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Stress-Strength Reliability of a Weibull-Standard Normal Distribution Based on Type-II Progressive Censored Samples

Ramin Kazemi

Department of Statistics, Imam Khomeini International University, Qazvin, Iran

Abstract: In this paper, under the Type-II progressive censored scheme, we obtain the point and interval estimates of stress-strength parameter (R), when stress and strength are two independent Weibull-standard normal variables. We study the problem in three cases. First, assuming that stress and strength have the different scale parameters and the common shape parameter, we obtain maximum likelihood estimation, approximation maximum likelihood estimation and two Bayesian approximation estimates due to the lack of explicit forms. Also, we construct the asymptotic and highest posterior density intervals for R. Second, assuming that common shape parameter is known, we derive the maximum likelihood estimation and Bayes estimate and uniformly minimum variance unbiased estimate of R. Third, assuming that all parameters are unknown and different, we achieve the statistical inference of R, namely maximum likelihood estimation, approximation maximum likelihood estimation and Bayesian inference of R. Furthermore, we use the Monte Carlo simulations to compare of the performance of different methods.

Keywords: Stress-strength model, Type-II progressive censored sample, Weibull-standard normal distribution, Bayesian inference, Monte Carlo simulation.

1. INTRODUCTION

One method of introducing distributions is to generalize previous distributions. In fact, the new distribution includes the previous distributions for different values of the parameters. Another method to introduce new distributions is to put a specific function of distribution function of a random variable into the distribution function of another random variable. The behavior of the distribution and hazard functions of these new random variables determine the importance of studing them. Various techinqes have been introduced, especially in recent years, to produce continuous distributions with the second method. If the support of these distributions is positive, they can be used in reliability analysis.

The Weibull distribution is one of the most widely used distributions in the reliability and survival studies. Some works on the stress-strength model of Weibull distribution and its related distribution under censored data can be found in [1, 10, 11] and some references therein. Consider a continuous distribution G and the Weibull cumulative density function (c.d.f.) $F_X(x) = 1 - e^{-ax^c}(x > 0)$ with positive parameters a and c. Based on this density, by replacing x with $G(x)/(1 - G(x))$, we can define the c.d.f. family by

$$F_X(x|a, c, \theta) = 1 - \exp\left(- a\left(\frac{G(x;\theta)}{1 - G(x;\theta)}\right)^c\right), \quad x \in D \subset \mathbb{R}, a, c > 0,$$

where $G(x;\theta)$ is a baseline c.d.f., which depends on a parameter vector θ. Henceforth, let G be a continuous baseline distribution. For each G distribution, the Weibull-G distribution with

*Corresponding author: r.kazemi@sci.ikiu.ac.ir

two extra parameters a and c is defined by the above c.d.f. [3]. For the normal distribution $n(\mu, \sigma^2)$,

$$\frac{G(x;\mu,\sigma)}{1-G(x;\mu,\sigma)} = \frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)},$$

where $\Phi(\cdot)$ is the c.d.f. of the standard normal distribution $n(0,1)$. Then, the Weibull-normal distribution (WND) is defined by

$$F(x) = 1-\exp\left\{-a\left(\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right)^c\right\}, \quad x,a,c,\sigma>0, \mu\in\mathbb{R}.$$

Thus, WND with scale parameter a and shape parameter c, respectively, which denoted by WN(a,c,μ,σ^2), has the probability density function (p.d.f.), and failure rate function as follows:

$$f(x) = \frac{ac\,\phi\left(\frac{x-\mu}{\sigma}\right)}{\sigma}\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)^{c-1}}{\left(1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right)^{c+1}}\exp\left\{-a\left(\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right)^c\right\},$$

$$h(x) = \frac{ac\,\phi\left(\frac{x-\mu}{\sigma}\right)}{\sigma}\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)^{c-1}}{\left(1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right)^{c+1}}, \quad x,a,c,\sigma>0,\ \mu\in\mathbb{R}$$

respectively.

In the stress-strength modelling, $R = P(X<Y)$ is a measure of component reliability when it is subjected to random stress X and has strength Y. Note that if X and Y are two independent random variables from $WN(a,c,\mu,\sigma^2)$ and $WN(b,c,\mu,\sigma^2)$ distributions, respectively, then with the change of variable

$$u = \left(\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right)^c,$$

we have

$$R = \int_{-\infty}^{+\infty} f_X(x)(1-F_Y(x))dx = \int_0^\infty ae^{-(a+b)u}du = \frac{a}{a+b}.$$

Because the our purpose is to analysis of the strees-strength reliability we focus on the standard normal distribution (SN) case. The reason for this is R is a function of a and b. Thus, for the WSND,

$$F_{\text{WSN}}(x) = 1-\exp\left\{-a\left(\frac{\Phi(x)}{1-\Phi(x)}\right)^c\right\},$$

$$f_{\text{WSN}}(x) = ac\,\phi(x)\frac{\Phi(x)^{c-1}}{\left(1-\Phi(x)\right)^{c+1}}\exp\left\{-a\left(\frac{\Phi(x)}{1-\Phi(x)}\right)^c\right\},$$

$$h_{\text{WSN}}(x) = ac\,\phi(x)\frac{\Phi(x)^{c-1}}{\left(1-\Phi(x)\right)^{c+1}}, \quad x,a,c>0.$$

It has been used very effectively for analyzing lifetime data, particularly when the data are censored. Among various censoring schemes, the Type II progressive censoring scheme has become very popular one in the last decade. It can be described as follows: Consider N units are placed under a study and only $n(< N)$ units are completely observed until failure. At the time of the first failure (the first stage), r_1 of the $N-1$ surviving units are randomly withdrawn (censored intentionally) from the experiment. At the time of the second failure (the second stage), r_2 of the $N-2-r_1$ surviving units are withdrawn and so on. Finally, at the time of the nth failure (the nth stage), all the remaining $r_n = N-n-r_1-...-r_{n-1}$ surviving units are withdrawn. We will refer to this as progressive Type-II right censoring with scheme $(r_1, r_2, ..., r_n)$. It is clear that this scheme includes the conventional Type-II right censoring scheme (when $r_1 = r_2 = ... = r_{n-1} = 0$ and $r_n = N-n$) and complete sampling scheme (when $N = n$ and $r_1 = r_2 = ... = r_n = 0$). For further details on progressively censoring and relevant references, the reader may refer to the book by Balakrishnan and Aggarwala [2].

Although, in complete sample case, many authors have been studied the stress-strength models, much attention has not been paid to censored sample case. Whereas in really applicable situations, for many reasons such as financial plane or limited time, the researchers confront censored data.

In this paper, based on Type-II progressive censoring, the reliability parameter $R = P(X < Y)$ is estimated, when X and Y are two independent random variables from the WSN distribution (WSND).

The rest of this paper is arranged as follows. In Section 2, under the Type-II progressive censoring, assuming $X \sim WSN(a, c)$ and $Y \sim WSN(b, c)$, we obtain the point and interval estimates of $R = P(X < Y)$, from the frequentist and Bayesian viewpoints. Because the maximum likelihood estimations (MLEs) of unknown parameters and R cannot be earned in the closed forms, we obtain the approximation maximum likelihood estimations (AMLEs) of parameters and R which have the explicit forms. Also, we develop the Bayes estimates of R, using Lindley's approximation and MCMC method due to the lack of explicit forms. Moreover, different confidence intervals such as asymptotic and HPD intervals of R are provided. In Section 3, assuming the common shape parameter is known, the MLE and Bayes estimate and uniformly minimum variance unbiased estimate (UMVUE) of R are earned. Because the assumption which we study in Section 2 is quite strong, we consider the statistical inference of R in general case. So, in Section 4, under the Type-II progressive censoring scheme, assuming $WSN(a, c)$ and $WSN(b, d)$, we provide the MLE, AMLE and Bayes estimate of R. In Section 5, we give the simulation results and conclude the paper in Section 6.

2. INFERENCE ON R WITH UNKNOWN COMMON c

2.1. Maximum likelihood estimation of R

Assume that X and Y are two independent random variables from $WSN(a, c) \equiv WN(a, c, 0, 1)$ and $WSN(b, c) \equiv WN(b, c, 0, 1)$ distributions, respectively. In this section, under the Type-II progressive censoring, we derive the MLE of R. Because R is a function of the unknown parameters, first we obtain the MLEs of a, b and c. Suppose $\mathbf{X} = (X_{1:N}, X_{2:N}, ..., X_{n:N})$ is a progressively Type-II censored sample from $WSN(a, c)$ with censored scheme $\mathbf{r} = (r_1, r_2, ..., r_n)$ and $\mathbf{Y} = (Y_{1:M}, Y_{2:M}, ..., Y_{m:M})$ is a progressively Type-II censored sample from $WSN(b, c)$ with censored scheme $\mathbf{s} = (s_1, s_2, ..., s_m)$. For notation simplicity, we will write $(X_1, X_2, ..., X_n)$ for $(X_{1:N}, X_{2:N}, ..., X_{n:N})$ and $(Y_1, Y_2, ..., Y_m)$ for $(Y_{1:M}, Y_{2:M}, ..., Y_{m:M})$. Therefore, the likelihood function of the unknown parameters a, b and

c, can be written as

$$L(a,b,c) = \left[k_1 \prod_{i=1}^{n} f_{\text{WSN}}(x_i)[1 - F_{\text{WSN}}(x_i)]^{r_i}\right] \times \left[k_2 \prod_{j=1}^{m} f_{\text{WSN}}(y_j)[1 - F_{\text{WSN}}(y_j)]^{s_j}\right],$$

where $k_1 = N(N-1-r_1)(N-2-r_1-r_2)...(N-n+1-r_1-...-r_{n-1})$ and $k_2 = M(M-1-s_1)(M-2-s_1-s_2)...(M-m+1-s_1-...-s_{m-1})$. Based on the observed data, the likelihood function can be obtained as:

$$L(\text{data}|a,b,c) = k_1 k_2 (ac)^n \left(\prod_{i=1}^{n} \phi(x_i) \frac{\Phi(x_i)^{c-1}}{\Big(1-\Phi(x_i)\Big)^{c+1}}\right) \exp\left\{ -a \sum_{i=1}^{n} (r_i+1) \left(\frac{\Phi(x_i)}{1-\Phi(x_i)}\right)^c \right\}$$
$$\times (bc)^m \left(\prod_{j=1}^{m} \phi(y_j) \frac{\Phi(y_j)^{c-1}}{\Big(1-\Phi(y_j)\Big)^{c+1}}\right) \exp\left\{ -b \sum_{j=1}^{m} (s_j+1) \left(\frac{\Phi(y_j)}{1-\Phi(y_j)}\right)^c \right\}.$$

