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# Data Envelopment Analysis in the Presence of Fuzzy Integer and Flexible Factors

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**Abstract:** In the traditional Data Envelopment Analysis (DEA) approaches, the inputs and outputs are usually regarded as exact and real values. Decision Making Units' (DMUs') relative efficiency is assessed and it is known that the factors are input and/or output. However, there are some conditions in which the DMUs' efficiency should be calculated while the data is integer-valued and ambiguous. So, different integer DEA models were proposed to determine the units' performance when integer-valued data and fuzzy factors are present. Furthermore, there are occasions which DMUs' efficiency score should be determined wherever integer-valued data and flexible factors are present. Hence, some integer DEA methods were suggested to calculate DMUs' performance and specify the role of flexible measures where some of the data are integer-valued and some of them are flexible factors. However, there are some situations that include integer data, fuzzy integer-valued measures and flexible factors. Hence, the current paper sheds on the kind of model that evaluate the entities' relative efficiency wherever integer-valued data, flexible factors and fuzzy integer-valued measures are present, and determines the role of factors with the uncertain input or output.

**Keywords:** DEA; relative efficiency; slacks-based model; flexible factor; fuzzy integer-valued measure; integer-valued data.

## 1. INTRODUCTION

Data envelopment analysis (DEA) is a non-parametric methodology which assesses the performance of a set of comparable Decision Making Units (DMUs). The conventional DEA models use the real-valued data and specify the roles of the data. Nevertheless, we face the cases in real application that inputs and/or outputs are integer-valued and the role of data is unknown. The factors that can be either inputs or outputs are called "flexible factors".

Cook and Zhu [1] considered the flexible factors to assess the DMUs' relative efficiency and specified the role of flexible measures. Amirteimoori, et al. [2] proposed a slacks-based measure with the flexible factors to evaluate the units' performance. Tohidi and Matroud [3] introduced a method to define the role of flexible factors. Lately, Toloo, et al. [4] suggested a non-radial directional distance model to categorize the flexible measures. Kordrostami, et al. [5] provided the methods to calculate the units' efficiency scores where integer-valued data and flexible factors are present. Lozano and villa [6] introduced models for examining integer-valued data in DEA. Afterwards, Kuosmanen and Kazemi Matin [7] provided a new axiomatic foundation for DEA models which are integer-valued. Jie, et al. [8] improved the Kuosmanen and Matin's model [7] and showed the model truly solves problems. Besides, Due, et al. [9] provided methods that examine the slacks to estimate the relative efficiency and super-efficiency scores of DMUs where integer-valued data are present.

Furthermore, the traditional DEA methods usually treat the data as continuous and exact measures. Nevertheless, there are situations that entities' performance should be estimated

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wherever integer-valued and imprecise data are present. Models with fuzzy measures can be detected in DEA context. Fuzzy set was used in [10] for the first time. Then, this theory was used to different problems; see [11,12,13] for more information.

Kordrostami, et al. [14] provided a number of models to assess the efficiency scores of DMUs and to identify the roles of fuzzy flexible measures. Saati and Imani [15] suggested a procedure to categorize shared factors using fuzzy concept and they determined the role of shared factors.

Kordrostami, et al. [16] introduced the approaches to estimate DMUs' efficiency score wherever integer-valued data and fuzzy factors are present. However, yet there is no study considering integer-valued data, fuzzy integer-valued measures and flexible factors in the texts related to DEA.

That's why, the current paper introduces the models to estimate DMUs' efficiency, and specify flexible factors' role, where integer-valued measures, fuzzy integer-valued data and flexible factors are present.

In section 2, the notations used in this paper are suggested and the main ideas of DEA with integer data, fuzzy integer-valued model and Flexible Slacks-Based Model (FSBM) are noted. In section 3, a new model for calculation of efficiency with integer-valued measures, fuzzy integer-valued data and flexible factors is presented. In section 4, the example is presented. In section 5, conclusion is drawn.

