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The Optimal Control Approach in Problems of the Income Distribution into Consumed and Invested Parts

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Abstract: In this paper, we investigate dynamic problems of the income distribution into consumed and invested parts using optimal control methods. Here it is assumed that the basic economic identity holds true, i.e., the income is the sum of the consumption and the investment. The income is understood either in the pure form or the gross domestic product, or the national income. The consumption is also understood in different ways: in its pure form or the aggregate consumption. We start with the analysis of the Harrod–Domar macromodel with the capital intensity of the income growth depending on the continuous time. In particular, it is shown that the balance equation for accumulated household savings also satisfies the basic economic identity and the capital intensity of the income growth can depend on time. Since households are the best savers and they demonstrate the best survival, we modify the considered macromodels replacing the given production functions with the problem of maximizing the integral discounted utility of the consumption. In this case, the utility function satisfies the Arrow–Pratt condition of the relative risk aversion.

Keywords: optimal control, income, consumption, investment

INTRODUCTION

Throughout the paper, we assume the basic economic identity: the income is equal to the sum of the consumption and the investment [1]. This identity has many various applications; see, e.g., [2]. For example, this identity holds true in the Harrod–Domar and the Solow macromodels [3]– [15], and also in the household micromodel [**?**], [19]. Hence, these models have some common properties. One of the main goals of the paper is to establish such properties.

It should be noted that in these models the income is understood in different ways. The income is understood either in the pure form or the gross domestic product, or the national income. The consumption is also understood in different ways: in its pure form or the aggregate consumption.

First, consider the Harrod–Domar macromodel with time-dependent capital intensity of the income growth. We also introduce a new variable, namely, the relative increase in the income of the Cobb–Douglas production function, and consider the balance equation for the capital. As a result, we establish the time dependence for the capital intensity of the income growth of the Solow model. Further, we show that the capital intensity of the income growth in the household model can also depend on time.

Since households are the best savers and they demonstrate the best survival, we shall modify the considered macromodels. Namely, we replace the production functions given a priori with the problem of maximizing the integral discounted utility of the consumption, as

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households usually do. Moreover, in all these models the utility function satisfies the Arrow–Pratt condition of the relative risk aversion; see [1, 2, 13, 14].

1. THE PRELIMINARY RESULTS

1.1. The main economic identity

In this paper, we shall always assume that the main economic identity holds true:

$$Y(t) = C(t) + I(t), \tag{1.1}$$

where t means the continuous time, $Y(t)$ is the net income or the gross domestic product or the national income, $C(t)$ is the consumption or the total consumption, $I(t)$ is the general investment.

1.2. The dependence of the income on the consumption in the Harrod–Domar model with varying capital intensity of the income growth

The main assumption in the Harrod–Domar model is that the investment linearly depends on the derivative of the income by t, that is, $I(t) = BY'(t)$; see [1, 15]. Here the coefficient B is a positive constant. Assuming that B depends on t, we have:

$$Y(t) = C(t) + B(t)Y'(t), \tag{1.2}$$

where $B(t)$ is the capital intensity of the income growth. Setting for (1.2) the initial condition

$$Y(t_0) = Y_0 > 0, \tag{1.3}$$

we obtain the Cauchy problem, whose solution is given by the following formula (see, e.g., [1, 8]):

$$Y(t) = \exp\left\{\int_{t_0}^{t} \frac{ds}{B(s)}\right\}\left(Y_0 - \int_{t_0}^{t} \frac{C(\tau)}{B(\tau)} \exp\left\{-\int_{t_0}^{\tau} \frac{ds}{B(s)}\right\} d\tau\right). \tag{1.4}$$

Formula (1.4) determines the dependence of the income on the consumption.

1.3. The relative increase of the income for the Cobb–Douglas production function

Consider the Cobb–Douglas production function [17, 18]:

$$Y = F(K, L) = AK^{\alpha}L^{1-\alpha}, \quad A = \text{const} > 0, \quad 0 < \alpha < 1, \tag{1.5}$$

where $K = K(t)$ is the capital, $L = L(t)$ is the labor, i.e., the number of employees. For convenience, let us introduce a new variable

$$\sigma = \sigma(t) = \frac{Y'(t)}{Y(t)} = \alpha\frac{K'(t)}{K(t)} + (1-\alpha)\frac{L'(t)}{L(t)},$$

which is called the relative increase of the income. Introduce the standard value

$$\nu = \nu(t) = \frac{L'(t)}{L(t)}, \tag{1.6}$$

which we call the rate of the growth of labor resources. Then, taking into account (1.6), for the relative increase in income we have the expression

$$\sigma = \sigma(t) = \frac{Y'(t)}{Y(t)} = \alpha\frac{K'(t)}{K(t)} + (1-\alpha)\nu. \tag{1.7}$$

