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On Order Covering Set-Valued Mappings and Their Applications to the Investigation of Implicit Differential Inclusions and Dynamic Models of Economic Processes

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Abstract: The present work is devoted to investigation of operator inclusions in partially ordered spaces and application of the obtained results to differential inclusions. We consider the inclusion $\Upsilon(x,x) \ni y$ with respect to the unknown $x \in X$, where $\Upsilon : X \times X \rightrightarrows Y$ is a set-valued mapping, X and Y are partially ordered spaces. It is assumed that the mapping Υ is order covering with respect to the first argument and antitone with respect to the second argument. We prove that for any $x_0 \in X$, if the set $G(x_0)$ contains an element y_0 such that $y \preceq y_0$, then there exists a solution to the inclusion under consideration, which satisfies the inequality $x \preceq x_0$. This statement is applied to investigation of a Cauchi problem for the differential inclusion $f(t,x,\dot{x},\dot{x}) \ni 0$ with a bound for the derivative of the unknown function $\dot{x}(t) \in B(t)$ (here $f : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, $B : [a,b] \rightrightarrows \mathbb{R}^n$). We obtain conditions of solvability in the space of absolutely continuous functions, conditions of existence of a solution with the least derivative, and derive the solutions estimates. The latter results are applied to the analysis of the dynamic Walrasian-Evans-Samuelson model of economic processes, which can be reduced to a system of implicit differential inclusions. We establish the existence of the equilibrium and obtain estimates of the equilibrium prices.

Keywords: operator inclusion, covering mapping of partially ordered spaces, implicit differential equation, supply-and-demand model

1. INTRODUCTION

The present work is devoted to the problem of solvability of operator inclusions in partially ordered spaces. The research uses the notion of order covering introduced for the "standard single-valued" mappings in [1, 2], and for set-valued mappings in [3, 4]. For a set-valued mapping G acting from a partially ordered space $(X,\preceq)$ to a partially ordered space $(Y,\preceq)$, the property of covering means that if for any $u \in X$ and $y \in Y$, the set $G(u)$ contains an element v such that $y \preceq v$, then there exists a solution x of the inclusion $G(x) \ni y$, which satisfies the inequality $x \preceq u$. In the cited papers, the existence of a coincidence point of a covering mapping and an isotone mapping (for the cases of single- and set-valued mappings) is demonstrated.

In the present work, the stability of the solvability property of inclusions with respect to antitone perturbations is investigated. This problem is formalised in terms of the inclusion

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$\Upsilon(x, x) \ni y$ with a set-valued mapping $\Upsilon : X \times X \rightrightarrows Y$ that is order covering with respect to the first argument and antitone with respect to the second argument. An analogous problem for "standard single-valued" mappings was considered in the papers [5–8]. Based on the statements on antitone perturbations proved in these works, comparison theorems (of the same type with the well-known Chaplygin theorem [9] on differential inequality) for the unresolved with respect to the derivative (i.e. implicit) differential equations were obtained (see [5, 10]). The results on antitone perturbations of order covering mappings presented in this article are applied to the investigation of implicit differential inclusions. Conditions of existence and estimates of solutions of a Cauchi problem are obtained. These results are applied to the analysis of the dynamic Walrasian-Evans-Samuelson supply-and-demand model (see [11–13]) that is reduced to a system of implicit differential inclusions. The existence of the equilibrium state is established and the estimates of the equilibrium prices are obtained. These statements extend the results on the models of economic processes of A.V. Arutyunov, N.G. Pavlova, S.E. Zhukovskiy, A.A. Shananin (see [14–17]) which employ single-valued functions of supply and demand. The fundamental mathematical tools for these works were represented by the theorems of the papers [18–21] on coincidence points of single-valued covering mapping of metric spaces.

2. A COMPARISON THEOREM FOR OPERATOR INCLUSIONS IN PARTIALLY ORDERED SPACES

Let partially ordered spaces $(X, \preceq)$, $(Y, \preceq)$ be given. For any $u, v \in X$ and $U \subset X$, we denote

$$[u, v]_X \doteq \{x \in X : v \preceq x \preceq u\}, \ \ \mathcal{O}_X(u) \doteq \{x \in X : x \preceq u\}, \ \ \mathcal{O}_X(U) \doteq \bigcup_{\forall u \in U} \mathcal{O}_X(u).$$

Consider a set-valued mapping $G : X \rightrightarrows Y$, i.e. a mapping that puts to any element $x \in X$ into the correspondence a non-empty set $G(x) \subset Y$. If for any $x \in X$, the set $G(x)$ contains only one element, then the mapping G becomes a "standard single-valued" mapping (we conventionally denote such a mapping as $G : X \to Y$). Thus, set-valued mappings naturally generalise "standard" mappings.

Let us remind definitions of some properties of set-valued mappings used in the present work.

Definition 2.1:
A mapping $G : X \rightrightarrows Y$ is called antitone (isotone) on the set $U \subset X$, if for any $x, u \in U$ such that $x \preceq u$ and for $z \in G(u)$, there exists $y \in G(x)$ satisfying the inequality $y \succeq z$ ($y \preceq z$). If a mapping is antitone (isotone) on the whole space X, it is called antitone (isotone).

The given definition of the antitone property for a mapping $G : X \to Y$ means that

$$\forall x, u \in U \ \ x \preceq u \ \Rightarrow \ G(x) \succeq G(u),$$

matches the definition of the antitone property for single-valued mappings. The given isotone property in the case of single-valued mappings also matches the "classical" isotone property.