## 2. PRELIMINARIES

### *2.1. Notations*

Suppose we deal with *n* DMUs. Symbols are introduced as follows:

$j = 1, \dots, n$: the set of DMUs
$i = 1, \dots, m$: the set of inputs
$r = 1, \dots, s$: the set of outputs
$k = 1, 2, \dots, K$: the set of flexible factors
$DMU_j$: the $j$-th unit, $j = 1, \dots, n$
$DMU_o$: the unit under consideration
$x_{ij}$: the $i$-th input resource of $j$-th unit
$x_{io}$: the input resource $i$ of $DMU_o$
$y_{rj}$ : the output product $r$ of $DMU_j$
$y_{ro}$ : the output product $r$ of $DMU_o$
$z_{kj}$ : the $k$-th flexible factor of $DMU_j$
$z_{ko}$: the $k$-th flexible factor of $DMU_o$
$s_i$ : $i$-th input slacks for $i = 1, \dots, m$
$s_r$ : $r$-th output slacks for $r = 1, \dots, s$
$g_k^{(3)}$: flexible factor slacks as the input for $k = 1, \dots, K$
$g_k^{(4)}$: flexible factor slacks as the output for $k = 1, \dots, K$
$d_k^{(1)}, d_k^{(2)}$: binary variables
$\lambda_j$: intensity vectors of $DMU_j$
$I^I$: the subset of inputs which are integer-valued
$O^I$: the subset of outputs which are integer-valued
$K^I$: the subset of flexible factors which are integer-valued
$\tilde{x}_{ij}$ : the $i$-th triangular fuzzy input of $DMU_j$
$\tilde{y}_{rj}$ : the $r$-th triangular fuzzy output of $DMU_j$
$\tilde{x}_{io}$ : the $i$-th triangular fuzzy input of $DMU_o$
$\tilde{y}_{ro}$ : the $r$-th triangular fuzzy output of $DMU_o$

### 2.2. Integer-Valued DEA

Suppose we deal with n DMUs, DMUj ($j = 1,\dots,n$), with m input resources $x_{ij}$ ($i = 1,\dots,m$) and s output products $y_{rj}$($r = 1,\dots,s$). In the traditional DEA methods, all data are regarded as real-valued measures. Thus, the performance of units is measured while the reference points of units obtain values which are real. However, in many real worlds, some inputs and/or outputs can only be integer-valued measures. Assume $x_{ij}$($i = 1,\dots,m$) and $y_{rj}$($r = 1,\dots,s$) are integer-valued data of DMU$_j$ ($j = 1,\dots,n$), thus some DEA methods were developed and improved to get integer-valued projections for integer measures. The suggested model is used to estimate the entities' performance where integer-valued data, fuzzy integer-valued factors and flexible measures are present. In the next subdivision, a fuzzy integer-valued number is determined.

Section 2.3 is similar to that of section 2.2 in Kordrostami, et al. [16].

### 2.3. Main Concepts of Fuzzy Integer-Valued Numbers

Let $R$ be the set of real numbers and $Z$ be the set of integer numbers.

**Definition 2.3.a.** Suppose $u : R \to [0,1]$ is a fuzzy set. it is called a fuzzy integer if its support is a closed integer interval (denoted as $\langle \underline{u}(0), \bar{u}(0) \rangle$) and satisfies the following:

1. $u$ is normal; i.e., there exists $x' \in \langle \underline{u}(0), \bar{u}(0) \rangle$ such that $u(x') = 1$,
2. $u(x_i) \le u(x_j)$ for any $x_i, x_j \in \langle \underline{u}(0), x' \rangle$ with $x_i \le x_j$,
3. $u(x_i) \ge u(x_j)$ for any $x_i, x_j \in \langle x', \bar{u}(0) \rangle$ with $x_i \le x_j$.

Note that an interval, which is closed and integer, is showed by $\langle s_1, s_2 \rangle = \{x \in Z : s_1 \le x \le s_2\}$ for any $s_1, s_2 \in Z$ and $s_1 \le s_2$.