1.4. The balance equation for capital

Obviously, the balance equation for capital has the form

$$K'(t) = I(t) - \mu K(t), \tag{1.8}$$

where μ is the share of the retired capital. The standard assumption

$$I(t) = \rho Y(t), \tag{1.9}$$

where $\rho = \rho(t)$ is the rate of accumulation [1, 15, 17, 18], leads to the equality

$$\frac{K'(t)}{K(t)} = \frac{\rho Y(t)}{K(t)} - \mu = \rho A \left(\frac{K(t)}{L(t)}\right)^{\alpha-1} - \mu. \tag{1.10}$$

Then, taking into account (1.7), we have

$$\sigma(t) = \alpha\rho A k(t)^{\alpha-1} + \nu - \alpha(\mu + \nu), \tag{1.11}$$

where $k(t) = K(t)/L(t)$ is the capital-labor ratio.

1.5. Capital capacity of increase in the Solow model

Sine equation (1.11) contains the capital-labor ratio, it leads to the Solow macromodel [12]–[15]. Using the equality $I(t) = B(t)Y'(t)$ for determination the incremental capital intensity and taking into account (1.9), we get

$$I(t) = B(t)Y'(t) = \rho Y(t), \quad B(t) = \frac{\rho}{\sigma(t)}. \tag{1.12}$$

The first equality has a deep economic meaning. When investing, an investor has to monitor not only the high capital intensity of the income growth. One must be sure that this incremental capital intensity is high not due to the small relative increase in income. Substituting expression (1.11) into the right-hand formula (1.12), we obtain

$$B(t) = \frac{\rho}{\alpha\rho A k(t)^{\alpha-1} + \nu - \alpha\lambda}, \quad \lambda = \mu + \nu. \tag{1.13}$$

In order to check that (1.13) is not constant and it really depends on time, one can use the exact solution of the Solow model for the Cobb–Douglas production function. For the capital-labor ratio, this yields the equations

$$k'(t) = \rho f(k(t)) - \lambda k(t) = \rho A k(t)^{\alpha} - \lambda k(t), \quad \lambda = \text{const}. \tag{1.14}$$

Setting the initial condition

$$k(t_0) = k_0 > 0, \tag{1.15}$$

we get the Cauchy problem, whose exact solution is given by the formula

$$k(t) = e^{-\lambda(t-t_0)} \left(\frac{\rho A}{\lambda} e^{\lambda(1-\alpha)(t-t_0)} - \frac{\rho A}{\lambda} + k_0^{1-\alpha}\right)^{\frac{1}{1-\alpha}}. \tag{1.16}$$

Finally, from (1.13) and (1.16) we have

$$B(t_0) = \frac{\rho}{\alpha\rho A k_0^{\alpha-1} + \nu - \alpha\lambda}, \quad B(+\infty) = \frac{\rho}{\nu}. \tag{1.17}$$

The obtained formula shows that the capital intensity of the income growth depends on time and in the general case it is not constant.

1.6. The balance equation for accumulated household savings

Now consider the household micromodel as an economic model governed by equation (1.1); see [**?**, 14]. Then we have the equation

$$x'(t) = p(t)x(t) + P(t) - C(t), \tag{1.18}$$

where $x(t)$ is the accumulated savings, $p(t)$ is the bank interest, $P(t)$ is determined by salaries, pensions and extra earnings. As before, $C(t)$ means the consumption.

If we assume that $Y(t) = p(t)x(t) + P(t)$ is the income and $I(t) = x'(t)$ is the investments, then we come to the basic economic identity (1.1). Similarly to [**?**, 1, 14], we maximize the integral discounted utility of consumption

$$J = \int_{t_0}^{t_1} u(C(t))e^{-\delta t}\, dt, \tag{1.19}$$

where u is the utility function, δ is the coefficient of discounting the future utility.