Definition 2.2:
A mapping $G : X \rightrightarrows Y$ order covers the set $V \subset Y$ if

$$\forall u \in X \ \ \mathcal{O}_Y\big(G(u)\big) \cap V \subset G\big(\mathcal{O}_X(u)\big). \tag{2.1}$$

The definition of order covering for set-valued mappings was introduced in [3, 4] for the case $V = Y$.

Note that (2.1) is equivalent to the relation

$$\big(\forall u \in X \ \ \forall v \in G(u) \ \ \forall y \in V \ \ y \preceq v\big) \ \ \Rightarrow \ \ \big(\exists x \in X \ \ y \in G(x) \ \text{ and } \ x \preceq u\big).$$

Moreover, if V is a one-element set, i.e. $V = \{\widehat{y}\}$, then the relation (2.1) is equivalent to the implication

$$\forall u \in X \ \ \widehat{y} \in \mathcal{O}_Y\big(G(u)\big) \ \Rightarrow \ \widehat{y} \in G\big(\mathcal{O}_X(u)\big).$$

Thus, the property of order covering of the set $\{\widehat{y}\}$ means that for any $u \in X$ such that the set $G(u)$ contains an element $v \succeq \widehat{y}$, the inclusion

$$\widehat{y} \in G(x) \tag{2.2}$$

has a solution $x \in X$ satisfying the inequality $x \preceq u$. In the same manner, a single-valued mapping $G : X \to Y$ order covers the set $\{\widehat{y}\}$ if

$$\forall u \in X \ \ G(u) \succeq \widehat{y} \ \Rightarrow \ \exists x \in \mathcal{O}_X(u) \ \ G(x) = \widehat{y}.$$

We consider the problem of stability of the solvability property of the inclusion (2.2) under antitone perturbations of the order covering mapping. Let a "perturbed" mapping $\Upsilon : X \times X \rightrightarrows Y$ that order covers the set $\{\widehat{y}\}$ with respect to the first argument and antitone with respect to the second argument be given. Define the mapping $F : X \rightrightarrows Y$ by the relation

$$\forall x \in X \ \ F(x) = \Upsilon(x, x)$$

and consider the inclusion

$$\widehat{y} \in F(x) \tag{2.3}$$

(with respect to $x \in X$).

Let $U \subset X$. In order to formulate a theorem on existence of solutions to (2.3), we define the set $\mathcal{S}(\Upsilon, U, \widehat{y})$ of all chains $S \subset U$ such that the following relations take place:

$$\begin{aligned} &\forall x \in S \ \ \exists y \in \Upsilon(x, x) \ \ y \succeq \widehat{y}, \\ &\forall x, u \in S \ \ x \prec u \ \Rightarrow \ \exists \xi \in [x, u] \ \ \widehat{y} \in \Upsilon(\xi, u). \end{aligned} \tag{2.4}$$

Theorem 2.1:
Let there exist $u_0 \in X$ and $y_0 \in Y$ such that

$$y_0 \in \Upsilon(u_0, u_0), \ \ y_0 \succeq \widehat{y}, \tag{2.5}$$

and the follwing conditions are satisfied:

(a1) *for any $x \in \mathcal{O}_X(u_0)$, the mapping $\Upsilon(\cdot, x) : X \rightrightarrows Y$ order covers the set $\{\widehat{y}\}$;*
(a2) *for any $x \in \mathcal{O}_X(u_0)$, the mapping $\Upsilon(x, \cdot) : X \rightrightarrows Y$ is antitone on the set $[x, u_0]_X$;*
(a3) *any infinite chain $S \in \mathcal{S}\big(\Upsilon, \mathcal{O}_X(u_0), \widehat{y}\big)$ is bounded from below, and for some its lower bound $\omega \in X$, one can find $z \in \Upsilon(\omega, \omega)$ satisfying the inequality $z \succeq \widehat{y}$.*

Then the inclusion (2.3) *has a solution and the set of solutions possesses a minimal element belonging to the set $\mathcal{O}_X(u_0)$.*

Proof
We introduce the set

$$U_0 = \{x \in \mathcal{O}_X(u_0) : \ \exists y \in F(x) \ \ y \succeq \widehat{y}\}.$$

Note that $U_0 \neq \emptyset$ as $u_0 \in U_0$. We define binary relations $\lhd$ and $\unlhd$ on U_0 as follows:

$$\begin{aligned} &\forall v, u \in U_0 \ \ v \lhd u \ \Leftrightarrow \ \big(v \prec u \ \text{and} \ \exists \xi \in [v, u] \ \ \widehat{y} \in \Upsilon(\xi, u)\big), \\ &\forall v, u \in U_0 \ \ v \unlhd u \ \Leftrightarrow \ \big(v = u \ \text{or} \ v \lhd u\big). \end{aligned}$$

Let us demonstrate that $\lhd$ and $\unlhd$ are strict and non-strict order relations, respectively. As the relation $\prec$ is antisymmetric, the relation $\lhd$ is antisymmetric as well. Prove that $\lhd$ is transitive. For $v, w, u \in U_0$ such that $v \lhd w \lhd u$, it holds true that

$$v \prec w \prec u \text{ and } \exists \xi \in [w, u] \subset [v, u] \ \widehat{y} \in \Upsilon(\xi, u). \tag{2.6}$$

Thus, we have $v \lhd u$,, which means that the relation $\lhd$ is transitive. Obviously, the relation $\unlhd$ is also antisymmetric and transitive, and, moreover, reflexive. We point out that the relation (2.6) also holds true for $v, w, u \in U_0$ such that $v \prec w \lhd u$,, i.e. the relation $v \lhd u$ takes place in this case as well.