**Definition 2.3.b.** Suppose $s_0, s_1, t_1$ and $t_0 \in I$ with $s_0 \le s_1 \le t_1 \le t_0$, and $\underline{m}, \bar{m} \in Z$. If the fuzzy set $u : R \to [0,1]$ is determined as:

$$u(x) = \begin{cases} 1, if\ x \in \langle s_1, t_1 \rangle; \\ \dfrac{x - s_0}{s_1 - s_0}, if\ x \in \langle \underline{m}, s_1 \rangle; \\ \dfrac{t_0 - x}{t_0 - t_1}, if\ x \in \langle t_1, \overline{m} \rangle; \\ 0, if\ x \in \langle \underline{m}, \overline{m} \rangle. \end{cases}$$

where $s_0 \le \underline{m} \le s_1$ and $t_1 \le \bar{m} \le t_0$; then $u$ is a trapezoidal fuzzy integer. A triangular fuzzy integer-valued number can obtain, if $s_1 = t_1$. See Kordrostami, et al. [16].

### 2.4. DEA Models with Fuzzy Integer-Valued Measures

In this part, the methods are suggested to assess the DMUs' performance wherever fuzzy integer-valued factors are present. Assume there exist n units, that produce s outputs by consuming m inputs. The $j$-th unit showed by DMU$_j$ ($j = 1,\dots,n$), whose $x_{ij}$ ($i = 1,\dots,m$) and $y_{rj}$ ($r = 1,\dots,s$) are $i$-th input and $r$-th output, respectively. See Kordrostami, et al. [16]. The following model, referred to as the CCR model, was introduced by Charnes et al. [17] for estimating the entities' relative efficiency with data that are precise and real numbers.

$$Minimum \quad \theta \tag{2.1}$$

$$\theta x_{io} \geq \sum_{j=1}^{n} x_{ij}\lambda_j\,, i = 1,\ldots,m, \lambda_j \geq 0, j = 1,\ldots,n;$$

$\theta$ shows the efficiency score. $\lambda_j (j = 1,\ldots,n)$ indicate intensity vectors. In this model,$x_{io}$and $y_{ro}$ are symbols of the inputs and outputs of DMU$_o$, respectively. $\tilde{x}_{ij} = (x_{ij1}, x_{ij2}, x_{ij3})$, $\tilde{y}_{rj} = (y_{rj1}, y_{rj2}, y_{rj3})$ are inputs and outputs that are triangular fuzzy numbers. Note that $\tilde{x}_{io}$ and $\tilde{y}_{ro}$ are inputs and outputs of DMU$_o$. The graded mean integration representation approach is used to calculate the DEA models with fuzzy data, as follows:

**Definition 2.4.a** Suppose $\tilde{A} = (a, b, c)$ is a triangular fuzzy number, the graded mean integration representation$\tilde{A}$ can be determined as $(a + 4b + c)/6$.

In fact, abovementioned models are used because of the easiness and rational calculation. Therefore, by considering Definition 2.4.a, the CCR model with fuzzy factors can be changed with the model (2.2) as follows:

$$\begin{gathered} Minimum\ \theta\ such\ that \\ \frac{1}{6}(4y_{ro2} + y_{ro1} + y_{ro3}) \leq \frac{1}{6}\sum_{j=1}^{n}(4y_{rj2} + y_{rj1} + y_{rj3})\lambda_j, r = 1, \ldots, s\,; \\ \frac{\theta}{6}(4x_{io2} + x_{io1} + x_{io3}) \geq \frac{1}{6}\sum_{j=1}^{n}(4x_{ij2} + x_{ij1} + x_{ij3})\lambda_j\,, i = 1, \ldots, m; \\ \lambda_j \geq 0, j = 1, \ldots, n. \end{gathered} \tag{2.3}$$

Definition 2.4.a will be true wherever integer-valued variables in the fuzzy linear programming are present. See [18,19,20] for more information.

Nevertheless, model (2.2) is not appropriate to assess DMUs' efficiency scores where fuzzy factors and integer-valued measures are present. Indeed, as non-integer values may be the reference point of a DMU with integer. The aim of preparing model (2.2) is to compare its outcomes with the models with fuzzy factors and integer measures.