1.7. The utility function and the risk avoidance

According to the Arrow–Pratt theory [**?**, 1, 14], the utility of consumption is estimated by the function $u(C)$, which describes the constant aversion to risk:

$$-\frac{u''(C)C}{u'(C)} = a \geq 0. \tag{1.20}$$

The relative measure of the Arrow–Pratt risk aversion is the consumption elasticity of the marginal utility taken with the opposite sign. Let us denote by $g(C) = u'(C)$ the marginal utility of consumption, and by

$$E_C(g) = \frac{g'(C)}{g(C)/C}$$

the elasticity of variation of the variable g with respect to the variable C. Then, similarly to [**?**, 1], one can consider (1.20) as a differential equation for $u(t)$. This yields $u'(C) = g(C) = \gamma C^{-a}$, where $\gamma > 0$ is the constant of integration. Finally, we have

$$u(C) = \begin{cases} \dfrac{\gamma C^{1-a}}{1-a} + \chi, & a \neq 1, \\ \gamma \ln C \; + \chi, & a = 1, \end{cases}$$

where χ is an arbitrary constant.

2. THE OPTIMAL CONTROL APPROACH

2.1. Optimal control in the modified Harrod–Domar model

According to [17], households are the best savers. This motivates the maximization of the functional (1.19) in the framework of the modified Harrod–Domar model, which is governed by the equation $Y(t) = C(t) + BY'(t)$.

Consider the maximization of the functional (1.19), where the utility function satisfies the following conditions:

$$-\frac{u''(C)C}{u'(C)} = a \geq 0, \;\; g(C) = u'(C), \;\; E_C(g) = -a, \tag{2.21}$$

under the following constrains:

$$Y(t) = C(t) + BY'(t), \;\; Y(t_0) = Y_0 > 0, \;\; Y(t_1) = Y_1 > 0. \tag{2.22}$$

This maximization problem can be solved using the standard variational method [**?**, 1]. A necessary condition of optimality is the Euler–Lagrange equation, which reads

$$\frac{d}{dt}\big(g(C(t))B(t)e^{-\delta t}\big) + g(C(t))e^{-\delta t} = 0. \tag{2.23}$$

Integrating (2.23), we have

$$C(t) = g^{-1}\bigg(\frac{D}{B(t)}\,\exp\bigg\{\delta t - \int_{t_0}^{t} \frac{d\tau}{B(\tau)}\bigg\}\bigg), \tag{2.24}$$

where D is the constant of integration. Finally, condition (2.21) gives a sufficient condition of optimality.

Theorem 2.1:
Under the above conditions, the total consumption $C(t)$ is given by the formula

$$C(t) = \big(Y_0 - Y_1 \exp\{-\beta(t_1)\}\big) \frac{B(t)^{\frac{1}{a}} \exp\{\frac{1}{a}\beta(t) - \frac{\delta}{a}t\}}{\displaystyle\int_{t_0}^{t_1} B(\tau)^{\frac{1}{a}-1} \exp\{(\frac{1}{a}-1)\beta(\tau) - \frac{\delta}{a}\tau\}\, d\tau}, \tag{2.25}$$

where the auxiliary function β is defined as

$$\beta(t) = \int_{t_0}^{t} \frac{ds}{B(s)}.$$

Proof
From (2.24) we have

$$g(C(t))B(t) = D\exp\{\delta t - \beta(t)\}, \;\; D = \text{const} > 0. \tag{2.26}$$

On the other hand, from (2.21) we get $g(C) = \gamma C^{-a}$, $\gamma = \text{const} > 0$. Substituting this expression into (2.21), we have

$$C(t) = \bigg(\frac{\gamma B(t)}{D}\bigg)^{\frac{1}{a}} \exp\{a^{-1}(\beta(t) - \delta t)\}. \tag{2.27}$$

Now from (1.4) and (2.22) we obtain the equality

$$\int_{t_0}^{t_1} \frac{C(\tau)}{B(\tau)} e^{-\beta(\tau)} d\tau = Y_0 - Y_1 e^{-\beta(t_1)}, \tag{2.28}$$

and from (2.27) and (2.28) one can express the constant:

$$\bigg(\frac{\gamma}{D}\bigg)^{\frac{1}{a}} = \frac{Y_0 - Y_1 e^{-\beta(t_1)}}{\displaystyle\int_{t_0}^{t_1} B(\tau)^{\frac{1}{a}-1} \exp\{(\frac{1}{a}-1)\beta(\tau) - \frac{\delta}{a}\tau\}\, d\tau}, \tag{2.29}$$