Let us consider the partially ordered set $(U_0, \unlhd)$. According to the Hausdorff maximal principle (see e.g. [22, Chapter 1]), this set possesses a maximal chain S that contains u_0. We first prove the theorem statement in the case when the chain S contains the least element ω, i.e. $\omega \unlhd x$ for any $x \in S$. If ω is not a solution of inclusion (2.3), then for some $z \in \Upsilon(\omega, \omega)$, it holds true that $z \succ \widehat{y}$ and, hence, by the virtue of the condition $(\mathrm{a}1)$, there exists an element $v \in X$ such that $\widehat{y} \in \Upsilon(v, \omega)$ and $v \prec \omega$. According to $(\mathrm{a}2)$, there exists $y \in \Upsilon(v, v)$ such that $y \succeq \widehat{y}$. So, $v \in U_0$ and $v \lhd \omega$. We obtained that $v \lhd x$ for any $x \in S$ which contradicts with the maximality of the chain S. Thus, ω is a solution to (2.3).

Consider now the case when the chain S does not possess the least element, i.e. for any $x \in S$, there exists $u \in S$, $u \lhd x$. Obviously, this chain is infinite. Note that S is a chain with respect to the initial order $\preceq$ in X and belongs to $\mathcal{S}\big(\Upsilon, \mathcal{O}_X(u_0), \widehat{y}\big)$. The assumption $(\mathrm{a}3)$ implies that this chain has a lower bound ω (with respect to the initial order $\preceq$), for which there exists $z \in \Upsilon(\omega, \omega)$ such that $z \succeq \widehat{y}$, i.e. $\omega \in U_0$ and $\omega \notin S$. For any $x \in S$, there is $u \in S$, $u \lhd x$. As $\omega \prec u$, we get $\omega \lhd x$. The latter contradicts with the maximality of the chain S, so the situation where the maximal chain S does not possess the least element is not possible.

Let us demonstrate that the obtained element $\omega \in \mathcal{O}_X(u_0)$ is minimal in the set of solutions of the inclusion (2.3). Assume the contrary. Then one can find $z \prec \omega$ such that $\widehat{y} \in \Upsilon(z, z)$. Hence, $z \in U_0$ and $z \lhd \omega$. However, this inequality is not possible by the virtue of the maximality of the chain S in the space $(U_0, \unlhd)$. □

The theorem 2.1 implies the theorem on perturbations of an order covering single-valued mapping $\Upsilon : X \times X \to Y$. The corresponding properties of order covering and monotonicity of a single-valued mapping are defined above, and the relations (2.4) can be written as:

$$\begin{aligned} &\forall x \in S \quad \Upsilon(x, x) \succeq \widehat{y}, \\ &\forall x, u \in S \ \ x \prec u \ \Rightarrow \ \exists \xi \in [x, u] \ \ \widehat{y} = \Upsilon(\xi, u). \end{aligned} \tag{2.7}$$

The statement on perturbations of order covering single-valued mappings is analogous to the results obtained in the papers [5, 6]. These works employed a less restrictive assumptions on the chains S compared to (2.7), namely:

$$\forall x, u \in S \ \ x \prec u \ \Rightarrow \ \widehat{y} \preceq \Upsilon(\xi, u),$$

so the condition corresponding to $(\mathrm{a}3)$ was more strict.

3. CONDITIONS OF ORDER COVERING FOR THE NEMYTSKII OPERATOR

In the application of Theorem 2.1 to concrete functional inclusions, the most of the difficulties connected with the verification of the order covering property of the corresponding set-valued operators, primarily the superposition operator that is also called the Nemytskii operator. Here we demonstrate that the Nemytskii operator is order covering provided that the corresponding set-valued function possesses this property with respect to the corresponding argument.

Typically, the verification of the order covering property for functions does not involve significant difficulties.

Below we assume that the space $\mathbb{R}^n$ is equipped with the natural order, i.e for $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $u = (u_1, ..., u_n) \in \mathbb{R}^n$, we have $x \le u$ if $x_i \le u_i$ for all $i = \overline{1, n}$. We denote by $\mathrm{C}(\mathbb{R}^n)$ and $\mathrm{K}(\mathbb{R}^n)$ the sets of all non-empty closed and all non-empty compact subsets of the space $\mathbb{R}^n$, respectively. We denote by $\mathrm{C}^{\mathrm{C}}(\mathbb{R}^n)$ and $\mathrm{K}^{\mathrm{C}}(\mathbb{R}^n)$ the sets of all non-empty connected closed and all non-empty connected compact subsets of the space $\mathbb{R}^n$, respectively. Let a set-valued mapping $B : [a, b] \rightrightarrows \mathbb{R}^n$ having closed images $B(t) \subset \mathbb{R}^n$ for any $t \in [a, b]$ be given. We denote such a mapping as $B : [a, b] \to \mathrm{C}(\mathbb{R}^n)$. Let this mapping be measurable. We define $W(B)$ to be the set of all Lebesgue measurable functions $x : [a, b] \to \mathbb{R}^n$ such that $x(t) \in B(t)$ for almost all $t \in [a, b]$. Thus, the set $W(B)$ is the set of all measurable selections of the set-valued mapping B. We endow the set $W(B)$ with an order by assuming that for any two its elements x, u, the inequality $x \le u$ holds true if $x(t) \le u(t)$ for almost all $t \in [a, b]$. In the case $B = \mathbb{R}^n$, this set of all measurable functions $x \in [a, b] \to \mathbb{R}^n$ we denote by W^n.