### *2.5. SBM Model with Flexible Factor (FSBM)*

Amirteimoori, et al. [2] provided the following model in terms of computing the DMUs' efficiency where the flexible measures are present:

$$\pi_o^* = Minimum \quad \frac{1 - (m + k)^{-1}\left[\sum_{i=1}^{m}\frac{s_i}{x_{io}} + \sum_{k=1}^{K}\frac{g_k^{(1)}}{z_{ko}}\right]}{1 + (s + K)^{-1}\left[\sum_{i=1}^{m}\frac{q_r}{y_{ro}} + \sum_{k=1}^{K}\frac{g_k^{(2)}}{z_{ko}}\right]};$$

$$\begin{gathered} such\ that \\ x_{io} = \sum_{j=1}^{n}\lambda_j\, x_{ij} + s_i, i = 1, \ldots, m; \\ y_{ro} = \sum_{j=1}^{n}\lambda_j\, y_{rj} - q_r, r = 1, \ldots, s; \\ z_{ko} = \sum_{j=1}^{n}\lambda_j\, z_{kj} + g_k^{(1)} - g_k^{(2)}, k = 1, \ldots, K; \\ g_k^{(1)}.\, g_k^{(2)} = 0, k = 1, \ldots, K; \\ \lambda_j, g_k^{(1)}, g_k^{(2)}, q_r, s_i \geq 0, \forall i, j, k, r. \end{gathered} \tag{2.3}$$

Aforementioned model evaluates the maximum of units' efficiency scores and determines the role of the flexible measures. Then, they changed model (2.3) into MILP model (2.4) using Charnes and Cooper's transformation [21] and some variables substitutions:

$$
\begin{aligned}
\pi_o^* = Minimum\ \rho - (m+k)^{-1}\left(\sum_{i=1}^{m}\frac{s_i}{x_{io}} + \sum_{k=1}^{K}\frac{g_k^{(1)}}{z_{ko}}\right)\ such\ that \\
\rho + (s+K)^{-1}\left(\sum_{r=1}^{s}\frac{q_r}{y_{ro}} + \sum_{k=1}^{K}\frac{g_k^{(2)}}{z_{ko}}\right) = 1, \\
\rho x_{io} = \sum_{j=1}^{n}\lambda_j\, x_{ij} + s_i, i = 1,\dots,m; \\
\rho y_{ro} = \sum_{j=1}^{n}\lambda_j\, y_{rj} - q_r, r = 1,\dots,s; \\
\rho z_{ko} = \sum_{j=1}^{n}\lambda_j\, z_{kj} + g_k^{(1)} - g_k^{(2)}, k = 1,\dots,K; \\
0 \le g_k^{(1)} \le Md_k^{(1)}, 0 \le g_k^{(2)} \le Md_k^{(2)}, d_k^{(1)} + d_k^{(2)} = 1, k = 1,\dots,K; \\
\lambda_j, g_k^{(1)}, g_k^{(2)}, q_r, s_i \ge 0, d_k^{(1)}, d_k^{(2)} \in \{0,1\}, \forall i,j,k,r,
\end{aligned}
\tag{2.4}
$$

That $\left(1 + (s+K)^{-1}\left[\sum_{i=1}^{m}\frac{q_r}{y_{ro}} + \sum_{k=1}^{K}\frac{g_k^{(2)}}{z_{ko}}\right]\right) \wedge \{-1\} = \rho$ and $M$is a large positive number.

Unit o is efficient if and only if $\pi_o^* = 1$.

if $g_k^{(1)} = g_k^{(2)} = 0$, the factor which is flexible, can be input or output.

### *2.6. DEA Model with Integer Data*

The following model is a slacks-based nonlinear model with the integer data to analyze the DMUs' performance where integer-valued measures are present. In this part, a brief explanation of this model is provided.

$$
\begin{aligned}
Minimum\ \rho = \frac{1 - (m)^{-1}\left(\sum_{i\in I}\frac{s_i^-}{x_{io}}\right)}{1 + (s)^{-1}\left(\sum_{r\in O}\frac{s_r^+}{y_{ro}}\right)}\ such\ that \\
\sum_{j=1}^{n}\lambda_j\, x_{ij} \le x_i\ \forall i, \\
x_i = x_{io} - s_i^-\ \forall i, \\
\sum_{j=1}^{n}\lambda_j\, y_{rj} \ge y_r\ \forall r, \\
y_r = y_{ro} + s_r^+\ \forall r, \\
y_r \in Z\ \forall r \in O^I.
\end{aligned}
\tag{2.5}
$$

In aforementioned formula, $s_i^-, s_r^+$ show the non-radial slacks. $x_i \in Z^+, y_r \in Z^+$ are the integer-valued projection points for inputs $I^I$and outputs $O^I$ respectively. $\lambda_j (j = 1,\dots,n)$ are the intensity vectors.