Therefore, the log-likelihood function is:

$$\begin{aligned}
\ell(a,b,c) &= \text{Constant} + n\log(a) + m\log(b) + (n+m)\log(c) \\
&+ \sum_{i=1}^{n} \log(\Phi(x_i)) + (c-1)\sum_{i=1}^{n} \log(\Phi(x_i)) - (c+1)\sum_{i=1}^{n} \log(1-\Phi(x_i)) \\
&+ \sum_{j=1}^{m} \log(\Phi(y_j)) + (c-1)\sum_{j=1}^{m} \log(\Phi(y_j)) - (c+1)\sum_{j=1}^{m} \log(1-\Phi(y_j)) \\
&- a\sum_{i=1}^{n} (r_i+1)\left(\frac{\Phi(x_i)}{1-\Phi(x_i)}\right)^c - b\sum_{j=1}^{m} (s_j+1)\left(\frac{\Phi(y_j)}{1-\Phi(y_j)}\right)^c. \qquad (2.1)
\end{aligned}$$

Set

$$w(x,t,c,k) := (t+1)\left(\frac{\Phi(x)}{1-\Phi(x)}\right)^c \log^k\left(\frac{\Phi(x)}{1-\Phi(x)}\right), \quad x>0, k \in \mathbb{N} \cup \{0\}.$$

So, to earn the MLEs of a, b and c, namely, $\widehat{a}$, $\widehat{b}$ and $\widehat{c}$, respectively, we should solve the following equations:

$$\frac{\partial \ell}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n} w(x_i, r_i, c, 0) = 0, \qquad (2.2)$$

$$\frac{\partial \ell}{\partial b} = \frac{m}{b} - \sum_{j=1}^{m} w(y_j, s_j, c, 0) = 0, \qquad (2.3)$$

$$\begin{aligned}
\frac{\partial \ell}{\partial c} &= \frac{m+n}{c} + \sum_{i=1}^{n} \log\left(\frac{\Phi(x_i)}{1-\Phi(x_i)}\right) + \sum_{j=1}^{m} \log\left(\frac{\Phi(y_j)}{1-\Phi(y_j)}\right) \\
&\quad - a\sum_{i=1}^{n} w(x_i, r_i, c, 1) - b\sum_{j=1}^{m} w(y_j, s_j, c, 1) = 0. \qquad (2.4)
\end{aligned}$$

From the equations (2.2) and (2.3), we have

$$\widehat{a}(c) = n\bigg(\sum_{i=1}^{n}(r_i+1)\bigg(\frac{\Phi(x_i)}{1-\Phi(x_i)}\bigg)^{c}\bigg)^{-1} = n\bigg(\sum_{i=1}^{n} w(x_i, r_i, c, 0)\bigg)^{-1},$$
$$\widehat{b}(c) = m\bigg(\sum_{j=1}^{m}(s_j+1)\bigg(\frac{\Phi(y_j)}{1-\Phi(y_j)}\bigg)^{c}\bigg)^{-1} = m\bigg(\sum_{j=1}^{m}(w(y_j, s_j, c, 0)\bigg)^{-1}.$$

Also, to derive $\widehat{c}$, we apply one numerical method such as Newton-Raphson on the equation (2.4). After obtaining the MLEs of a, b and c, by using the invariance property, the MLE of R can be derived as

$$\widehat{R}^{\text{MLE}} = \frac{\widehat{a}}{\widehat{a}+\widehat{b}}. \tag{2.5}$$

2.2. Approximation maximum likelihood estimation of R

From the Section 2.1, we see that the MLEs of unknown parameters and R cannot be earned in the closed forms. So in this section, we obtain the AMLEs of the parameters which have the explicit forms. Let Z' and Z'' be Weibull and Extreme value distributions, in symbols $Z' \sim W(a,\theta)$ and $Z'' \sim EV(\xi,\sigma)$, if they have the following cumulative distribution functions respectively as:

$$F_{Z'}(z) = 1 - e^{-\frac{z^a}{\theta}}, \qquad z>0,\ \ a,\theta>0,$$
$$F_{Z''}(z) = 1 - e^{-e^{\frac{z-\xi}{\sigma}}}, \quad z \in \mathbb{R},\ \ \xi\in\mathbb{R}, \sigma>0.$$

The following simple theorem is critical to obtain the AMLEs of the parameters.

Theorem 2.1: *(i) If $Z \sim WSN(a,c)$, then*

$$Z' = \frac{\Phi(Z)}{1-\Phi(Z)} \sim W(c, 1/a).$$

(ii) If $Z' \sim W(c,1/a)$ and $Z'' = \log(Z')$, then $Z'' \sim EV(\xi,\sigma)$, where

$$\xi = -\frac{1}{c}\log(a) \ \ and \ \ \sigma = \frac{1}{c}.$$

Proof
The proof is obvious. □ □

Suppose that $\{X_1,\ldots,X_n\}$ and $\{Y_1,\ldots,Y_m\}$ be two Type-II progressive censoring samples with the above censoring schemes and

$$X_i' = \frac{\Phi(x_i)}{1-\Phi(x_i)}, \quad U_i = \log(X_i'),$$
$$Y_j' = \frac{\Phi(y_j)}{1-\Phi(y_j)}, \quad V_j = \log(Y_j').$$

Applying Theorem 2.1, we have $U_i \sim EV(\xi_1,\sigma)$ and $V_j \sim EV(\xi_2,\sigma)$, where

$$\xi_1 = -\frac{1}{c}\log(a),\ \xi_2 = -\frac{1}{c}\log(b), \quad \sigma = \frac{1}{c}.$$

After earning $\tilde{\xi}_1$, $\tilde{\xi}_2$ and $\tilde{\sigma}$, the values of $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ can be evaluated by (see Appendix A):

$$\tilde{a} = e^{-\frac{\tilde{\xi}_1}{\tilde{\sigma}}}, \quad \tilde{b} = e^{-\frac{\tilde{\xi}_2}{\tilde{\sigma}}}, \quad \tilde{c} = \frac{1}{\tilde{\sigma}}.$$

So, the AMLE of R, namely $\tilde{R}$, is

$$\tilde{R} = \frac{\tilde{a}}{\tilde{a} + \tilde{b}}. \tag{2.6}$$

2.3. Asymptotic confidence interval

In this section, we earn the asymptotic confidence interval of R by the asymptotic distribution of $\widehat{R}$, which obtain from the asymptotic distribution of $\widehat{\lambda} = (\widehat{a}, \widehat{b}, \widehat{c})$. We denote the observed Fisher information matrix by $I(\lambda) = [I_{ij}] = \left[-\frac{\partial^2 \ell}{\partial\lambda_i \partial\lambda_j}\right]$, $i, j = 1, 2, 3$. By differentiating twice from (2.1) with respect to a, b and c, the inlines of $I(\lambda)$ matrix can be obtained as:

$$\begin{aligned}
&I_{11} = \frac{n}{a^2}, \;\; I_{22} = \frac{m}{b^2}, \quad I_{12} = I_{21} = 0, \\
&I_{13} = I_{31} = \sum_{i=1}^{n} w(x_i, r_i, c, 1), \;\; I_{23} = I_{32} = \sum_{j=1}^{m} w(y_j, s_j, c, 1), \\
&I_{33} = \frac{m+n}{c^2} + a\sum_{i=1}^{n} w(x_i, r_i, c, 2) + b\sum_{j=1}^{m} w(y_j, s_j, c, 2).
\end{aligned}$$

Theorem 2.2:
Let $\widehat{a}, \widehat{b}$ and $\widehat{c}$ be the MLEs of a, b and c. Then

$$[(\widehat{a} - a) \;\; (\widehat{b} - b) \;\; (\widehat{c} - c)]^T \xrightarrow{D} N_3(0, \mathbf{I^{-1}}(a, b, c)),$$

where $\mathbf{I}(a, b, c)$ and $\mathbf{I}^{-1}(a, b, c)$ are symmetric matrices and

$$\mathbf{I}(a,b,c) = \begin{pmatrix} I_{11} & 0 & I_{13} \\ & I_{22} & I_{23} \\ & & I_{33} \end{pmatrix}, \;\; \mathbf{I}^{-1}(a,b,c) = \frac{1}{|\mathbf{I}(a,b,c)|}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ & b_{22} & b_{23} \\ & & b_{33} \end{pmatrix},$$

in which $|\mathbf{I}(a, b, c)| = I_{11}I_{22}I_{33} - I_{11}I_{23}^2 - I_{13}^2 I_{22}$,

$$\begin{aligned}
b_{11} &= I_{22}I_{33} - I_{23}^2, \quad & b_{12} &= I_{13}I_{23}, \quad & b_{13} &= -I_{13}I_{22}, \\
b_{22} &= I_{11}I_{33} - I_{13}^2, \quad & b_{23} &= -I_{11}I_{23}, \quad & b_{33} &= I_{11}I_{22}.
\end{aligned}$$

Proof
From the asymptotic normality of the MLE, the theorem resulted. □ □

Theorem 2.3:
Let $\widehat{R}^{MLE}$ be the MLE of R. Then,

$$(\widehat{R}^{MLE} - R) \xrightarrow{D} N(0, B),$$

where

$$B = \frac{1}{|\mathbf{I}(a,b,c)|}\left[\Big(\frac{\partial R}{\partial a}\Big)^2 b_{11} + \Big(\frac{\partial R}{\partial b}\Big)^2 b_{22} + 2\Big(\frac{\partial R}{\partial a}\Big)\Big(\frac{\partial R}{\partial b}\Big) b_{12}\right]. \tag{2.7}$$

Proof
Using Theorem 2.2 and applying delta method, the asymptotic distribution of $\widehat{R} = \frac{\widehat{a}}{\widehat{a}+\widehat{b}}$ can be obtained as follows:

$$(\widehat{R}^{MLE} - R) \xrightarrow{D} N(0, B),$$

where $B = \mathbf{b}^T \mathbf{I}^{-1}(a, b, c)\mathbf{b}$, with $\mathbf{b} = [\frac{\partial R}{\partial a}, \frac{\partial R}{\partial b}, \frac{\partial R}{\partial c}]^T = [\frac{\partial R}{\partial a}, \frac{\partial R}{\partial b}, 0]^T$, in which

$$\frac{\partial R}{\partial a} = \frac{b}{(a+b)^2}, \qquad \frac{\partial R}{\partial b} = -\frac{a}{(a+b)^2}. \tag{2.8}$$

Also, $\mathbf{I}^{-1}(a, b, c)$ is defined in Theorem 2.2. Therefore, B can be represented as (2.7) and the theorem is resulted. □ □

Using Theorem 2.3, the asymptotic confidence interval of R can be derived. It is notable that B should be estimated by the MLEs of a, b and c. So, a $100(1-\gamma)\%$ asymptotic confidence interval of R can be constructed as,

$$(\widehat{R}^{MLE} - z_{1-\frac{\gamma}{2}}\sqrt{\widehat{B}}, \widehat{R}^{MLE} + z_{1-\frac{\gamma}{2}}\sqrt{\widehat{B}}),$$

where z_γ is 100γ-th percentile of $N(0, 1)$.