Consider a set-valued mapping $g : [a, b] \times \mathbb{R}^n \to \mathrm{C}(\mathbb{R}^m)$. Assume that it satisfies the Caratheodori conditions (i.e. it is measurable with respect to the first argument and continuous in the Hausdorff metric in the second argument). These assumptions allow to define the set-valued Nemytskii operator

$$N_g : W^n \rightrightarrows W^m,\ \forall x \in W^n\ N_g x = \big\{y \in W^m :\ y(t) \in g(t, x(t))\ \text{ for almost all }\ t \in [a, b]\big\}.$$

We denote $\Delta = \big\{(t, x) :\ t \in [a, b],\ \ x \in B(t)\big\}$ and define $g_\Delta : \Delta \to \mathrm{C}(\mathbb{R}^m)$ to be the restriction of the set-valued mapping g to the set Δ, and by $N_{g_\Delta} : W(B) \rightrightarrows W^m$ — the restriction of the operator N_g to $W(B)$.

The following statement establishes connection between the property of order covering of the set-valued mapping g_Δ with respect to the second argument and the order covering property of the corresponding Nemytskii operator N_{g_Δ}.

Let a measurable function $\widehat{y} : [a, b] \to \mathbb{R}^m$ be given.

Theorem 3.1:
If at almost all $t \in [a, b]$, a set-valued mapping $g_\Delta(t, \cdot) : B(t) \to \mathrm{C}(\mathbb{R}^m)$ order covers the set $\{\widehat{y}(t)\} \subset \mathbb{R}^m$, then the operator $N_{g_\Delta} : W(B) \rightrightarrows W^m$ order covers the set $\{\widehat{y}\} \subset W^m$.

Proof
Let the mapping $g_\Delta(t, \cdot)$ order cover the set $\{\widehat{y}(t)\} \subset \mathbb{R}^m$. assume that for some measurable function $u \in W(B)$, there exists $y \in N_{g_\Delta} u$ such that $y \ge \widehat{y}$. Thus, $y(t) \in g_\Delta\big(t, u(t)\big)$ and $y(t) \ge \widehat{y}(t)$ for almost all $t \in [a, b]$. We define the set-valued mappings $O, U : [a, b] \rightrightarrows \mathbb{R}^n$ by the relations:

$$O(t) \doteq \mathcal{O}_{\mathbb{R}^n}(u(t)),\ \ U(t) \doteq O(t) \bigcap B(t) = \big\{x \in B(t) :\ x \le u(t)\big\},\ \ t \in [a, b].$$

For almost all $t \in [a, b]$, the closedness of the sets $\mathcal{O}_{\mathbb{R}^n}(u(t))$ and $B(t)$ in $\mathbb{R}^n$implies the closedness of the set $U(t)$, i.e. $U : [a, b] \to \mathrm{C}(\mathbb{R}^n)$. The measurability of $O, B : [a, b] \to \mathrm{C}(\mathbb{R}^n)$ guarantees the measurability of the mapping U (see [23, § 1.5.1, § 1.5.8]).

As $g_\Delta(t, \cdot)$ order covers the set $\{\widehat{y}(t)\} \subset \mathbb{R}^m$, for almost all $t \in [a, b]$, the inclusion $\widehat{y}(t) \in g_\Delta\big(t, U(t)\big)$ takes place for almost all $t \in [a, b]$. Therefore, according to Filippov's lemma on implicit function (see [23, § 1.5.15]), there exists $\widehat{x} \in W^n$ such that $\widehat{x}(t) \in U(t)$ and $\widehat{y}(t) \in g_\Delta\big(t, \widehat{x}(t)\big)$ for almost all $t \in [a, b]$. Thus, the relations $\widehat{y} \in N_{g_\Delta} \widehat{x}$ and $\widehat{x} \le u$ are fulfilled, i.e. the operator N_{g_Δ} order covers the set $\{\widehat{y}\} \subset W^m$. □

Example 2.1:
Let ordinary single-valued" functions $q_0, q_1 : [a, b] \to \mathbb{R}$ be given. Consider the function

$g : [a, b] \times \mathbb{R} \to \mathbb{R}$ defined by the formula

$$g(t, x) \doteq q_0(t) + 2q_1(t)x - x^2, \;\; t \in [a, b], \;\; x \in \mathbb{R}. \tag{3.8}$$

Obviously, g satisfies the Caratheodori conditions. Let $r > 0$, $B(t) \doteq [q_1(t) - r,\, q_1(t) + r]$ and $\Delta = \big\{(t, x) : t \in [a, b],\ |x - q_1(t)| \le r\big\}$, respectively. The values of $g_\Delta(t, \cdot) : B(t) \to \mathbb{R}$ constitute the segment $\big[q_0(t) + q_1^2(t) - r^2,\, q_0(t) + q_1^2(t)\big]$. If for the measurable function $\widehat{y} : [a, b] \to \mathbb{R}$, it holds true that

$$\widehat{y}(t) \ge q_0(t) + q_1^2(t) - r^2, \tag{3.9}$$

then $g_\Delta(t, \cdot) : B(t) \to \mathbb{R}$ order covers the set $\{\widehat{y}(t)\} \subset \mathbb{R}$ at almost all $t \in [a, b]$ and, hence, by the virtue of Theorem 3.1, the Nemytskii operator $N_{g_\Delta} : W(B) \rightrightarrows W$ order covers the set $\{\widehat{y}\} \subset W$. In the particular case when $B(t) \equiv \mathbb{R}$, the operator $N_g : W \rightrightarrows W$ is order covering (i.e. order covers the whole space W).