Finally, substituting the expression (2.29) into (2.27) and taking into account (2.22), we obtain (2.25). □

2.2. Optimal control in the modified Solow model

Similarly to the above, here we consider the maximization of the functional

$$J = \int_{t_0}^{t_1} u(c(t))e^{-\delta t}\,dt, \text{ where } c(t) = \frac{C(t)}{L(t)}, \tag{2.30}$$

and the utility function u satisfies the conditions similar to (2.21):

$$-\frac{u''(c)c}{u'(c)} = a \geq 0,\ \ g(c) = u'(c),\ \ E_c(g) = -a, \tag{2.31}$$

under the following constrains (see (1.14)):

$$k'(t) = \rho(t)f(k(t)) - \lambda k(t),\ \ k(t_0) = k_0 \geq 0,\ \ k(t_1) = k_1 \geq 0. \tag{2.32}$$

Taking into account the formulas

$$c(t) = \frac{C}{L} = \frac{Y-I}{L} = \frac{Y-\rho Y}{L} = (1-\rho)\frac{Y}{L} = (1-\rho)f(k),$$

$$k'(t) = \frac{c\rho}{1-\rho} - \lambda k,\ \ c(t) = \frac{1-\rho}{\rho}(k' + \lambda k),\ \ 0 < \rho < 1,$$

the Euler–Lagrange equation (a necessary condition of optimality) reads

$$-\frac{d}{dt}\left(g(c(t))\frac{1-\rho}{\rho}e^{-\delta t}\right) + g(c(t))\frac{1-\rho}{\rho}\lambda e^{-\delta t} = 0. \tag{2.33}$$

Integrating (2.33) (see [1]), we have

$$c(t) = g^{-1}\left(\frac{D\rho}{1-\rho}\ \exp\left\{\delta t + \int_{t_0}^{t}\lambda(s)ds\right\}\right). \tag{2.34}$$

The condition (2.31) gives a sufficient condition of optimality.

Theorem 2.2:
Under the above conditions, the total consumption $c(t)$ is given by the formula

$$c(t) = \left(k_1 e^{\Lambda(t_1)} - k_0\right)\frac{R(t)^{\frac{1}{a}}\exp\left\{-\frac{1}{a}(\Lambda(t)+\delta t)\right\}}{\displaystyle\int_{t_0}^{t_1} R(\tau)^{\frac{1}{a}-1}\exp\left\{(1-\frac{1}{a})\Lambda(\tau) - \frac{\delta}{a}\tau\right\}d\tau} \tag{2.35}$$

where the auxiliary functions l and R are defined as

$$\Lambda(t) = \int_{t_0}^{t}\lambda(s)ds,\ \ R(t) = \frac{1-\rho(t)}{\rho(t)}.$$

Proof

From (2.34) we have

$$g(c(t))R(t)e^{-\delta t} = De^{\Lambda(t)},\ \ D = \text{const} > 0. \tag{2.36}$$

On the other hand, from (2.31) we get $g(c) = \gamma c^{-a}$, $\gamma = \text{const} > 0$. Substituting this expression into (2.36), we have

$$c(t) = \left(\frac{\gamma}{D}R(t)\right)^{\frac{1}{a}} \exp\{-a^{-1}(\Lambda(t) + \delta t)\}. \tag{2.37}$$

Substituting (2.37) into the equation

$$k'(t) = \frac{c\rho}{1-\rho} - \lambda k,$$

we get a linear differential equation for $k(t)$. Integrating it, we found

$$k(t) = e^{-\Lambda(t)} \int_{t_0}^{t} \frac{c(\tau)}{R(\tau)} e^{\Lambda(\tau)} d\tau + k_0 e^{-\Lambda(t)}. \tag{2.38}$$

From (2.37) and (2.38) it follows

$$k_1 e^{\Lambda(t_1)} - k_0 = \left(\frac{\gamma}{D}\right)^{\frac{1}{a}} \int_{t_0}^{t_1} R(\tau)^{\frac{1}{a}-1} \exp\{(\tfrac{1}{a}-1)\Lambda(\tau) - \tfrac{\delta}{a}\tau\} d\tau. \tag{2.39}$$

The obtained equality allows us to express the constant $(\gamma/D)^{\frac{1}{a}}$. Therefore, from the equalities (2.37) and (2.39) we obtain (2.35). □