### *2.7. FSBM Method with Integer-Valued Data*

To assess the DMUs' relative efficiency where integer data and flexible measures are present, Kordrostami et al. [5] suggested the following model:

$$
\begin{gathered}
Maximum \quad \sum_{i \notin I^I} \frac{s_i^-}{x_{io}} + \sum_{i \in I^I} \frac{\tilde{s}_i^-}{x_{io}} + \sum_{k \in K^I} \frac{g'^{(1)}_k}{z_{ko}} + \sum_{k \notin K^I} \frac{\bar{g}_k^{(1)}}{z_{ko}} + \\
\sum_{r \notin O^I} \frac{s_r^+}{y_{ro}} + \sum_{r \in O^I} \frac{\tilde{s}_r^+}{y_{ro}} + \sum_{k \in K^I} \frac{g'^{(2)}_k}{z_{ko}} + \sum_{k \notin K^I} \frac{\bar{g}_k^{(2)}}{z_{ko}} \\
such \quad that \\
x_{io} - s_i^- = \sum_{j=1}^{n} \lambda_j x_{ij}, i \notin I^I \\
y_{ro} + s_r^+ = \sum_{j=1}^{n} \lambda_j y_{rj}, r \notin O^I, \\
\tilde{x}_{io} \geq \sum_{j=1}^{n} \lambda_j x_{ij}, \quad x_{io} - \tilde{s}_i^- = \tilde{x}_{io} \qquad i \in I^I, \\
\tilde{y}_{ro} \leq \sum_{j=1}^{n} \lambda_j y_{rj}, \quad y_{ro} + \tilde{s}_r^+ = \tilde{y}_{ro}, \qquad r \in O^I, \\
z_{ko} - \bar{g}_k^{(1)} + \bar{g}_k^{(2)} = \sum_{j=1}^{n} \lambda_j z_{kj}, \qquad k \notin K^I, \\
0 \leq \bar{g}_k^{(1)} \leq M d_k^{(1)}, \quad 0 \leq \bar{g}_k^{(2)} \leq M d_k^{(2)}, \\
\tilde{z}_{ko} - \hat{g}_k^{(1)} + \hat{g}_k^{(2)} = \sum_{j=1}^{n} \lambda_j z_{kj}, \quad z_{ko} - g'^{(1)}_k + g'^{(2)}_k = \tilde{z}_{ko}, \qquad k \in K^I, \\
0 \leq g'^{(1)}_k \leq M d_k^{(1)}, \quad 0 \leq g'^{(2)}_k \leq M d_k^{(2)}, \quad 0 \leq \hat{g}_k^{(1)} \leq M d_k^{(1)}, \quad 0 \leq \hat{g}_k^{(2)} \leq M d_k^{(2)}, \\
d_k^{(1)} + d_k^{(2)} = 1, \\
\tilde{x}_{io}, \tilde{y}_{ro}, \tilde{z}_{ko} \in z^+, \quad d_k^{(1)}, d_k^{(2)} \in \{0,1\}, \quad i \in I^I, r \in O^I, k \in K^I, \lambda_j \geq 0 \\
s_i^- \geq 0, i \notin I^I, s_r^+ \geq 0, r \notin O^I, \tilde{s}_i^- \geq 0, i \in I^I, \tilde{s}_r^+ \geq 0, r \in O^I, \\
g'^{(1)}_k, g'^{(2)}_k \geq 0, \hat{g}_k^{(1)}, \hat{g}_k^{(2)} \geq 0, \qquad k \in K^I, \quad \bar{g}_k^{(1)} + \bar{g}_k^{(2)} \geq 0, \qquad k \notin K^I,
\end{gathered} \tag{2.6}
$$