Now we pick $\delta > 0$ and define two set-valued mappings $g^+, g^- : [a, b] \times \mathbb{R} \to \mathrm{K}(\mathbb{R})$ by the formulas:

$$g^+(t, x) \doteq \big[g(t, x),\, g(t, x) + \delta\big],\; g^-(t, x) \doteq \big[g(t, x) - \delta,\, g(t, x)\big],\; t \in [a, b],\; x \in \mathbb{R}. \tag{3.10}$$

The images of the set-valued functions $g^+_\Delta(t, \cdot) : B(t) \to \mathrm{K}(\mathbb{R})$ and $g^-_\Delta(t, \cdot) : B(t) \to \mathrm{K}(\mathbb{R})$ are the sets $\big[q_0(t) + q_1^2(t) - r^2,\, q_0(t) + q_1^2(t) + \delta\big]$, and $\big[q_0(t) + q_1^2(t) - r^2 - \delta,\, q_0(t) + q_1^2(t)\big]$, respectively.

Using Theorem 3.1 one can easily demonstrate that if the measurable function $\widehat{y} : [a, b] \to \mathbb{R}$ satisfies the inequality (3.9) for almost all $t \in [a, b]$, then the operator $N_{g^+_\Delta} : W(B) \rightrightarrows \mathrm{C}(W)$ order covers the set $\{\widehat{y}(t)\} \subset \mathbb{R}$. Theorem 3.1 also implies that the operator $N_{g^-_\Delta} : W(B) \rightrightarrows \mathrm{C}(W)$ order covers the set $\{\widehat{y}(t)\} \subset \mathbb{R}$ provided that the inequality

$$\widehat{y}(t) \ge q_0(t) + q_1^2(t) - r^2 - \delta,$$

is valid almost everywhere on $[a, b]$,

4. IMPLICIT DIFFERENTIAL INCLUSION

In this section, based on Theorems 2.1 and 3.1 we investigate differential inclusions not resolved with respect the derivative (which are also called implicit differential inclusions).

Let a measurable set-valued mapping $B : [a, b] \to \mathrm{C}(\mathbb{R}^n)$ be given. Denote by $L(B)$ the space of all integrable selections of the set-valued mapping B and denote by $AC(B)$ the space of all absolutely continuous functions $x : [a, b] \to \mathbb{R}^n$ such that $\dot{x} \in L(B)$. In the case $B(t) \equiv \mathbb{R}^n$, we denote these spaces these spaces by L^n and AC^n.

We will require the fulfillment of the following analogue of the one-sided continuity property known for "ordinary single-valued" functions applicable to a set-valued mapping $g : \mathbb{R} \to \mathrm{K}(\mathbb{R}^m)$, which was defined in [24]. Such mapping is called *right continuous at the point* $x_0 \in \mathbb{R}$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in (x_0, x_0 + \delta)$, it holds true that $h_{\mathbb{R}^m}\big(G(x_0), G(x)\big) < \varepsilon$ (hereinafter the symbol $h_{\mathbb{R}^m}$ denotes the Hausdorff distance between the sets in the space $\mathbb{R}^m$).

Let a set-valued function $f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{K}(\mathbb{R}^m)$ and a vector $\gamma \in \mathbb{R}^n$ be given. We assume that for any $x, v, u \in \mathbb{R}^n$, the function $f(\cdot, x, v, u) : [a, b] \to \mathrm{K}(\mathbb{R}^m)$ is measurable; for almost all $t \in [a, b]$ and all $v, u \in \mathbb{R}^n$, the function $f(t, \cdot, v, u) : \mathbb{R}^n \to \mathrm{K}(\mathbb{R}^m)$ is right continuous in each of the arguments $x_1, \ldots, x_n$; for almost all $t \in [a, b]$ and any $x, u \in \mathbb{R}^n$, the function $f(t, x, \cdot, u) : \mathbb{R}^n \to \mathrm{K}(\mathbb{R}^m)$ is right continuous in each of the arguments $v_1, \dots, v_n$; for almost all $t \in [a,b]$ and any $x, v \in \mathbb{R}^n$, the function $f(t,x,v,\cdot): \mathbb{R}^n \to \mathrm{K}(\mathbb{R}^m)$ is continuous.

We consider the differential inclusion

$$f(t, x, \dot{x}, \dot{x}) \ni 0, \ \ t \in [a,b], \tag{4.11}$$

with the following additional restriction on the derivative of the unknown function:

$$\dot{x}(t) \in B(t), \ \ t \in [a,b]. \tag{4.12}$$

A solution of the system of inclusions (4.11), (4.12) is understood to be a function $x \in AC(B)$ that satisfies the inclusion (4.11) for almost all $t \in [a,b]$. Let us formulate a statement on solvability and estimates of solutions to the Cauchi problem for the system (4.11), (4.12) with the initial condition

$$x(a) = \gamma. \tag{4.13}$$

Denote $\Omega = \big\{(t,x,v,u): \ t \in [a,b], \ x \in \mathbb{R}^n, \ v \in B(t), \ u \in B(t)\big\}$ and define $f_\Omega : \Omega \to \mathrm{C}(\mathbb{R}^m)$ to be the restriction of the set-valued mapping f to the set Ω.

