rho)+\rho)^2}{2}\right)^{1-1/\rho} \left(\frac{1}{2} + \frac{\chi(\rho)}{2(1-\rho)}\right),$$

where $\chi(\rho) = (\rho^2 - 4\rho + 2)^{1/2}$. It is easy to check that $\chi(\rho) > \sqrt{2} - \rho$ and, consequently, $\chi(\rho) > 1 - \rho$ hold for any $\rho < 0$. Thus, (3.9) follows. □

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