## 3. DEA MODELS WITH INTEGER-VALUED DATA, FLEXIBLE FACTORS AND FUZZY INTEGER-VALUED MEASURES

Suppose $n$ DMUs (i.e. DMU$_j$ $(j = 1, \ldots, n)$) is present that produces $s$ outputs (i.e. $y_{rj}(r = 1, \ldots, s)$) by consuming $m$ inputs (i.e. $x_{ij}(i = 1, \ldots, m)$). The following model is provided to assess the DMUs' performance wherever integer-valued data, the flexible measures and fuzzy integer-valued factors are present and specify the role of flexible factors.

$$\begin{aligned}
&Minimum \quad \frac{1-(m+K)^{-1}\left(\sum_{i=1}^{m}\frac{s_i^-}{\tilde{x}_{io}}+\sum_{k=1}^{K}\frac{g_k^{(3)}}{\tilde{z}_{ko}}\right)}{1+(s+K)^{-1}\left(\sum_{r=1}^{s}\frac{s_r^+}{\tilde{y}_{ro}}+\sum_{k=1}^{K}\frac{g_k^{(4)}}{\tilde{z}_{ko}}\right)} \; such \quad that\\
&\sum_{j=1}^{n}\lambda_j\,\tilde{x}_{ij}\le x_i, \quad x_i=\tilde{x}_{io}-s_i^- \qquad \forall i,\\
&\sum_{j=1}^{n}\lambda_j\,\tilde{y}_{rj}\ge y_r, \quad y_r=\tilde{y}_{ro}+s_r^+ \qquad \forall r,\\
&\sum_{j=1}^{n}\lambda_j\,\tilde{z}_{kj}=z_k-g_k^{(1)}+g_k^{(2)}, \qquad k\in K^I,\\
&z_k=\tilde{z}_{ko}-g_k^{(3)}+g_k^{(4)}, \qquad k\in K^I,\\
&0\le g_k^{(1)}\le Md_k^{(1)}, \quad 0\le g_k^{(2)}\le Md_k^{(2)}, \quad 0\le g_k^{(3)}\le Md_k^{(1)}, \quad 0\le g_k^{(4)}\le Md_k^{(2)}\\
&d_k^{(1)}+d_k^{(2)}=1,\\
&x_i,y_r,z_k\in z, i\in I^I, r\in O^I, k\in K^I, \quad d_k^{(1)},d_k^{(2)}\in\{0,1\}\\
&s_i^-\ge 0, s_r^+\ge 0, \quad g_k^{(1)},g_k^{(2)},g_k^{(3)},g_k^{(4)}\ge 0, \quad \lambda_j\ge 0, \quad x_i\ge 0, \quad y_r\ge 0, \quad \forall i,j,r,k
\end{aligned} \tag{3.1}$$

$\lambda_j (j=1,\dots,n)$ are intensity vectors. When imprecise data as triangular fuzzy numbers are present (i.e. $\tilde{x}_{io}=(x_{ij1},x_{ij2},x_{ij3})$, $\tilde{y}_{rj}=(y_{rj1},y_{rj2},y_{rj3})$ in the model (3.1) that $x_{ij1}\ge 0$ and $y_{rj1}\ge 0$), a fuzzy model should be used to assess the units' efficiency score. In above model $\tilde{x}_{io}$ and $\tilde{y}_{ro}$ are inputs and outputs of DMU$_o$ . The data of DMUo are fuzzy integer-valued measures. Consider that in this case inputs and outputs of DMU$_o$ are showed by $\tilde{x}_{io}$ and $\tilde{y}_{ro}$, respectively.

The graded mean integration representation method is used to calculate the DEA models with fuzzy data and handle fuzzy factors. Based on Definition 2.4.a, the model with integer-valued data, fuzzy integer-valued measures and flexible factors can be written in (3.2).

In this formula, $\lambda_j$ indicates intensity weights. $M$is a large positive number. $x_i, y_r, z_k$ are variables that are positive and integer-valued. They represent integer-valued projection points for data that are integer-valued. Moreover, if $g_k^{(3)}>0$ then $z_k$ is considered as an input, and it will be an output if $g_k^{(4)}>0$. On the other hands if $d_k^{(1)}=0$, the flexible factor is called output and otherwise (i.e. $d_k^{(2)}=0$), it is considered as an input.