Theorem 4.1:
Let a function $v_0 \in AC(B)$ such that $v_0(a) \geq \gamma$ and

$$f\big(t, v_0(t), \dot{v}_0(t), \dot{v}_0(t)\big) \cap \mathbb{R}^n_+ \neq \emptyset \textit{ for almost all } \ t \in [a,b] \tag{4.14}$$

be given. Let the set of measurable selections of the set-valued mapping $B(\cdot) \cap \mathcal{O}_{\mathbb{R}^n}\big(\dot{v}_0(\cdot)\big) : [a,b] \to \mathrm{C}(\mathbb{R}^n)$ be integrally bounded from below (i.e. there exists a number C such that for any measurable function $u \in W^n$, satisfying the relations $u(t) \in B(t)$ and $u(t) \leq \dot{v}_0(t)$ for almost all $t \in [a,b]$, it holds true that $\int_a^b u(t)dt \geq C$) and the following conditions are fulfilled:

(b1) *for almost all $t \in [a,b]$, any $x \in \mathbb{R}^n$ and $v \in B(t)$, the mapping $f_\Omega(t,x,v,\cdot) : B(t) \to \mathrm{K}(\mathbb{R}^m)$ order covers the set $\{0\} \subset \mathbb{R}^m$;*
(b2) *for almost all $t \in [a,b]$ and any $u \in B(t)$, the mapping $f_\Omega(t,\cdot,\cdot,u) : \mathbb{R}^n \times B(t) \to \mathrm{K}(\mathbb{R}^m)$ is antitone.*

Then there exists a solution $x \in AC(B)$ to the problem (4.11), (4.12), (4.13) *such that $\dot{x}(t) \leq \dot{v}_0(t)$ for almost all $t \in [a,b]$.*

Proof
The problem (4.11), (4.12), (4.13) with respect to the unknown function $x \in AC(B)$ can be rewritten in the form of the following inclusion

$$f_\Omega\big(t, \gamma + \int_a^t u(s)ds, u(s), u(s)\big) \ni 0, \ \ t \in [a,b], \tag{4.15}$$

with respect to the unknown function $u = \dot{x} \in L(B)$. Let us demonstrate that the inclusion (4.15) can be represented in the form of the operator inclusion (2.3), that can be investigated using Theorem 2.1.

According to [24, Theorem 2.1], the assumptions made for the set-valued function f provide its superpositional measurability, hence, for any $x \in AC^n$, the set-valued function $f\big(\cdot, x(\cdot), \dot{x}(\cdot), \dot{x}(\cdot)\big) : [a,b] \to \mathrm{K}(\mathbb{R}^m)$ is measurable. Superpositional measurability of f allows to define the mapping $\Upsilon : L(B) \times L(B) \rightrightarrows W^m$,

$$\forall u, v \in L(B) \ \ \Upsilon(u,v) \doteq f_\Omega\big(t, \gamma + \int_a^t v(s)ds, v(t), u(t)\big), \ \ t \in [a,b], \tag{4.16}$$

and the corresponding mapping $F : L(B) \rightrightarrows W^m$, $F(u) \doteq \Upsilon(u,u)$. Let us verify the conditions of Theorem 2.1 (where we put $\widehat{y} = 0 \in W^m$) for these mappings.

First of all, there exists a measurable function $y \in W^m$ satisfying the relations

$$y(t) \in f\big(t, v_0(a) + \int_a^t \dot v_0(s)ds, \dot v_0(t), \dot v_0(t)\big) = f_\Omega\big(t, v_0(a) + \int_a^t \dot v_0(s)ds, \dot v_0(t), \dot v_0(t)\big),$$

$$y(t) \geq 0 \text{ for almost all } t \in [a,b].$$

These relations, according to (b2), imply the existence of a measurable function $y_0 \in W^m$ such that

$$y_0(t) \in f_\Omega\big(t, \gamma + \int_a^t \dot v_0(s)ds, \dot v_0(t), \dot v_0(t)\big), \ \ y_0(t) \geq y(t) \geq 0 \text{ for almost all } t \in [a,b].$$

Thus, for the mapping Υ defined by (4.16), we have $\Upsilon(\dot v_0, \dot v_0) \ni y_0$, $y_0 \geq 0$, i.e. the relations (2.5) where $u_0 = \dot v_0 \in L(B)$ are fulfilled.

Secondly, for any $v \in L(B)$, the mapping $\Upsilon(\cdot, v) : L(B) \rightrightarrows W^m$ is the Nemytskii operator generated by the restriction g_Δ to the set $\Delta = \big\{(t,u) : \ t \in [a,b], \ u \in B(t)\big\}$ of the function $g : [a,b] \times \mathbb{R}^n \to \mathrm{K}(\mathbb{R}^m)$,

$$g(t,u) \doteq f_\Omega\big(t, \gamma + \int_a^t v(s)ds, v(t), u\big) \text{ for almost all } t \in [a,b] \text{ and any } u \in \mathbb{R}^n.$$

The assumption (b1) implies that for almost all $t \in [a,b]$, the function $g_\Delta(t,\cdot) : B(t) \to \mathrm{K}(\mathbb{R}^m)$ order covers the set $\{0\} \in \mathbb{R}^m$. According to Theorem 3.1, for any $v \in L(B)$, the mapping $\Upsilon(\cdot, v)$ order covers the set $\{0\} \in W^m$. Thus, the condition (a1) of Theorem 2.1 is satisfied.

Next, due to the assumption (b2), for any $u \in L(B)$, the mapping $\Upsilon(u, \cdot) : L(B) \rightrightarrows W^m$ is antitone, so the condition (a2) of Theorem 2.1 is fulfilled as well.