2.4. Bayes estimation

In this section, under the squared error loss function, we infer the Bayesian estimation and corresponding credible interval of the stress-strength parameter, when $a \sim \Gamma(a_1, b_1)$, $b \sim \Gamma(a_2, b_2)$ and $c \sim \Gamma(a_3, b_3)$ are independent random variables. Based on the observed censoring samples, the joint posterior density function of a, b and c are given by:

$$\pi(a, b, c|\text{data}) = \frac{L(\text{data}|a, b, c)\pi_1(a)\pi_2(b)\pi_3(c)}{\int_0^\infty \int_0^\infty \int_0^\infty L(\text{data}|a, b, c)\pi_1(a)\pi_2(b)\pi_3(c)dadbdc}, \tag{2.9}$$

where

$$\begin{aligned}
\pi_1(a) &\propto a^{a_1-1}e^{-b_1 a}, \qquad a > 0, \;\; a_1, b_1 > 0,\\
\pi_2(b) &\propto b^{a_2-1}e^{-b_2 b}, \qquad b > 0, \;\; a_2, b_2 > 0,\\
\pi_3(c) &\propto c^{a_3-1}e^{-b_3 c}, \qquad c > 0, \;\; a_3, b_3 > 0.
\end{aligned}$$

As we see from (2.9), the Bayes estimates cannot be derived in the closed form. So, we approximate them by applying two methods:

- Lindley's approximation,
- MCMC method.

2.4.1. Lindley's approximation One of the most numerical methods to approximate the Bayes estimate has been introduced by Lindley in [8]. This method has explained in Appendix B. Based on this approximation, the Bayes estimate of R is:

$$\begin{aligned}
\widehat{R}^{Lin} = R + [u_1 d_1 + u_2 d_2 + d_4 + d_5] + \frac{1}{2}[A(u_1\sigma_{11} + u_2\sigma_{12})\\
+ B(u_1\sigma_{21} + u_2\sigma_{22}) + C(u_1\sigma_{31} + u_2\sigma_{32})]. \qquad (2.10)
\end{aligned}$$

As we see, constructing the HPD credible interval is not possible, using the Lindley's approximation. So, we apply the Markov Chain Monte Carlo (MCMC) method to approximate the Bayes estimate and construct the corresponding HPD credible intervals.

2.4.2. MCMC method By (2.9), we get the posterior p.d.fs of a, b and c as:

$$a|c,\text{data} \sim \Gamma\Big(n+a_1, \sum_{i=1}^{n} w(x_i, r_i, c, 0) + b_1\Big),$$

$$b|c,\text{data} \sim \Gamma\Big(n+a_2, \sum_{j=1}^{m} w(y_j, s_j, c, 0) + b_2\Big),$$

$$\pi(c|a,b,\text{data}) \propto c^{n+m+a_3-1} \left(\prod_{i=1}^{n} \frac{\Phi(x_i)^{c-1}}{\Big(1-\Phi(x_i)\Big)^{c+1}}\right) \left(\prod_{j=1}^{m} \frac{\Phi(y_j)^{c-1}}{\Big(1-\Phi(y_j)\Big)^{c+1}}\right)$$
$$\times \exp\Big\{ -a\sum_{i=1}^{n} w(x_i, r_i, c, 0) - b\sum_{j=1}^{m} w(y_j, s_j, c, 0) - b_3 c\Big\}$$

It is obvious that the posteriors p.d.f. of c are not the well known distributions. So, we utilize the Metropolis-Hastings method with normal proposal distribution to generate random samples from it. Therefore, the Gibbs sampling algorithm can be proposed as follows:

1. Start with the begin value $(a_{(0)}, b_{(0)}, c_{(0)})$.
2. Set $t = 1$.
3. Generate $c_{(t)}$ from $\pi(c|a_{(t-1)}, b_{(t-1)}, \text{data})$, using Metropolis-Hastings method.
4. Generate $a_{(t)}$ from $\Gamma\Big(n+a_1, \sum_{i=1}^{n} w(x_i, r_i, c, 0) + b_1\Big)$.
5. Generate $b_{(t)}$ from $\Gamma\Big(n+a_2, \sum_{j=1}^{m} w(y_j, s_j, c, 0) + b_2\Big)$.
6. Calculate $R_t = \frac{a_t}{a_t+b_t}$.
7. Set $t = t+1$.
8. Repeat steps 3-7, for T times.

Applying this algorithm, the Bayes estimate of R, under the squared error loss function is given by

$$\widehat{R}^{MC} = \frac{1}{T}\sum_{t=1}^{T} R_t. \tag{2.11}$$

Moreover, a $100(1-\gamma)\%$ HPD credible interval of R can be constructed by utilizing the method of Chen and Shao [4].

3. INFERENCE ON R WITH KNOWN COMMON c

3.1. Maximum likelihood estimation of R

Suppose $\mathbf{X} = (X_{1:N}, X_{2:N}, ..., X_{n:N})$ is a progressively Type-II censored sample from $WSN(a,c)$ with censored scheme $\mathbf{r} = (r_1, r_2, ..., r_n)$ and $\mathbf{Y} = (Y_{1:M}, Y_{2:M}, ..., Y_{m:M})$ is a progressively Type-II censored sample from $WSN(b,c)$ with censored scheme $\text{s} = (s_1, s_2, ..., s_m)$. Based on Section 2.1, when the common shape parameter c is known, the MLE of R can be earned easily by

$$\widehat{R}^{MLE} = \left(1 + \frac{m\sum_{i=1}^{n} w(x_i, r_i, c, 0)}{n\sum_{j=1}^{m} w(y_j, s_j, c, 0)}\right)^{-1}. \tag{3.12}$$

In a similar manner as Section 2.3, $(\widehat{R}^{MLE} - R) \xrightarrow{D} N(0, C)$, where $C = (\frac{\partial R}{\partial a})^2 \frac{1}{I_{11}} + (\frac{\partial R}{\partial b})^2 \frac{1}{I_{22}}$, and $\frac{\partial R}{\partial a}$ and $\frac{\partial R}{\partial b}$ are given in (2.8). So, a $100(1-\gamma)\%$ asymptotic confidence interval for R can be constructed as,

$$(\widehat{R}^{MLE} - z_{1-\frac{\gamma}{2}}\sqrt{\widehat{C}}, \widehat{R}^{MLE} + z_{1-\frac{\gamma}{2}}\sqrt{\widehat{C}}), \tag{3.13}$$

where z_γ is 100γ-th percentile of $N(0, 1)$.

3.2. Bayes estimation

In this section, we infer the Bayesian estimation and corresponding credible interval of the stress-strength parameter, when $a \sim \Gamma(a_1, b_1)$ and $b \sim \Gamma(a_2, b_2)$ are independent random variables. Based on the observed censoring samples, the joint posterior density function of a and b are given by:

$$\pi(a, b|c, \text{data}) = \frac{(V + b_1)^{n+a_1}(U + b_2)^{m+a_2}}{\Gamma(n + a_1)\Gamma(m + a_2)} a^{n+a_1-1} b^{m+a_2-1} e^{-a(V+b_1)-b(U+b_2)}, \tag{3.14}$$

where $V = \sum_{i=1}^{n} w(x_i, r_i, c, 0)$ and $U = \sum_{j=1}^{m} w(y_j, s_j, c, 0)$. Under the squared error loss function, to obtain R Bayes estimate, we solve the following integral

$$\widehat{R}^B = \int_0^\infty \int_0^\infty \frac{a}{a+b} \times \pi(a, b|c, \text{data}) da db.$$

Now, we use the idea of Kizilaslan and Nadar [5] and obtain the R Bayes estimate as

$$\widehat{R}^B = \begin{cases} \dfrac{(1-z)^{n+a_1}(n+a_1)}{w} F_1^2(w, n+a_1+1; w+1, z), & |z| < 1, \\[2ex] \dfrac{(n+a_1)}{w(1-z)^{m+a_2}} F_1^2(w, m+a_2; w+1, z(1-z)^{-1}), & z < -1, \end{cases} \tag{3.15}$$

where $w = n + m + a_1 + a_2$, $z = 1 - \dfrac{V + b_1}{U + b_2}$ and

$$F_1^2(a, b; c, z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt, \quad |z| < 1,$$

is the hypergeometric series, which is quickly evaluated and readily available in standard software such as MATLAB. Moreover, we construct a $100(1-\gamma)\%$ Bayesian interval for the stress-strength parameter by (L, U), where L and U are the lower and upper bounds, respectively which satisfy

$$\int_0^L f_R(R) dR = \frac{\gamma}{2}, \quad \int_0^U f_R(R) dR = 1 - \frac{\gamma}{2}, \tag{3.16}$$

where $f_R(R)$ is the probability density function of R which obtained from (3.14) as

$$f_R(R) = \frac{1}{B(n + a_1, m + a_2)} (1-z)^{n+a_1} R^{n+a_1-1}(1-R)^{m+a_2-1}(1-Rz)^{-w}, \quad 0 < R < 1.$$

3.3. Uniformly minimum variance unbiased estimate of R

Suppose $\mathbf{X} = (X_{1:N}, X_{2:N}, ..., X_{n:N})$ is a progressively Type-II censored sample from WSN(a, c) with censored scheme $\mathbf{r} = (r_1, r_2, ..., r_n)$ and $\mathbf{Y} = (Y_{1:M}, Y_{2:M}, ..., Y_{m:M})$ is a progressively Type-II censored sample from WSN(b, c) with censored scheme $\mathbf{s} = (s_1, s_2, ..., s_m)$. When the common shape parameter c is known, based on the observed data, the likelihood function is:

$$
\begin{aligned}
L(\text{data}, c|a, b) \propto (ac)^n & \left(\prod_{i=1}^{n} \phi(x_i) \frac{\Phi(x_i)^{c-1}}{\Big(1-\Phi(x_i)\Big)^{c+1}} \right) \exp\{-aU\} \\
\times (bc)^m & \left(\prod_{j=1}^{m} \phi(y_j) \frac{\Phi(y_j)^{c-1}}{\Big(1-\Phi(y_j)\Big)^{c+1}} \right) \exp\{-bV\}.
\end{aligned}
$$

From the above equation, we conclude that V and U are complete sufficient statistics for a and b, respectively. We can verify that $X_i^* = \left(\frac{\Phi(X_i)}{1-\Phi(X_i)}\right)^c$, $i = 1, \ldots, n$ is one Type-II progressive censoring samples from an exponential distribution with mean a^{-1}. Now, using the transformations

$$
\begin{aligned}
& Z_1 = N X_1^*, \\
& Z_2 = (N - r_1 - 1)(X_2^* - X_1^*), \\
& \quad \vdots \\
& Z_n = (N - \sum_{i=1}^{n-1} r_i - n + 1)(X_n^* - X_{n-1}^*).
\end{aligned}
$$

From Balakrishnan and Aggarwala [2], we conclude that $Z_1, \ldots, Z_n$ are independent and identically distributed as an exponential distribution with mean a^{-1}. So, $V = \sum_{i=1}^{n} Z_i \sim \Gamma(n, a)$.