2.3. Optimum consumption management in household model

In this section, we study a household model in more detail, see [17, 21]. As in the Solow model, the study is based on the maximization of the functional (2.30), and the utility function $u(c)$ satisfies the conditions (2.31). However, the constrains now have the form of the balance equation for accumulated household savings (see (1.19)), that is,

$$x'(t) = p(t)x(t) + P(t) - c(t), \;\; x(t_0) = x_0 \geq 0, \;\; x(t_1) = x_1 \geq 0. \tag{2.40}$$

Then the Euler–Lagrange equation (a necessary condition of optimality) reads

$$\frac{d}{dt}\Big(g(c(t))e^{-\delta t}\Big) + g(c(t))p(t)e^{-\delta t} = 0. \tag{2.41}$$

Integrating (2.41), we have

$$c(t) = g^{-1}\left(D \exp\left\{\delta t - \int_{t_0}^{t} p(s)ds\right\}\right). \tag{2.42}$$

As before, the condition (2.31) gives a sufficient condition of optimality. Moreover, for the Cauchy problem obtained from the condition (2.40) by elimination the last equality (for $x(t_1)$) one can write the solution:

$$x(t) = e^{W(t)}\left(x_0 + \int_{t_0}^{t} (P(\tau) - c(\tau))e^{-W(\tau)} d\tau\right), \quad W(t) = \int_{t_0}^{t} p(s)ds. \tag{2.43}$$

Theorem 2.3:
Under the above conditions, the total consumption $c(t)$ is given by the formula

$$c(t) = \frac{\displaystyle\int_{t_0}^{t_1} P(\tau)e^{-W(\tau)}d\tau + x_0 - x_1 e^{-W(t_1)}}{\displaystyle\int_{t_0}^{t_1} \exp\big\{(\tfrac{1}{a}-1)W(\tau) - \tfrac{\delta}{a}\tau\big\} d\tau} \exp\big\{\tfrac{1}{a}(W(t) - \delta t)\big\}. \tag{2.44}$$

Proof
From (2.34) we have

$$g(c(t)) = D\exp\{\delta t - W(t)\}, \ \ D = \text{const} > 0. \tag{2.45}$$

From (2.31) we get $g(c) = \gamma c^{-a}$, $\gamma = \text{const} > 0$. Substituting this expression into (2.45), we obtain the equation

$$c(t) = c_0 \exp\{\tfrac{1}{a}(W(t) - \delta t)\}, \ \ c_0 = \Big(\frac{\gamma}{D}\Big)^{\frac{1}{a}}. \tag{2.46}$$

After that, from (2.40), (2.43), (2.46) we have the equality

$$x_1 e^{-W(t_1)} - x_0 = \int_{t_0}^{t_1} \big(P(\tau) - c_0 \exp\{\tfrac{1}{a}(W(\tau) - \delta\tau)\}\big) e^{-W(\tau)} d\tau, \tag{2.47}$$

which yields

$$c_0 = \frac{\displaystyle\int_{t_0}^{t_1} P(\tau)e^{-W(\tau)}d\tau + x_0 - x_1 e^{-W(t_1)}}{\displaystyle\int_{t_0}^{t_1} \exp\{\tfrac{1}{a}(W(\tau) - \delta\tau) - W(\tau)\}d\tau}. \tag{2.48}$$

Finally, from (2.46) and (2.48) we obtain (2.44). □

2.4. Restrictions on the accumulated savings in household model

In this section, we obtain conditions for no debt in the in the household model considered above. In our notations, no debt is the inequality

$$x(t) \geq 0, \ \ t_0 \leq t \leq t_1. \tag{2.49}$$

Theorem 2.4:
A sufficient condition for (2.49) *is given by the inequality*

$$\Phi(t) - \Phi(t_1) \geq 0, \ \ t_0 \leq t \leq t_1. \tag{2.50}$$

Proof
First, let us prove that the inequality

$$x(t) \geq x_0 + \int_{t_0}^{t} p(\tau)x(\tau)d\tau, \ \ t_0 \leq t \leq t_1, \tag{2.51}$$

is equivalent to

$$\int_{t_0}^{t} (P(\tau) - c(\tau))d\tau, \ \ t_0 \leq t \leq t_1. \tag{2.52}$$

Indeed, integrating the both sides of the equation (2.40), we have the equality

$$x(t) - x_0 = \int_{t_0}^{t} p(\tau)x(\tau)d\tau + \int_{t_0}^{t} (P(\tau) - c(\tau))d\tau,$$

which shows the equivalence of (2.51) and (2.52).