In order to verify the assumption (a1) of Theorem 2.1, we consider an arbitrary chain $S \in \mathcal{S}\big(\Upsilon, \mathcal{O}_{L(B)}(\dot v_0), 0\big)$ and demonstrate that it possesses an infimum $\underline{u} \doteq \inf S$ and, moreover, there exists a decreasing sequence $\{u_n\} \subset S$ having the same lower bound $\inf\{u_n\} = \inf S = \underline{u}$. For any $u \in S$, we define the number $lu \doteq \int_0^a u(s)ds$. The set $lS \doteq \{lu, \ u \in S\}$ is a chain in $\mathbb{R}$, which is bounded from below due to the integral boundedness of the set of measurable selections of the set-valued mapping $B(\cdot) \cap \mathcal{O}_{\mathbb{R}^n}\big(\dot v_0(\cdot)\big) : [a,b] \to \mathrm{C}(\mathbb{R}^n)$ from below. Thus, there exists $\lambda \doteq \inf lS$, and, hence, exists a decreasing sequence $\{u_n\} \subset S$ such that $\lim_{n\to\infty} lu_n = \lambda$. In the space L, any decreasing norm bounded sequence has an infimum (see e.g. [25, p. 257]), so there exists $\underline{u} = \inf\{u_n\}$, and, moreover, for almost all $t \in [a,b]$, it holds true that $\underline{u}(t) = \inf\{u_n(t)\} = \lim_{n\to\infty} u_n(t)$. Therefore, for almost all $t \in [a,b]$, due to closedness of the set $B(t) \subset \mathbb{R}^n$, the inclusion $\underline{u}(t) \in B(t)$ takes place, so $\underline{u} \in L(B)$. Let us now demonstrate that $\inf S = \underline{u}$. For any $u \in S$, it holds true that $lu > \lambda$, so for some number n, we get $lu > lu_n$, hence $u > u_n > \underline{u}$. Thus, $\underline{u}$ is a lower bound of S, and any function that is greater than $\underline{u}$, is not a lower bound of the sequence $\{u_n\}$ and moreover, not a lower bound of the chain S. So, $\underline{u} = \inf S$ and $\underline{u}(t) = \inf\{u_n(t)\} = \lim_{n\to\infty} u_n(t)$ for almost all $t \in [a,b]$.

It remains now to prove the existence of a non-negative measurable selection of the set-valued mapping $\Upsilon(\underline{u}, \underline{u}) : [a,b] \to \mathrm{K}(\mathbb{R}^m)$ to verify the condition (a3). By the definition of the chain S, for any natural n, there exists a measurable function $y_n \in W^m$ such that $y_n \in \Upsilon(u_n, u_n)$ and $y_n \geq 0$. As the mapping $\Upsilon(u_n, \cdot)$ is antitone, there exists a measurable function $\zeta_n \in W^m$ such that $\zeta_n \in \Upsilon(u_n, \underline{u})$ and $\zeta_n \geq 0$. The mapping $\Upsilon(\cdot, \underline{u}) : L(B) \rightrightarrows W^m$ is the restriction to $L(B)$ of the Nemytskii operator generated by the function $\underline{g} : [a,b] \times \mathbb{R}^n \to \mathrm{K}(\mathbb{R}^m)$,

$$\underline{g}(t,u) \doteq f\big(t, \gamma + \int_a^t \underline{u}(s)ds, \underline{u}(t), u\big) \text{ for almost all } t \in [a,b] \text{ and any } u \in \mathbb{R}^n,$$

that satisfies the Caratheodori conditions. Therefore

$$h_{\mathbb{R}^m}\big(\underline{g}(t,u_n(t)),\underline{g}(t,\underline{u}(t))\big)\to 0. \tag{4.17}$$

For almost all $t\in[a,b]$, it holds true that $\underline{g}(t,u_n(t))\cap\mathbb{R}^m_+\neq\emptyset$ (as $\zeta_n(t)\in\underline{g}(t,u_n(t))$). Assume that the set-valued mapping $\underline{g}(\cdot,\underline{u}(\cdot))$ does not possess non-negative selections. Then for some set $T\subset[a,b]$ of a positive measure, it holds true that $\underline{g}(t,\underline{u}(t))\cap\mathbb{R}^m_+=\emptyset$. In this case, for $t\in T$ there is $\varepsilon>0$ such that the ε-neighborhood of the set $\underline{g}(t,\underline{u}(t))$ does not have intersections with the cone $\mathbb{R}^m_+$. The latter implies that $h_{\mathbb{R}^m}\big(\underline{g}(t,u_n(t)),\underline{g}(t,\underline{u}(t))\big)>\varepsilon$ for any $n,$ which contradicts with the convergence (4.17). Thus, the condition $(\mathrm{a}3)$ is also fulfilled.

Theorem 2.1 implies the existence of a solution $u\in L(B)$ of the inclusion (4.15) and, hence, the existence of a solution $x\in AC(B)$ of the problem (4.11), (4.12), (4.13) (which is defined by the formula $x=\gamma+\int_a^t u(s)ds$). □

Let us now illustrate applications of Theorem 4.1 to investigation of concrete differential equations and inclusions.