Lemma 3.1:
Let $X_i^* = \left(\frac{\Phi(X_i)}{1-\Phi(X_i)}\right)^c$, $i = 1, \ldots, n$ *and* $Y_j^* = \left(\frac{\Phi(Y_j)}{1-\Phi(Y_j)}\right)^c$, $j = 1, \ldots, m$. *Then the conditional p.d.fs of* X_1^* *given* V *and* Y_1^* *given* U *are, respectively, as follows:*

$$
\begin{aligned}
f_{X_1^*|V=v}(x) &= N(n-1)\frac{(v - Nx)^{n-2}}{v^{n-1}}, \qquad 0 < x < \frac{v}{N}, \\
f_{Y_1^*|U=u}(y) &= M(m-1)\frac{(u - My)^{m-2}}{u^{m-1}}, \quad 0 < y < \frac{u}{M}.
\end{aligned}
$$

Proof
The proof of the lemma is similar to the proof of Lemma 2 in [7]. □

Theorem 3.1:
For the complete sufficient statistics V and U, the UMVUE of R, say $\hat{\hat{R}}$, is as follows:

$$\hat{\hat{R}} = \begin{cases} 1 - \sum_{k=0}^{n-1} (-1)^k (\frac{u}{v})^k \frac{\binom{n-1}{k}}{\binom{m-1+k}{k}}, & u < v, \\ \sum_{k=0}^{m-1} (-1)^k (\frac{v}{u})^k \frac{\binom{m-1}{k}}{\binom{n-1+k}{k}}, & u > v. \end{cases} \tag{3.17}$$

Proof
It is observable that X_1^* and Y_1^* are exponential variables with mean $(Na)^{-1}$ and $(Mb)^{-1}$, respectively. By this, we can easily show that

$$\phi(X_1^*, Y_1^*) = \begin{cases} 1, & MY_1^* > NX_1^*, \\ 0, & MY_1^* < NX_1^*, \end{cases}$$

is an unbiased estimate for R. So,

$$\begin{aligned} \hat{\hat{R}} &= \mathbb{E}(\phi(X_1^*, Y_1^*)|V = v, U = u) \\ &= \iint_{\mathcal{A}} f_{X_1^*|V=v}(x) f_{Y_1^*|U=u}(y) dx dy, \end{aligned}$$

where $\mathcal{A} = \{(x, y) : 0 < x < v/N, 0 < y < u/M, My > Nx\}$. Moreover, $f_{X_1^*|V=v}(x)$ and $f_{Y_1^*|U=u}(y)$ are given in Lemma 3.1. Now, for $u < v$, we have

$$\begin{aligned} \hat{\hat{R}} &= \frac{NM(n-1)(m-1)}{v^{n-1}u^{m-1}} \int_0^{\frac{u}{M}} \int_0^{\frac{My}{N}} (v - Nx)^{n-2}(u - My)^{m-2} dx dy \\ &= 1 - \frac{M(m-1)}{v^{n-1}u^{m-1}} \int_0^{\frac{u}{M}} (v - My)^{n-1}(u - My)^{m-2} dy \\ &= 1 - (m-1) \int_0^1 (1-t)^{m-2} \left(1 - \frac{u}{v} t\right)^{n-1} dt \\ &= 1 - (m-1) \int_0^1 (1-t)^{m-2} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \left(\frac{u}{v}\right)^k t^k dt \\ &= 1 - \sum_{k=0}^{n-1} (-1)^k \left(\frac{u}{v}\right)^k \frac{\binom{n-1}{k}}{\binom{m-1+k}{k}} \end{aligned}$$

By a similar method, for $u > v$, the result can be verified. □

4. ESTIMATION OF R IN GENERAL CASE

4.1. Maximum likelihood estimation of R

Assume that X and Y are two independent random variables from $WSN(a,c)$ and $WSN(b,d)$ distributions, respectively. We have

$$R = a\int_0^\infty \exp\{-au - bu^{\frac{d}{c}}\}du.$$

So, the likelihood function, based on the observed data can be obtained as:

$$L(\text{data}|a,b,c,d) = k_1k_2(ac)^n \left(\prod_{i=1}^{n} \frac{\phi(x_i)\Phi(x_i)^{c-1}}{\Big(1-\Phi(x_i)\Big)^{c+1}}\right) \exp\left\{-a\sum_{i=1}^{n}(r_i+1)\Big(\frac{\Phi(x_i)}{1-\Phi(x_i)}\Big)^c\right\}$$
$$\times (bd)^m \left(\prod_{j=1}^{m} \phi(y_j)\frac{\Phi(y_j)^{d-1}}{\Big(1-\Phi(y_j)\Big)^{d+1}}\right) \exp\left\{-b\sum_{j=1}^{m}(s_j+1)\Big(\frac{\Phi(y_j)}{1-\Phi(y_j)}\Big)^d\right\}.$$

Therefore, the log-likelihood function is as:

$$\begin{aligned}\ell(a,b,c,d) &= \text{Constant} + n\log(a) + m\log(b) + n\log(c) + m\log(d)\\ &\quad + \sum_{i=1}^{n}\log(\Phi(x_i)) + (c-1)\sum_{i=1}^{n}\log(\Phi(x_i)) - (c+1)\sum_{i=1}^{n}\log(1-\Phi(x_i))\\ &\quad + \sum_{j=1}^{m}\log(\Phi(y_j)) + (d-1)\sum_{j=1}^{m}\log(\Phi(y_j)) - (d+1)\sum_{j=1}^{m}\log(1-\Phi(y_j))\\ &\quad - a\sum_{i=1}^{n}(r_i+1)\Big(\frac{\Phi(x_i)}{1-\Phi(x_i)}\Big)^c - b\sum_{j=1}^{m}(s_j+1)\Big(\frac{\Phi(y_j)}{1-\Phi(y_j)}\Big)^d. \end{aligned} \tag{4.18}$$

So, to earn the MLEs of a, b, c and d, namely, $\widehat{a}$, $\widehat{b}, \widehat{c}$ and $\widehat{d}$, respectively, we should solve the following equations:

$$\frac{\partial \ell}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n} w(x_i, r_i, c, 0) = 0, \tag{4.19}$$

$$\frac{\partial \ell}{\partial b} = \frac{m}{b} - \sum_{j=1}^{m} w(y_j, s_j, d, 0) = 0, \tag{4.20}$$

$$\frac{\partial \ell}{\partial c} = \frac{m}{c} + \sum_{i=1}^{n}\log\left(\frac{\Phi(x_i)}{1-\Phi(x_i)}\right) - a\sum_{i=1}^{n} w(x_i, r_i, c, 1) = 0, \tag{4.21}$$

$$\frac{\partial \ell}{\partial d} = \frac{m}{d} + \sum_{j=1}^{m}\log\left(\frac{\Phi(y_j)}{1-\Phi(y_j)}\right) - b\sum_{j=1}^{m} w(y_j, s_j, d, 1) = 0. \tag{4.22}$$

After obtaining the MLEs of a, b, c and d, by using the invariance property, the MLE of R can be derived as

$$\widehat{R}^{MLE} = \widehat{a}\int_0^\infty \exp\{-\widehat{a}u - \widehat{b}u^{\frac{\widehat{d}}{\widehat{c}}}\}du. \tag{4.23}$$

4.2. Approximation maximum likelihood estimation of R

Suppose that $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_m\}$ be two Type-II progressive censoring samples from $WSN(a, c)$ and $WSN(b, d)$ distributions and

$$X_i' = \frac{\Phi(x_i)}{1 - \Phi(x_i)}, \quad U_i = \log(X_i'),$$
$$Y_j' = \frac{\Phi(y_j)}{1 - \Phi(y_j)}, \quad V_j = \log(Y_j').$$

Applying Theorem 2.1, we have $U_i \sim EV(\xi_1, \sigma_1)$ and $V_j \sim EV(\xi_2, \sigma_2)$, where

$$\xi_1 = -\frac{1}{c}\log(a),\ \xi_2 = -\frac{1}{d}\log(b), \quad \sigma_1 = \frac{1}{c}, \text{ and } \sigma_2 = \frac{1}{d}.$$

In a similar manner as Section 2.2, we derive the AMLEs of ξ_1, ξ_2, σ_1 and σ_2, say $\tilde{\xi}_1$, $\tilde{\xi}_2$, $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$, respectively, by

$$\tilde{\xi}_1 = A_1 - \tilde{\sigma}_1 B_1,$$
$$\tilde{\xi}_2 = A_2 - \tilde{\sigma}_2 B_2,$$
$$\tilde{\sigma}_1 = \frac{-D_1 + \sqrt{D_1^2 + 4C_1E_1}}{2C_1},$$
$$\tilde{\sigma}_2 = \frac{-D_2 + \sqrt{D_2^2 + 4C_2E_2}}{2C_2},$$

where A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, E_1, E_2 are given in Appendix A. After earning $\tilde{\xi}_1$, $\tilde{\xi}_2$, $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$, the values of $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ can be evaluated by

$$\tilde{c} = \frac{1}{\tilde{\sigma}_1}, \quad \tilde{d} = \frac{1}{\tilde{\sigma}_2}, \quad \tilde{a} = \exp\left(\frac{-\tilde{\xi}_1}{\tilde{\sigma}_1}\right), \quad \tilde{b} = \exp\left(\frac{-\tilde{\xi}_2}{\tilde{\sigma}_2}\right).$$

Then,

$$\tilde{R} = \tilde{a}\int_0^\infty \exp\{-\tilde{a}u - \tilde{b}u^{\frac{\tilde{d}}{\tilde{c}}}\}du. \tag{4.24}$$

4.3. Bayes estimation

In this section, under the squared error loss function, we infer the Bayesian estimation and corresponding credible interval of the stress-strength parameter, when the unknown parameters $a \sim \Gamma(a_1, b_1)$, $b \sim \Gamma(a_2, b_2)$, $c \sim \Gamma(a_3, b_3)$ and $d \sim \Gamma(a_4, b_4)$ are independent random variables. In a similar manner as Section 2.4, as the Bayesian estimation of R has not a closed form, we approximate it by MCMC method. After simplify the joint posterior density function of the unknown parameters, we get the posterior p.d.fs of a, b, c and d as:

$$a|c, \text{data} \sim \Gamma\Big(n + a_1, \sum_{i=1}^{n} w(x_i, r_i, c, 0) + b_1\Big),$$

$$b|d, \text{data} \sim \Gamma\Big(n + a_2, \sum_{j=1}^{m} w(y_j, s_j, d, 0) + b_2\Big),$$

$$\pi(c|a, \text{data}) \propto c^{n+a_3-1}\left(\prod_{i=1}^{n} \frac{\Phi(x_i)^{c-1}}{\Big(1-\Phi(x_i)\Big)^{c+1}}\right) \times \exp\Big\{-a\sum_{i=1}^{n} w(x_i, r_i, c, 0) - b_3 c\Big\},$$

$$\pi(d|b, \text{data}) \propto d^{m+a_4-1}\left(\prod_{j=1}^{m} \frac{\Phi(y_j)^{d-1}}{\Big(1-\Phi(y_j)\Big)^{d+1}}\right) \times \exp\Big\{-b\sum_{j=1}^{m} w(y_j, s_j, d, 0) - b_4 d\Big\}.$$

It is recognized that the posterior p.d.fs of c and d are not well known distributions. So, we utilize the Metropolis-Hastings method with normal proposal distribution to generate random samples from them. Therefore, the Gibbs sampling algorithm can be proposed as follows:

1. Start with the begin value $(a_{(0)}, b_{(0)}, c_{(0)}, d_{(0)})$.
2. Set $t = 1$.
3. Generate $c_{(t)}$ from $\pi(c|a_{(t-1)}, \text{data})$, using Metropolis-Hastings method.
4. Generate $d_{(t)}$ from $\pi(d|b_{(t-1)}, \text{data})$, using Metropolis-Hastings method.
5. Generate $a_{(t)}$ from $\Gamma\Big(n + a_1, \sum_{i=1}^{n} w(x_i, r_i, c_{(t-1)}, 0) + b_1\Big)$.
6. Generate $b_{(t)}$ from $\Gamma\Big(n + a_2, \sum_{j=1}^{m} w(y_j, s_j, d_{(t-1)}, 0) + b_2\Big)$.
7. Calculate
$$R_t = a_{(t)} \int_0^{\infty} \exp\Big\{-a_{(t)}u - b_{(t)}u^{\frac{d_{(t)}}{c_{(t)}}}\Big\}du.$$
8. Set $t = t + 1$.
9. Repeat steps 3-8, for T times.