It should be noted that the inequality (2.51) has an obvious economic meaning: The right-hand side of (2.51) is the amount of money that would be on the deposit account at the moment t if the amount x_0 was deposited on the account at the moment t_0.

Second, let us present (2.52) in the form

$$x(t) = \Psi(t)e^{W(t)}(\Phi(t) - \Phi(t_1)) + x_1\frac{\Phi(t)}{\Phi(t_1)}\, e^{W(t)-W(t_1)}, \tag{2.53}$$

with the functions

$$\begin{aligned}\Psi(t) &= \int_{t_0}^{t} \exp\{(\tfrac{1}{a}-1)W(\tau) - \tfrac{\delta}{a}\tau\}d\tau,\\ \Phi(t) &= \left(x_0 + \int_{t_0}^{t} P(\tau)e^{-W(\tau)}d\tau\right)\Psi^{-1}(t).\end{aligned} \tag{2.54}$$

Since the function Ψ is strictly positive and $x_1 \geq 0$, the equality (2.53) obviously shows that (2.50) implies (2.49). □

The function Φ defined above is called the *indicator function* of the model. Theorem 2.4 states that if the indicator function decreases, then the condition (2.49) holds true.

3. EXAMPLES OF INDICATOR FUNCTIONS

Here we discuss several examples of monotonically decreasing indicator functions in the household model. Let

$$P(t) = P_0e^{r(t-t_0)}, \;\; P_0 = \text{const} > 0, \;\; p(t) = p = \text{const} > 0. \tag{3.55}$$

Then from (2.54) we have

$$\Phi(t) = \frac{x_0e^{p(t-t_0)} + P_0e^{pt-rt_0}\displaystyle\int_{t_0}^{t} e^{(r-p)\tau}d\tau}{e^{p(t-t_0/a)}\displaystyle\int_{t_0}^{t} \exp\{((\tfrac{1}{a}-1)p - \tfrac{\delta}{a})\tau\}d\tau}. \tag{3.56}$$

To simplify further calculations, we introduce new notations:

$$\Theta = (\tfrac{1}{a}-1)p - \tfrac{\delta}{a}, \;\; \theta = r - p.$$

First, consider the generic case: $\theta \neq 0$ and $\Theta \neq 0$. Then formula (3.56) yields the indicator function

$$\Phi(t) = e^{\frac{\delta}{a}t_0}\left(x_0\frac{\Theta}{e^{\Theta(t-t_0)}-1} + P_0\frac{\Theta(e^{\theta(t-t_0)}-1)}{\theta(e^{\Theta(t-t_0)}-1)}\right). \tag{3.57}$$

Now consider the partial cases:

1. Assume that $\Theta = \theta$ and $\theta \neq 0$. Then we have the function

$$\Phi(t) = e^{\frac{\delta}{a}t_0}\left(x_0\frac{\Theta}{e^{\Theta(t-t_0)}-1} + P_0\right),$$

which is monotonically decreasing.

2. Assume that $\Theta > 0$ and $\theta = 0$. Then taking to the limit $\theta \to 0$ in (3.57), we get

$$\Phi(t) = e^{\frac{\delta}{a}t_0}\left(x_0\frac{\Theta}{e^{\Theta(t-t_0)}-1} + P_0\frac{\Theta(t-t_0)}{e^{\Theta(t-t_0)}-1}\right),$$

which is also a monotonically decreasing function.

3. Assume that $\Theta = \theta = 0$. Then taking to the limit $\Theta, \theta \to 0$ in (3.57), we obtain the function

$$\Phi(t) = e^{\frac{\delta}{a}t_0}\left(\frac{x_0}{t-t_0} + P_0\right).$$

This is a monotonically decreasing function as well.

4. CONCLUSION

The validity of the basic economic identity in all the models considered in the paper allows to determine general laws. The introduced new variable – the relative increase of the income – makes it possible to establish a correspondence between the Harrod–Domar models with variable incremental capital intensity and the Solow model with variable rate of accumulation. Since households are the best savers and they have the best survival rates, the savings regimes in the Harrod–Domar model and the Solow model are examined.

Finally, we analyze a modification of the above macromodels replacing the given production functions by setting the problem of maximizing the integral discounted utility of the consumption. In this case, the utility function satisfies the Arrow–Pratt condition of the relative risk aversion.

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