In fact, fuzzy sets $\tilde{x}_{ij},\tilde{y}_{rj}$ and $\tilde{z}_{kj}$ are replaced with $(4x_{ij2}+x_{ij1}+x_{ij3})/6$, $(4y_{rj2}+y_{rj1}+y_{rj3})/6$ and $(4z_{kj2}+z_{kj1}+z_{kj3})/6$, respectively. The pessimistic and optimistic targets are used to show the fuzzy produced outputs and the fuzzy consumed inputs based on [20]; that is $w_1x_{ij2}+w_2x_{ij1}+w_3x_{ij3}, w_1y_{rj2}+w_2y_{rj1}+w_3y_{rj}$ and $w_1z_{kj2}+w_2z_{kj1}+w_3z_{kj3}$ where $w_1+w_2+w_3=1$.

As explained in [20], $x_{ij3}$ and $y_{rj3}$ are too optimistic and $x_{ij1}$ and $y_{rj1}$ are too pessimistic. Therefore, we use the weights$w_1=1/6, w_2=4/6$and $w_3=1/6$ that can be substituted subjectively. Thus, these boundary values provide us boundary solutions.

Notice that$(a,b,c)$ is a triangular integer fuzzy number, while $(a+4b+c)/6$is gained as non-integer value, we will round it to the closest integer value. Indeed, we consider $\lfloor(a+4b+c)/6\rceil$ in terms of the effect of rounding $(a+4b+c)/6$is almost insignificant.

$$\begin{aligned}
&Minimum \quad \frac{1-\frac{6}{m+K}\left(\sum_{i=1}^{m}\frac{s_i^-}{4x_{io2}+x_{io1}+x_{io3}}+\sum_{k=1}^{K}\frac{g_k^{(3)}}{4z_{ko2}+z_{ko1}+z_{ko3}}\right)}{1+\frac{6}{s+K}\left(\sum_{r=1}^{s}\frac{s_r^+}{4y_{ro2}+y_{ro1}+y_{ro3}}+\sum_{k=1}^{K}\frac{g_k^{(4)}}{4z_{ko2}+z_{ko1}+z_{ko3}}\right)}\\
&such \quad that\\
&\frac{1}{6}\sum_{j=1}^{n}\lambda_j\,(4x_{ij2}+x_{ij}\ +x_{ij3})\le x_i \qquad \forall i,\\
&x_i=\frac{1}{6}(4x_{io2}+x_{io1}+x_{io3})-s_i^- \qquad \forall i,\\
&\frac{1}{6}\sum_{j=1}^{n}\lambda_j\,(4y_{rj2}+y_{rj1}+y_{rj3})\ge y_r \qquad \forall r,\\
&y_r=\frac{1}{6}(4y_{ro2}+y_{ro1}+y_{ro3})+s_r^+ \qquad \forall r,\\
&\frac{1}{6}\sum_{j=1}^{n}\lambda_j\,(4z_{kj2}+z_{kj1}+z_{kj3})=z_k-g_k^{(1)}+g_k^{(2)}, \qquad \forall k,\\
&z_k=\frac{1}{6}(4z_{ko2}+z_{ko1}+z_{ko3})-g_k^{(3)}+g_k^{(4)}, \qquad \forall k,\\
&0\le g_k^{(1)}\le Md_k^{(1)},\quad 0\le g_k^{(2)}\le Md_k^{(2)},\quad 0\le g_k^{(3)}\le Md_k^{(1)},\quad 0\le g_k^{(4)}\le Md_k^{(2)},\\
&d_k^{(1)}+d_k^{(2)}=1,\quad d_k^{(1)},d_k^{(2)}\in\{0,1\},\\
&x_i,y_r,z_k\in z, i\in I^I, r\in O^I, k\in K^I,\\
&s_i^-\ge 0, s_r^+\ge 0,\quad g_k^{(1)},g_k^{(2)},g_k^{(3)},g_k^{(4)}\ge 0,\quad x_i\ge 0,\quad y_r\ge 0,\quad \lambda_j\ge 0,\quad \forall i,r,k,j.
\end{aligned} \tag{3.2}$$