Using this algorithm, under the squared error loss function, the R Bayes estimate is given by

$$\widehat{R}^{MC} = \frac{1}{T}\sum_{t=1}^{T} R_t. \tag{4.25}$$

Moreover, a $100(1-\gamma)\%$ HPD credible interval of R can be constructed by utilizing the method of Chen and Shao [4].

5. SIMULATION STUDY

Using Monte Carlo simulations, we compare the behavior of different methods, in this section. To compare of point estimates, we compute the mean squared errors (MSEs). Also, to comparing of interval estimates, we compute the average lengths and coverage percentages. Different schemes, parameters and hyper parameters are employed to obtain the simulation results. We report all results, based on 3000 replications and the nominal level is 0.95. The censoring schemes which we used are as follows:

$$\text{Scheme 1: } r_1 = \ldots = r_n = \frac{N-n}{n},$$
$$\text{Scheme 2: } r_1 = \ldots = r_{\frac{n}{2}} = 0,\ R_{\frac{n}{2}+1} = \ldots = R_n = \frac{2(N-n)}{n},$$
$$\text{Scheme 3: } r_1 = \frac{N-n}{2},\ r_2 = \ldots = r_{n-1} = 0,\ r_n = \frac{N-n}{2}.$$

First, when the common scale parameter c is unknown, without loss of generality, we put $a = 3,\ b = 2,\ c = 4$. Also, Bayesian inference are given based on three priors as: Prior 1: $a_j = 0,\ b_j = 0,\ j = 1, 2, 3$, Prior 2: $a_j = 1,\ b_j = 0.1,\ j = 1, 2, 3$, and Prior 3: $a_j = 2,\ b_j = 0.2,\ j = 1, 2, 3$. We derive the MLE using (2.5), AMLE using (2.6), Bayes estimates of R via Lindley's approximation and MCMC method using (2.10) and (2.11), respectively. Further, we obtain the asymptotic confidence and HPD credible intervals of R. The results are given in Tables 5.1-5.2.

Second, when the common scale parameter c is known, without loss of generality, we put $a = 2,\ b = 3,\ c = 4$. Also, Bayesian inference are given based on three priors as: Prior 4: $a_j = 0,\ b_j = 0,\ j = 1, 2$, Prior 5: $a_j = 1,\ b_j = 0.1,\ j = 1, 2$, and Prior 6: $a_j = 2,\ b_j = 0.2,\ j = 1, 2$. We derive the MLE using (3.12), Bayes estimates using (3.15) and UMVUE of R using (3.17). Further, we obtain the asymptotic and Bayesian intervals of R using (3.13) and (3.16), respectively. The results are given in Table 5.3.

Third, when all parameters are unknown and different, without loss of generality, we put $a = 1.5,\ b = 3,\ c = 2\ , d = 4$. Also, Bayesian inference are given based on three priors as: Prior 7: $a_j = 0,\ b_j = 0,\ j = 1, 2, 3, 4$, Prior 8: $a_j = 1,\ b_j = 0.1,\ j = 1, 2, 3, 4$, and Prior 9: $a_j = 2,\ b_j = 0.2,\ j = 1, 2, 3, 4$. We derive the MLE, AMLE and Bayes estimates via MCMC method using (4.23), (4.24) and (4.25), respectively. The results are given in Table 5.4.

To monitor the convergence of MCMC method, all three cases, we considered the trace plots for different censoring schemes and parameters. In all cases, it is observed that the MCMC method is converged. Some of this plots are shown in Figures 5.1-5.2.

From Tables 5.1-5.2, we observe that Bayes estimates and AMLE have the best and worst performance, based on MSEs, respectively. Also, in Bayesian inference, the informative priors perform better than non-informative ones, in point and interval estimates. Furthermore, the Lindley's approximation performs worse that the MCMC method.

From Table 5.3, we observe that Bayes estimates and UMVUEs have the best and worst performance based on MSEs, respectively. Also, in Bayesian inference, the informative priors perform better than non-informative ones, in point and interval estimates.

From Table 5.4, we observe that the Bayes estimates perform better that the MLEs based on MSEs. Also, in Bayesian inference, the informative priors perform better than non-informative ones, in point and interval estimates.

As a fact, from Tables 5.1-5.4, for fixed N, with increasing n, the MSEs of all estimates decrease, the average confidence lengths decrease and the associated coverage percentages increase, in all cases. This can be due to the fact, with increasing n, some additional information is gathered.

6. CONCLUSION

In this paper, we obtain different estimates of stress-strength parameter, under the hybrid progressive censored scheme, when stress and strength are two independent Kumaraswamy random variables. The problem is solved in three cases. First, when $X \sim WSN(a, c)$ and $Y \sim WSN(b, c)$, we derive ML, AML and two approximated Bayes estimates using Lindley's approximation and MCMC method, due to the lack of explicit forms. Also, we

Table 5.1. Biases and MSEs for estimates of R when c is unknown.

(N,n)	C.S	AMLE		MLE		Prior 1				Prior 2				Prior 3			
						MCMC		Lindley		MCMC		Lindley		MCMC		Lindley	
		Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE
(30,10)	(1,1)	0.6182	0.0182	0.5607	0.0170	0.6071	0.0097	0.6212	0.0134	0.5884	0.0083	0.6011	0.0110	0.5857	0.0068	0.5752	0.0088
	(2,2)	0.6073	0.0172	0.6032	0.0147	0.5561	0.0093	0.6047	0.0156	0.5912	0.0085	0.5659	0.0087	0.5642	0.0067	0.6022	0.0086
	(3,3)	0.6056	0.0182	0.5943	0.0171	0.5522	0.0089	0.5923	0.0144	0.6148	0.0079	0.6049	0.0093	0.6149	0.0068	0.6027	0.0087
	(1,2)	0.5815	0.0196	0.6109	0.0152	0.6123	0.0096	0.5767	0.0149	0.6089	0.0086	0.5937	0.0093	0.6129	0.0075	0.5862	0.0083
	(1,3)	0.5716	0.0179	0.6085	0.0138	0.6022	0.0086	0.5771	0.0157	0.6102	0.0078	0.5895	0.0093	0.5884	0.0065	0.5979	0.0087
	(2,3)	0.6049	0.0198	0.5858	0.0139	0.5861	0.0091	0.6033	0.0129	0.5969	0.0073	0.5877	0.0097	0.5984	0.0065	0.5985	0.0088
(50,20)	(1,1)	0.5866	0.0177	0.5891	0.0168	0.5981	0.0098	0.6137	0.0142	0.6038	0.0083	0.5937	0.0096	0.6101	0.0066	0.6109	0.0087
	(2,2)	0.6059	0.0186	0.5894	0.0150	0.5791	0.0097	0.6102	0.0145	0.6020	0.0082	0.5898	0.0096	0.5987	0.0066	0.5952	0.0092
	(3,3)	0.5807	0.0172	0.5900	0.0151	0.5984	0.0099	0.6014	0.0123	0.5987	0.0081	0.6091	0.01102	0.6162	0.0065	0.5865	0.0086
	(1,2)	0.5915	0.0179	0.6171	0.0169	0.6138	0.0092	0.5841	0.0134	0.5812	0.0087	0.5907	0.0096	0.5923	0.0066	0.5987	0.0078
	(1,3)	0.5981	0.0194	0.5810	0.0132	0.6162	0.0097	0.6070	0.0141	0.6071	0.0088	0.6086	0.0097	0.5983	0.0070	0.5923	0.0086
	(2,3)	0.6118	0.0179	0.6187	0.0142	0.6052	0.0098	0.5976	0.0133	0.6012	0.0079	0.6103	0.0115	0.5967	0.0068	0.5935	0.0087
(50,30)	(1,1)	0.6101	0.0164	0.6082	0.0141	0.6155	0.0087	0.6082	0.0107	0.6057	0.0064	0.5984	0.0081	0.5913	0.0042	0.6014	0.0069
	(2,2)	0.6162	0.0175	0.5897	0.0137	0.6177	0.0073	0.6179	0.0108	0.6100	0.0065	0.5977	0.0089	0.6033	0.0043	0.5896	0.0068
	(3,3)	0.6137	0.0172	0.6195	0.0147	0.5939	0.0077	0.6093	0.0106	0.5978	0.0062	0.5974	0.0089	0.5977	0.0044	0.6011	0.0065
	(1,2)	0.5938	0.0171	0.6178	0.0148	0.5982	0.0078	0.6165	0.0106	0.5974	0.0068	0.5985	0.0087	0.5858	0.0041	0.59687	0.0058
	(1,3)	0.5879	0.0166	0.5862	0.0141	0.6174	0.0071	0.5975	0.0108	0.5987	0.0065	0.6016	0.0085	0.5931	0.0043	0.6071	0.0067
	(2,3)	0.6154	0.0170	0.5882	0.0141	0.6122	0.0077	0.6098	0.0125	0.5949	0.0062	0.6022	0.0081	0.6093	0.0049	0.6186	0.0057
(70,30)	(1,1)	0.6107	0.0163	0.5981	0.0145	0.5915	0.0075	0.5952	0.0121	0.5981	0.0065	0.5943	0.0079	0.5986	0.0039	0.6079	0.0061
	(2,2)	0.6081	0.0162	0.6161	0.0145	0.6161	0.0077	0.5940	0.0108	0.6042	0.0065	0.6031	0.0081	0.6004	0.0051	0.6091	0.0058
	(3,3)	0.5851	0.0167	0.6143	0.0146	0.6180	0.0077	0.6108	0.0109	0.5939	0.0066	0.6048	0.0079	0.6075	0.0042	0.6079	0.0063
	(1,2)	0.5932	0.0161	0.6108	0.0137	0.6077	0.0071	0.5967	0.0106	0.6017	0.0064	0.5918	0.0087	0.5905	0.0041	0.5965	0.0058
	(1,3)	0.6170	0.0176	0.6140	0.0145	0.6126	0.0074	0.6107	0.0106	0.6046	0.0061	0.6084	0.0083	0.5804	0.0044	0.5987	0.0065
	(2,3)	0.6148	0.0170	0.5931	0.0139	0.5911	0.0074	0.5958	0.0108	0.6015	0.0063	0.5977	0.0088	0.6062	0.0042	0.6037	0.0063

Table 5.2. Average confidence/credible lengths and coverage percentages for estimates of R when c is unknown.

(N, n)	C.S	AMLE		MLE		Prior 1		Prior 2		Prior 3	
		length	C.P	length	C.P	length	C.P	length	C.P	length	C.P
(30,10)	(1,1)	0.5499	0.912	0.5391	0.925	0.3680	0.935	0.4253	0.942	0.2906	0.946
	(2,2)	0.5400	0.911	0.5294	0.920	0.3687	0.935	0.4329	0.943	0.3830	0.945
	(3,3)	0.5414	0.913	0.5311	0.927	0.4577	0.936	0.4172	0.942	0.3847	0.944
	(1,2)	0.5534	0.912	0.5204	0.921	0.4539	0.935	0.4178	0.943	0.3856	0.945
	(1,3)	0.5421	0.911	0.5302	0.924	0.3617	0.936	0.4351	0.942	0.3820	0.946
	(2,3)	0.5500	0.911	0.5279	0.924	0.3626	0.935	0.4237	0.942	0.2902	0.945
(50,20)	(1,1)	0.5415	0.913	0.5289	0.920	0.4568	0.935	0.4299	0.942	0.3800	0.944
	(2,2)	0.5588	0.911	0.5296	0.923	0.4581	0.935	0.4155	0.943	0.2950	0.943
	(3,3)	0.5532	0.910	0.5374	0.921	0.3626	0.937	0.4228	0.941	0.2971	0.945
	(1,2)	0.5583	0.910	0.5206	0.927	0.3679	0.937	0.4317	0.942	0.3886	0.945
	(1,3)	0.5569	0.910	0.5305	0.920	0.3639	0.936	0.4311	0.944	0.2939	0.944
	(2,3)	0.5551	0.911	0.5374	0.923	0.3690	0.935	0.4297	0.943	0.2993	0.946
(50,30)	(1,1)	0.4001	0.928	0.3740	0.939	0.3014	0.943	0.2622	0.947	0.3323	0.955
	(2,2)	0.5194	0.928	0.3719	0.936	0.3022	0.944	0.2612	0.946	0.3205	0.952
	(3,3)	0.4012	0.926	0.4507	0.937	0.3086	0.941	0.2589	0.947	0.3221	0.957
	(1,2)	0.4085	0.927	0.3699	0.935	0.4156	0.944	0.2756	0.949	0.3305	0.950
	(1,3)	0.5229	0.924	0.4569	0.936	0.4155	0.941	0.2749	0.946	0.3296	0.957
	(2,3)	0.5147	0.925	0.3759	0.936	0.4150	0.943	0.2627	0.947	0.3359	0.960
(70,30)	(1,1)	0.4079	0.924	0.3614	0.935	0.2956	0.940	0.2653	0.946	0.3344	0.959
	(2,2)	0.5109	0.925	0.3625	0.938	0.3066	0.942	0.2503	0.947	0.3214	0.956
	(3,3)	0.4067	0.924	0.4525	0.936	0.4124	0.940	0.2633	0.948	0.3257	0.954
	(1,2)	0.4017	0.928	0.3696	0.935	0.2913	0.944	0.2552	0.946	0.3346	0.950
	(1,3)	0.5126	0.926	0.3799	0.936	0.2981	0.943	0.2630	0.946	0.3336	0.956
	(2,3)	0.5255	0.925	0.3626	0.938	0.4105	0.944	0.2773	0.945	0.3224	0.953

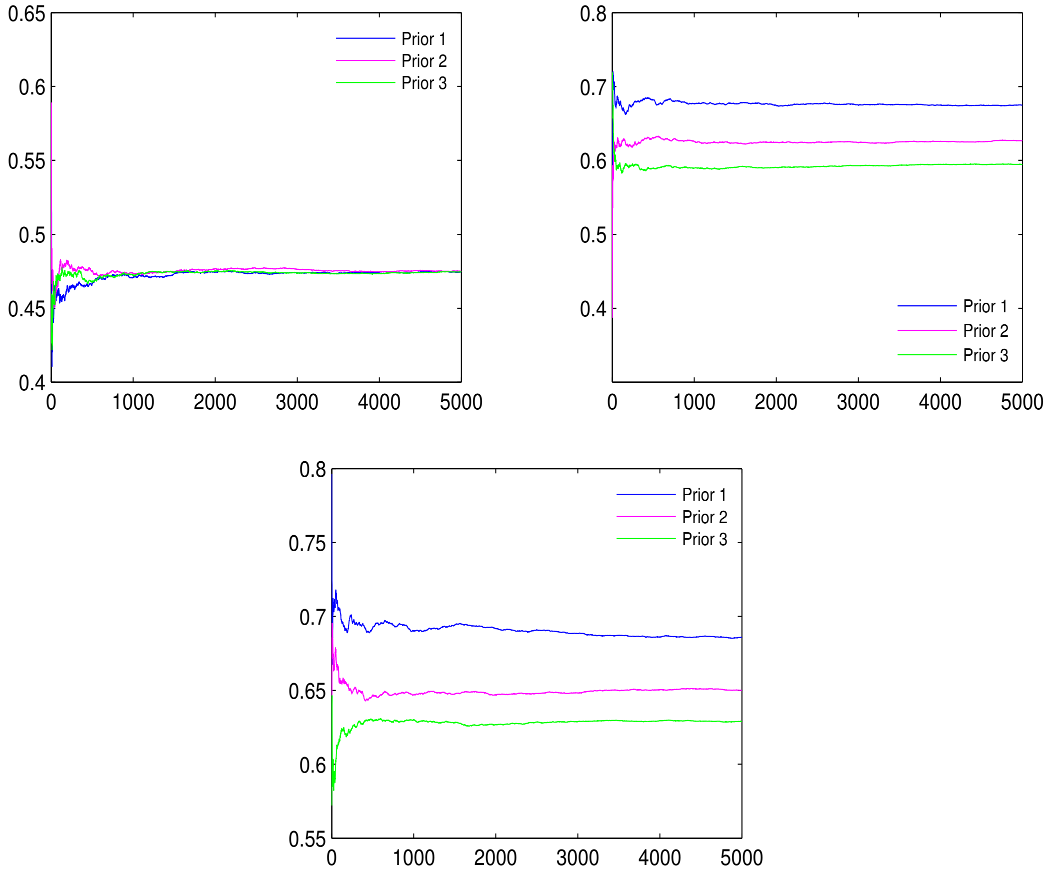

Fig. 5.1. Trace plots with C.S $(2, 2)$ with $(N, n) = (30, 10)$ (top left), $(1, 2)$ with $(N, n) = (70, 30)$ (top right) and $(1, 3)$ with $(N, n) = (50, 30)$(down), in unknown common c.

Table 5.3. Biases, MSEs, Average confidence/credible lengths and coverage percentages for estimates of R when c is known.

(N,n)	C.S	MLE		Asymp. C.I		Prior 4				Prior 5				Prior 6				UMVUE	
						Exact		C.I		Exact		C.I		Exact		C.I			
		Est.	MSE	length	C.P	Est.	MSE	length	C.P	Est.	MSE	length	C.P	Est.	MSE	length	C.P	Est.	MSE
(30,10)	(1,1)	0.5146	0.0153	0.5322	0.929	0.6024	0.0135	0.4637	0.937	0.6016	0.0090	0.4336	0.943	0.6006	0.0067	0.4025	0.946	0.4964	0.0159
	(2,2)	0.5103	0.0156	0.5368	0.929	0.5106	0.0133	0.4699	0.936	0.5175	0.0081	0.4455	0.944	0.4856	0.0065	0.4010	0.947	0.4994	0.0158
	(3,3)	0.6053	0.0158	0.5275	0.928	0.4992	0.0136	0.5766	0.936	0.4983	0.0085	0.4410	0.943	0.5142	0.0064	0.2041	0.947	0.5152	0.0155
	(1,2)	0.6054	0.0153	0.5245	0.930	0.6084	0.0132	0.4611	0.937	0.6071	0.0093	0.4464	0.943	0.5114	0.0065	0.4007	0.946	0.6050	0.0159
	(1,3)	0.4948	0.0155	0.5239	0.929	0.5154	0.0135	0.4685	0.936	0.6091	0.0099	0.4381	0.944	0.5138	0.0067	0.3985	0.946	0.4960	0.0154
	(2,3)	0.6075	0.0150	0.5227	0.930	0.5132	0.0130	0.5761	0.936	0.4884	0.0086	0.4330	0.943	0.4924	0.0062	0.3905	0.946	0.5141	0.0152
(50,20)	(1,1)	0.6037	0.0152	0.5325	0.929	0.6057	0.0131	0.4634	0.937	0.4975	0.0085	0.4462	0.943	0.4977	0.0060	0.3959	0.946	0.4980	0.0158
	(2,2)	0.4937	0.0154	0.5376	0.928	0.4852	0.0138	0.4610	0.937	0.4891	0.0097	0.4475	0.943	0.4918	0.0066	0.4084	0.947	0.4967	0.0152
	(3,3)	0.6004	0.0158	0.5215	0.930	0.6047	0.0135	0.5755	0.936	0.4860	0.0091	0.4489	0.943	0.4885	0.0062	0.3900	0.946	0.4910	0.0157
	(1,2)	0.5169	0.0154	0.5342	0.931	0.4884	0.0132	0.4698	0.937	0.5143	0.0088	0.4498	0.944	0.6080	0.0069	0.3984	0.947	0.6065	0.0155
	(1,3)	0.4876	0.0157	0.5324	0.928	0.4973	0.0133	0.5743	0.937	0.4934	0.0086	0.4360	0.944	0.4918	0.0062	0.4038	0.947	0.6090	0.0159
	(2,3)	0.4949	0.0156	0.5330	0.928	0.4963	0.0134	0.4661	0.936	0.6060	0.0099	0.4340	0.944	0.5170	0.0060	0.3906	0.947	0.4982	0.0152
(50,30)	(1,1)	0.5165	0.0135	0.4839	0.934	0.6005	0.0110	0.4364	0.941	0.6070	0.0070	0.4179	0.946	0.5107	0.0048	0.3777	0.952	0.6019	0.0150
	(2,2)	0.4962	0.0136	0.4883	0.934	0.4904	0.0109	0.4394	0.941	0.6030	0.0067	0.4006	0.947	0.5133	0.0047	0.3695	0.954	0.5110	0.0150
	(3,3)	0.6068	0.0136	0.4949	0.934	0.4932	0.0107	0.4414	0.942	0.6093	0.0066	0.4094	0.947	0.5141	0.0055	0.3670	0.950	0.5107	0.0153
	(1,2)	0.6020	0.0137	0.4817	0.933	0.6075	0.0103	0.4401	0.943	0.6080	0.0071	0.4163	0.948	0.4997	0.0052	0.3701	0.959	0.4976	0.0152
	(1,3)	0.5175	0.0135	0.4916	0.934	0.6034	0.0102	0.4316	0.941	0.4904	0.0065	0.4192	0.947	0.4882	0.0047	0.3688	0.950	0.5154	0.0155
	(2,3)	0.5107	0.0130	0.4849	0.933	0.4991	0.0111	0.4462	0.942	0.5102	0.0078	0.4169	0.948	0.5146	0.0049	0.3710	0.953	0.6093	0.0150
(70,30)	(1,1)	0.6092	0.0130	0.4844	0.933	0.6018	0.0109	0.4439	0.942	0.6009	0.0066	0.4073	0.946	0.5119	0.0050	0.3619	0.955	0.4993	0.0139
	(2,2)	0.5149	0.0137	0.4800	0.934	0.4993	0.0111	0.4415	0.942	0.5157	0.0069	0.4157	0.948	0.4861	0.0046	0.3798	0.956	0.4935	0.0152
	(3,3)	0.4959	0.0134	0.4949	0.933	0.5119	0.0109	0.4476	0.941	0.4998	0.0067	0.13	0.947	0.6060	0.0053	0.3712	0.953	0.6076	0.0139
	(1,2)	0.5139	0.0138	0.4800	0.934	0.6081	0.0108	0.4492	0.942	0.5157	0.0073	0.4041	0.947	0.5128	0.0052	0.3623	0.955	0.6059	0.0150
	(1,3)	0.6022	0.0133	0.4921	0.933	0.5135	0.0109	0.4452	0.942	0.4875	0.0074	0.4066	0.947	0.6097	0.0047	0.3753	0.950	0.5100	0.0154
	(2,3)	0.4969	0.0132	0.4903	0.933	0.4922	0.0110	0.4465	0.942	0.4854	0.0065	0.4103	0.948	0.5156	0.0043	0.3704	0.958	0.4999	0.0137