Table 1. Data of an example

| DMU | $x_1$ | $x_2$ | $\tilde{y}_1$ | $\tilde{y}_2$ | Flexible Measure |
|---|---|---|---|---|---|
| 1 | 280 | 182 | (100,121,160) | (160,182,195) | (15,12,30) |
| 2 | 370 | 280 | (180,210,232) | (145,156,160) | (10,10,10) |
| 3 | 230 | 124 | (102,120,136) | (150,175,190) | (12,10,9) |
| 4 | 430 | 210 | (150,170,190) | (50,60,70) | (9,6,18) |
| 5 | 325 | 122 | (102,130,160) | (280,286,293) | (6,8,12) |
| 6 | 315 | 240 | (190,213,234) | (72,85,94) | (16,30,13) |
| 7 | 253 | 170 | (130,151,167) | (260,275,286) | (15,10,4) |
| 8 | 305 | 185 | (201,225,246) | (76,87,93) | (10,5,8) |
| 9 | 245 | 129 | (130,146,160) | (230,242,251) | (3,5,10) |

## 4. NUMERICAL EXAMPLE

Suppose 9 DMU exist. Inputs are numbers which are integer-valued and outputs are measures that are fuzzy integer-valued. The flexible measures are fuzzy integer-valued factors which are showen by triangular fuzzy numbers. The columns of table 1 indicate inputs and outputs. Column 2 shows the first input and column 3 shows second input while columns 4 and 5 display outputs and column 6 indicates the flexible measures. Our purpose is to assess the units' efficiency score in the presence of the abovementioned measures. For defining the role of flexible measures and estimating the DMUs' efficiency, model (3.2) is used. The targets that are obtained from proposed method, can be seen in table 2. The efficiency score of model (3.2) are shown in column 2 from Table 2. Moreover, the scores are specified in columns 3,4,5 and 6. It is clear that the scores of integer data are assessed as non-integer scores.

Consider that firstly we have used one method to defuzzify the fuzzy numbers. Secondly we have assessed the DMUs' performance. Therefore, the reference points are evaluated as continuous and crisp values.

Based on the columns 5 and 6 of table 2, DMU 1, 3, 7 are regarded as inputs and the others are considered as outputs.

Consider that all integer-valued scores in model (3.2) have integer-valued projections. Besides, unit 4 is the most inefficient DMU in model (3.2). However, it is reminded in Lozano and villa [6] that rounding the real reference point may not be suitable. Thus, apparently, the model proposed (model (3.2)) is reasonable for the conditions that all inputs and outputs are integer-valued numbers, fuzzy integer-valued factors and flexible measures.

## 5. CONCLUSIONS

In the conventional DEA models, all data are usually regarded as exact and real numbers. Classifying data is an important subject to analyze the performance. There are the situations in real application that the units' efficiency score with integer-valued data, fuzzy integer-valued measures and flexible factors should be evaluated. Some studies have examined the units' efficiency scores wherever integer-valued data, fuzzy integer-valued measures are present. Furthermore, some researches studied the DMUs' efficiency where integer-valued data and flexible factors are present. This paper was suggested a slacks-based nonlinear programming problem to estimate the entities' efficiency score and specify the roles of flexible factors in the presence of integer data, flexible factors and fuzzy integer-valued measures. The graded mean integration representation method was used on fuzzy data in terms of defuzzify them. An example was stated to describe and illustrate approaches.

Table 2. Results of the proposed models

| DMU | Efficiency | $g_k^{*(3)}$ | $g_k^{*(4)}$ | $d_k^{*(1)}$ | $d_k^{*(2)}$ |
|---|---|---|---|---|---|
| 1 | 0.644 | 4.5 | 0 | 1 | 0 |
| 2 | 0.524 | 0 | 5 | 0 | 1 |
| 3 | 0.738 | 4.167 | 0 | 1 | 0 |
| 4 | 0.279 | 0 | 4.5 | 0 | 1 |
| 5 | 0.852 | 0 | 4.667 | 0 | 1 |
| 6 | 0.395 | 0 | 4.167 | 0 | 1 |
| 7 | 0.862 | 2.833 | 0 | 1 | 0 |
| 8 | 0.463 | 0 | 4.667 | 0 | 1 |
| 9 | 0.846 | 0 | 4.5 | 0 | 1 |

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