Table 5.4. Biases, MSEs, Average credible lengths and coverage percentages for estimates of R in general case.

(N,n)	C.S	AMLE		MLE		Prior 7				Prior 8				Prior 9			
						MCMC		C.I		MCMC		C.I		MCMC		C.I	
		Est.	MSE	Est.	MSE	Est.	MSE	length	C.P	Est.	MSE	length	C.P	Est.	MSE	length	C.P
(30,10)	(1,1)	0.5862	0.0158	0.6113	0.0133	0.5954	0.0116	0.4840	0.937	0.5869	0.0105	0.4582	0.942	0.5821	0.0081	0.4023	0.943
	(2,2)	0.5858	0.0153	0.5914	0.0135	0.6146	0.0119	0.4846	0.939	0.6195	0.0104	0.4524	0.942	0.6107	0.0085	0.4165	0.949
	(3,3)	0.6081	0.0157	0.6092	0.0130	0.6167	0.0106	0.5705	0.937	0.5854	0.0102	0.4594	0.943	0.6097	0.0089	0.3987	0.945
	(1,2)	0.6086	0.0159	0.5961	0.0126	0.5915	0.0119	0.5732	0.937	0.6119	0.0101	0.4459	0.943	0.5886	0.0071	0.4065	0.945
	(1,3)	0.6080	0.0151	0.6183	0.0133	0.5806	0.0115	0.5726	0.938	0.6128	0.0100	0.4404	0.941	0.6003	0.0077	0.3945	0.946
	(2,3)	0.5898	0.0156	0.6026	0.0138	0.6170	0.0107	0.5708	0.938	0.6008	0.0097	0.4527	0.942	0.5837	0.0077	0.3975	0.947
(50,20)	(1,1)	0.5915	0.0148	0.6117	0.0136	0.6014	0.0112	0.4880	0.936	0.5895	0.0098	0.4529	0.942	0.6135	0.0088	0.4044	0.947
	(2,2)	0.5990	0.0151	0.5949	0.0138	0.6179	0.0115	0.5766	0.939	0.5913	0.0097	0.4514	0.943	0.6178	0.0087	0.4145	0.944
	(3,3)	0.6197	0.0153	0.6194	0.0132	0.6004	0.0118	0.4720	0.936	0.6042	0.0096	0.4594	0.942	0.5889	0.0075	0.4176	0.945
	(1,2)	0.6014	0.0155	0.5973	0.0131	0.5909	0.0120	0.4707	0.937	0.6120	0.0100	0.4441	0.944	0.5956	0.0072	0.3965	0.945
	(1,3)	0.6050	0.0152	0.6026	0.0133	0.6016	0.0110	0.4730	0.938	0.6107	0.0100	0.4509	0.942	0.6014	0.0072	0.4075	0.945
	(2,3)	0.5903	0.0158	0.6061	0.0137	0.5924	0.0115	0.4846	0.936	0.5901	0.0103	0.4616	0.944	0.5854	0.0085	0.3965	0.943
(50,30)	(1,1)	0.6108	0.0141	0.5994	0.0117	0.6154	0.0098	0.4405	0.941	0.5887	0.0065	0.4092	0.946	0.6141	0.0053	0.3529	0.956
	(2,2)	0.5863	0.0131	0.5871	0.0102	0.6113	0.0091	0.4383	0.941	0.6005	0.0067	0.3944	0.946	0.6186	0.0044	0.3556	0.955
	(3,3)	0.6100	0.0135	0.6114	0.0114	0.5896	0.0097	0.4445	0.942	0.5966	0.0067	0.4090	0.943	0.5994	0.0048	0.3537	0.960
	(1,2)	0.6028	0.0139	0.5923	0.0115	0.6159	0.0095	0.4369	0.942	0.5912	0.0070	0.4105	0.946	0.5865	0.0061	0.3575	0.955
	(1,3)	0.5900	0.0135	0.6050	0.0109	0.5837	0.0098	0.4381	0.941	0.5953	0.0075	0.4162	0.947	0.6196	0.0054	0.3540	0.953
	(2,3)	0.5823	0.0134	0.6088	0.0107	0.6138	0.0086	0.4355	0.940	0.5921	0.0069	0.4054	0.944	0.5931	0.0041	0.3614	0.955
(70,30)	(1,1)	0.6145	0.0140	0.5916	0.0101	0.6087	0.0090	0.4302	0.941	0.5965	0.0061	0.4069	0.943	0.5901	0.0047	0.3561	0.955
	(2,2)	0.6097	0.0140	0.6150	0.0114	0.6074	0.0095	0.4308	0.943	0.6071	0.0066	0.4104	0.945	0.6018	0.0042	0.3623	0.957
	(3,3)	0.6005	0.0135	0.6126	0.0117	0.6150	0.0091	0.4374	0.942	0.5822	0.0064	0.4164	0.945	0.5833	0.0053	0.3661	0.960
	(1,2)	0.5806	0.0132	0.6085	0.0108	0.5946	0.0098	0.4359	0.943	0.5881	0.0072	0.4154	0.945	0.5932	0.0043	0.3524	0.955
	(1,3)	0.6152	0.0135	0.5851	0.0103	0.5878	0.0097	0.4335	0.941	0.5931	0.0072	0.4080	0.946	0.6080	0.0043	0.3522	0.954
	(2,3)	0.5908	0.0132	0.6117	0.0118	0.5915	0.0083	0.4470	0.941	0.5868	0.0071	0.3964	0.946	0.6130	0.0051	0.3622	0.956

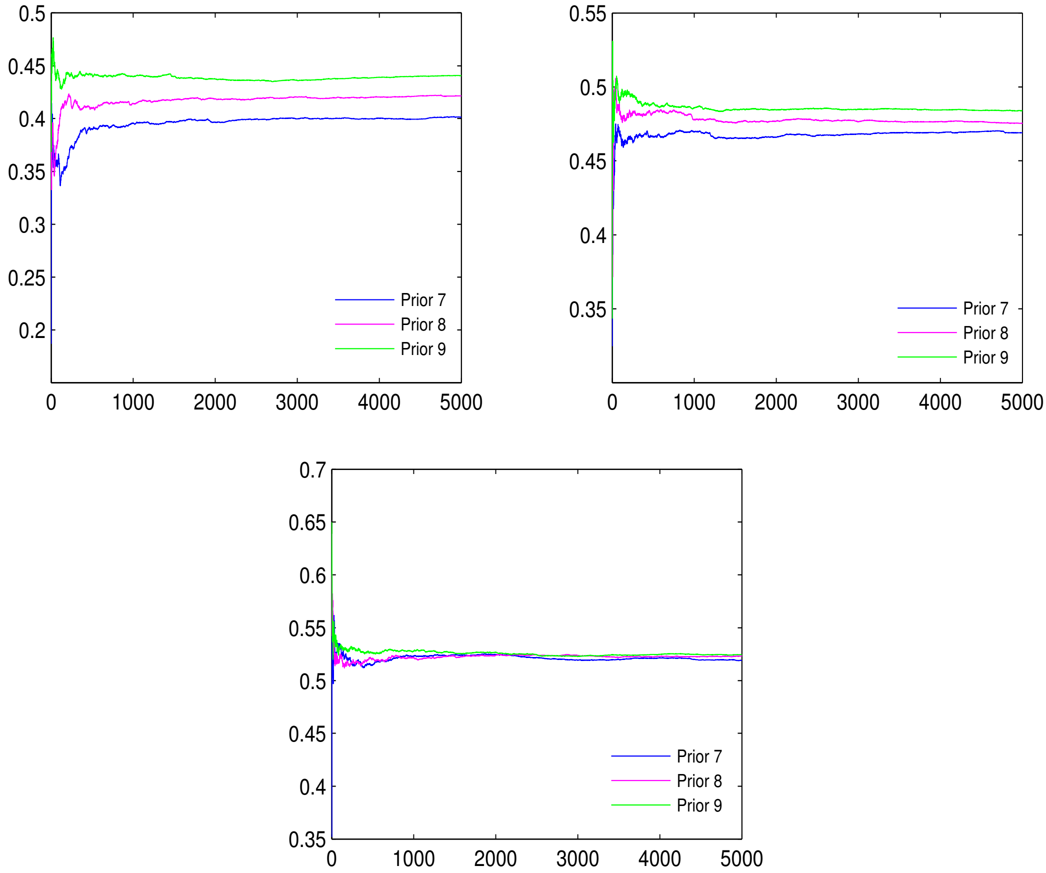

Fig. 5.2. Trace plots with C.S $(1, 1)$ with $(N, n) = (50, 20)$ (top left), $(2, 3)$ with $(N, n) = (50, 30)$ (top right), $(3, 3)$ with $(N, n) = (70, 30)$ (down) in general case.

consider the existence and uniqueness of the MLE and construct the asymptotic and HPD intervals for R. Second, when the common second shape parameter, c, is known, we obtain the MLE and exact Bayes estimate of R. Third, in general case, when $X \sim WSN(a, c_1)$ and $Y \sim WSN(b, c_2)$, we provide ML, AML and Bayesian inferences of R.

From the simulation results, which obtained by the Monte Carlo method, in point estimates, we observed that the Bayes estimates have the minimum value of MSEs. Also, in Bayesian inference, the informative priors perform better than non-informative ones. Furthermore, the MCMC method performs better than Lindley's approximation. In interval estimates, we observed that the HPD credible intervals have the better performance than the asymptotic confidence intervals. Also, in Bayesian inference, the HPD credible intervals based on informative priors have the smallest average lengths and largest coverage percentages.

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APPENDIX A

The standard extreme value distribution has the p.d.f. and c.d.f. as

$$g(v) = e^{v-e^v}, \quad G(v) = 1 - e^{-e^v}.$$

Therefore, based on the observed data $\{U_1, \ldots, U_n\}$ and $\{V_1, \ldots, V_m\}$, ignoring the constant value, the log-likelihood function is as follows:

$$\ell^*(\xi_1, \xi_2, \sigma) \propto -n\log(\sigma) + \sum_{i=1}^{n} t_i - \sum_{i=1}^{n}(r_i+1)e^{t_i} - m\log(\sigma) + \sum_{j=1}^{m} z_j - \sum_{j=1}^{m}(s_j+1)e^{z_j}, \tag{6.26}$$

where

$$t_i = \frac{u_i - \xi_1}{\sigma},\ z_j = \frac{v_j - \xi_2}{\sigma}.$$

Now by taking derivatives with respect to ξ_1, ξ_2 and σ from (6.26), we obtain the following equations:

$$\frac{\partial \ell^*}{\partial \xi_1} = -\frac{1}{\sigma}\left[n - \sum_{i=1}^{n}(r_i+1)e^{t_i}\right] = 0,$$

$$\frac{\partial \ell^*}{\partial \xi_2} = -\frac{1}{\sigma}\left[m - \sum_{j=1}^{m}(s_j+1)e^{z_j}\right] = 0,$$

$$\frac{\partial \ell^*}{\partial \sigma} = -\frac{1}{\sigma}\left[n + m + \sum_{i=1}^{n} t_i - \sum_{i=1}^{n}(r_i+1)t_i e^{t_i} + \sum_{j=1}^{m} z_j - \sum_{j=1}^{m}(s_j+1)z_j e^{z_j}\right] = 0.$$

To obtain the AMLEs, let

$$q_i = 1 - \prod_{j=n-i+1}^{n} \frac{j + \sum\limits_{k=n-j+1}^{n} R_k}{j + 1 + \sum\limits_{k=n-j+1}^{n} R_k}, \ i = 1, \ldots, n,$$

$$\bar{q}_j = 1 - \prod_{i=m-j+1}^{m} \frac{i + \sum\limits_{k=m-i+1}^{m} S_k}{i + 1 + \sum\limits_{k=m-i+1}^{m} S_k}, \ j = 1, \ldots, m.$$

By expanding the functions e^{t_i} and e^{z_j} in Taylor series around the points

$$\nu_i = \log\big(-\log(1 - q_i)\big), \qquad \bar{\nu}_j = \log\big(-\log(1 - \bar{q}_j)\big),$$

respectively and keeping the first order derivatives, we have $e^{t_i} = a_i + b_i t_i$ and $e^{z_j} = \bar{a}_j + \bar{b}_j z_j$, where $a_i = e^{\nu_i}(1 - \nu_i), \ b_i = e^{\nu_i}, \ \bar{a}_j = e^{\bar{\nu}_j}(1 - \bar{\nu}_j)$ and $\bar{b}_j = e^{\bar{\nu}_j}$. With the similar manner to [1], we derive the AMLEs of ξ_1, ξ_2 and σ, say $\tilde{\xi}_1$, $\tilde{\xi}_2$, and $\tilde{\sigma}$, respectively, by

$$\begin{aligned}
\tilde{\xi}_1 &= A_1 - \tilde{\sigma} B_1, \\
\tilde{\xi}_2 &= A_2 - \tilde{\sigma} B_2, \\
\tilde{\sigma} &= \frac{-(D_1 + D_2) + \sqrt{(D_1 + D_2)^2 + 4(C_1 + C_2)(E_1 + E_2)}}{2(C_1 + C_2)},
\end{aligned}$$

where A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, E_1, E_2 are given as follows:

$$\begin{aligned}
A_1 &= \frac{\sum\limits_{i=1}^{n} (r_i + 1) b_i u_i}{\sum\limits_{i=1}^{n} (r_i + 1) b_i}, \quad B_1 = \frac{\sum\limits_{i=1}^{n} a_i - \sum\limits_{i=1}^{n} r_i (1 - a_i)}{\sum\limits_{i=1}^{n} (r_i + 1) b_i}, \quad C_1 = n, \\
A_2 &= \frac{\sum\limits_{j=1}^{m} (s_j + 1) \bar{b}_j v_j}{\sum\limits_{j=1}^{m} (s_j + 1) \bar{b}_j}, \quad B_2 = \frac{\sum\limits_{j=1}^{m} \bar{a}_j - \sum\limits_{j=1}^{m} s_j (1 - \bar{a}_j)}{\sum\limits_{j=1}^{m} (s_j + 1) \bar{b}_j}, \quad C_2 = m, \\
D_1 &= \sum_{i=1}^{n} a_i u_i - A_1 B_1 \left(\sum_{i=1}^{n} (r_i + 1) b_i \right) - \sum_{i=1}^{n} r_i u_i (1 - a_i), \\
D_2 &= \sum_{j=1}^{J_2} \bar{a}_j v_j - A_2 B_2 \left(\sum_{j=1}^{m} (s_j + 1) \bar{b}_j \right) - \sum_{j=1}^{m} s_j v_j (1 - \bar{a}_j), \\
E_1 &= \sum_{i=1}^{n} (r_i + 1) b_i (u_i - A_1)^2, \quad E_2 = \sum_{j=1}^{m} (s_j + 1) \bar{b}_j (v_j - A_2)^2.
\end{aligned}$$

APPENDIX B

Let $U(\lambda)$ be a function of parameter value. The Bayes estimate of $U(\lambda)$, under the squared error loss function is

$$\mathbb{E}\big(u(\lambda)|\text{data}\big) = \frac{\int u(\lambda)e^{Q(\lambda)}d\lambda}{\int e^{Q(\lambda)}d\lambda},$$

where $Q(\lambda) = \ell(\lambda) + \rho(\lambda)$, $\ell(\lambda)$ and $\rho(\lambda)$ are the logarithm of likelihood function and prior density of λ, respectively. Lindley has been approximated $\mathbb{E}(u(\lambda)|\text{data})$ as

$$\mathbb{E}\big(u(\lambda)|\text{data}\big) = u + \frac{1}{2}\sum_i\sum_j(u_{ij} + 2u_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_i\sum_j\sum_k\sum_p \ell_{ijk}\sigma_{ij}\sigma_{kp}u_p\Big|_{\lambda=\widehat{\lambda}},$$

where $\lambda = (\lambda_1, \dots, \lambda_m)$, $i, j, k, p = 1, \dots, m$, $\widehat{\lambda}$ is the MLE of λ, $u = u(\lambda)$, $u_i = \partial u/\partial\lambda_i$, $u_{ij} = \partial^2 u/\partial\lambda_i\partial\lambda_j$, $\ell_{ijk} = \partial^3\ell/\partial\lambda_i\partial\lambda_j\partial\lambda_k$, $\rho_j = \partial\rho/\partial\lambda_j$, and $\sigma_{ij} = (i, j)$th element in the inverse of matrix $[-\ell_{ij}]$ all calculated at the MLE of parameters.

When we confront the case of three parameter $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, Lindley's approximation conducts to

$$\begin{aligned}\mathbb{E}\big(u(\lambda)|\text{data}\big) = u + (u_1d_1 + u_2d_2 + u_3d_3 + d_4 + d_5) + \frac{1}{2}[A(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13})\\ + B(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) + C(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33})],\end{aligned}$$

calculated at $\widehat{\lambda} = (\widehat{\lambda}_1, \widehat{\lambda}_2, \widehat{\lambda}_3)$, where

$$\begin{aligned}
d_i &= \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3}, \quad i = 1, 2, 3,\\
d_4 &= u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23},\\
d_5 &= \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33}),\\
A &= \ell_{111}\sigma_{11} + 2\ell_{121}\sigma_{12} + 2\ell_{131}\sigma_{13} + 2\ell_{231}\sigma_{23} + \ell_{221}\sigma_{22} + \ell_{331}\sigma_{33},\\
B &= \ell_{112}\sigma_{11} + 2\ell_{122}\sigma_{12} + 2\ell_{132}\sigma_{13} + 2\ell_{232}\sigma_{23} + \ell_{222}\sigma_{22} + \ell_{332}\sigma_{33},\\
C &= \ell_{113}\sigma_{11} + 2\ell_{123}\sigma_{12} + 2\ell_{133}\sigma_{13} + 2\ell_{233}\sigma_{23} + \ell_{223}\sigma_{22} + \ell_{333}\sigma_{33}.
\end{aligned}$$

In our case, for $(\lambda_1, \lambda_2, \lambda_3) \equiv (a, b, c)$ and $u = R = \frac{a}{a+b}$, we have

$$\rho_1 = \frac{a_1 - 1}{a} - b_1, \quad \rho_2 = \frac{a_2 - 1}{b} - b_2, \quad \rho_3 = \frac{a_3 - 1}{c} - b_3,$$

σ_{ij}, $i, j = 1, 2, 3$ are obtained by using ℓ_{ij}, $i, j = 1, 2, 3$ and

$$\begin{aligned}
\ell_{111} &= \frac{2n}{a^3}, \quad \ell_{222} = \frac{2m}{b^3},\\
\ell_{133} &= \ell_{331} = \ell_{313} = -\sum_{i=1}^{n} w(x_i, r_i, c, 2),\\
\ell_{233} &= \ell_{332} = \ell_{323} = -\sum_{j=1}^{m} w(y_j, s_j, c, 2),\\
\ell_{333} &= \frac{2(m+n)}{c^3} - a\sum_{i=1}^{n} w(x_i, r_i, c, 3) - b\sum_{j=1}^{m} w(y_j, s_j, c, 3),
\end{aligned}$$

and other $\ell_{ijk} = 0$. Moreover, $u_3 = u_{i3} = 0$, $i = 1, 2, 3$, and u_1, u_2 are given in (2.8). Also, $u_{11} = \frac{-2b}{(a+b)^3}$, $u_{12} = u_{21} = \frac{a-b}{(a+b)^3}$, $u_{22} = \frac{2a}{(a+b)^3}$. So,

$$
\begin{aligned}
d_4 &= u_{12}\sigma_{12}, \\
d_5 &= \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22}), \\
A &= \ell_{111}\sigma_{11} + 2\ell_{131}\sigma_{13} + \ell_{331}\sigma_{33}, \\
B &= 2\ell_{232}\sigma_{23} + \ell_{222}\sigma_{22} + \ell_{332}\sigma_{33}, \\
C &= \ell_{113}\sigma_{11} + 2\ell_{133}\sigma_{13} + 2\ell_{233}\sigma_{23} + \ell_{223}\sigma_{22} + \ell_{333}\sigma_{33}.
\end{aligned}
$$

It is notable that all parameters are evaluated at $(\widehat{a}, \widehat{b}, \widehat{c})$.