Example 3.1:
We denote by χ the Heaviside function, i.e. $\chi(x)=1$ for $x\geq 0$ and $\chi(x)=0$ for $x<0$. Let a measurable function $q_0:[a,b]\to\mathbb{R}$, a measurable integrable function $q_1:[a,b]\to\mathbb{R}$, and a positive number r be given. We consider the differential equation

$$\dot{x}^2-2q_1(t)\dot{x}+\chi(x)-q_0(t)=0,\ \ t\in[a,b], \tag{4.18}$$

together with the additional condition

$$\dot{x}\in B(t)\doteq[q_1(t)-r,q_1(t)+r],\ \ t\in[a,b]. \tag{4.19}$$

Let us demonstrate that if the inequalities

$$-1\leq q_0(t)+q_1^2(t)\leq r^2,\ \ t\in[a,b], \tag{4.20}$$

are fulfilled, the Cauchi problem for the system (4.18), (4.19) with the initial condition (4.13) for any γ has a solution $x\in AC(B)$ such that $\dot{x}(t)\leq q_1(t)$, $t\in[a,b]$.

We treat the equation (4.18) as the inclusion (4.11) with the single-valued mapping $f:[a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ defined by the relation

$$f(t,x,u)=g(t,u)-\chi(x),\ \ t\in[a,b],\ \ x,u\in\mathbb{R},$$

where g is given in the Example 2.1 by the formula (3.8). Obviously, for all x,u, the function $f(\cdot,x,u)$ is measurable, for almost all t and any u, the function $f(t,\cdot,u)$ decreases (i.e. the condition (b2) of Theorem 4.1 is fulfilled) and right continuous, and for almost all t and any x, the function $f(t,x,\cdot)$ is continuous. As it was demonstrated in Example 2.1, the inequality $q_0(t)+q_1^2(t)\leq r^2$ implies the condition (b1) of Theorem 4.1. If one takes $v_0(t)=\gamma+\int_a^b p(s)ds$, $t\in[a,b]$, the relations

$$f\big(t,v_0(t),\dot{v}_0(t)\big)=-\dot{v}_0^2(t)+2q_1(t)\dot{v}_0(t)-\chi(v_o(t))+q_0(t)\geq q_1^2(t)-1+q_0(t)\geq 0$$

take place for all $t\in[a,b]$. Thus, the condition (4.14) is also satisfied, and, according to Theorem 4.1, the problem (4.18), (4.19), (4.13) has a solution $x\in AC(B)$ satisfying the estimate $\dot{x}(t)\leq q_1(t)$, $t\in[a,b]$.

Consider now the following differential inclusion

$$g^+(t,\dot{x})-\chi(x)\ni 0,\ \ t\in[a,b], \tag{4.21}$$

where the set-valued mapping $g^+ : [a,b] \times \mathbb{R} \to \mathrm{K}(\mathbb{R})$ is given by the first formula in (3.10) (see Example 2.1). In the investigation of this inclusion, one can also make use of Theorem 4.1. In order to do this, one should use the condition that guarantees that the mapping $g^+(t,\cdot)$ order covers the set $\{0\} \subset \mathbb{R}$, which was derived in Example 2.1. One thus concludes that if the inequalities

$$-1-\delta \le q_0(t) + q_1^2(t) \le r^2, \ \ t \in [a,b], \tag{4.22}$$

are satisfied, the problem (4.21), (4.19), (4.13) possesses a solution for any γ.

Finally, we consider the differential inclusion

$$g^-(t,\dot{x}) - \chi(x) \ni 0, \ \ t \in [a,b], \tag{4.23}$$

with the set-valued function $g^- : [a,b] \times \mathbb{R} \to \mathrm{K}(\mathbb{R})$ given by the second formula in (3.10). For this inclusion, we analogously find that the validity of the inequalities

$$-1 \le q_0(t) + q_1^2(t) \le r^2 + \delta, \ \ t \in [a,b], \tag{4.24}$$

guarantees the solvability of the problem (4.23), (4.19), (4.13) for any γ.

Consider a particular case of the inclusion (4.11), which allows to obtain the existence of a lower solution to the corresponding Cauchi problem.

Let $m = n$ and the components of f be given by the functions $f_i : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathrm{K}^{\mathrm{C}}(\mathbb{R}),\ i = \overline{1,n}$ (having compact and connected values, i.e. the real line segments). Let us assume that for any $x, v \in \mathbb{R}^n,\ z \in \mathbb{R}$, the function $f_i(\cdot, x, v, z) : [a,b] \to \mathrm{K}^{\mathrm{C}}(\mathbb{R})$ is measurable; for almost all $t \in [a,b]$, any $v \in \mathbb{R}^n$ and $z \in \mathbb{R}$, the function $f_i(t,\cdot,v,z) : \mathbb{R}^n \to \mathrm{K}^{\mathrm{C}}(\mathbb{R})$ is right continuous in each of the arguments $x_1, \ldots, x_n$; for almost all $t \in [a,b]$, any $x \in \mathbb{R}^n$ and $z \in \mathbb{R}$, the function $f_i(t,x,\cdot,z) : \mathbb{R}^n \to \mathrm{K}^{\mathrm{C}}(\mathbb{R})$ is right continuous in each of the arguments $v_1, \ldots, v_n$; for almost all $t \in [a,b]$ and any $x, v \in \mathbb{R}^n$, the function $f_i(t,x,v,\cdot) : \mathbb{R} \to \mathrm{K}^{\mathrm{C}}(\mathbb{R})$ is continuous. We investigate the following system:

$$f_i(t, x, \dot{x}, \dot{x}_i) \ni 0, \ \ t \in [a,b], \ \ i = \overline{1,n}. \tag{4.25}$$

Theorem 4.2:
Let functions $u_0, v_0 \in AC^n$ such that $u_0(a) \le \gamma \le v_0(a)$,

$$-f_i\big(t, u_0(t), \dot{u}_0(t), \dot{u}_{0i}(t)\big) \cap \mathbb{R}_+ \ne \emptyset \ \textit{for almost all} \ t \in [a,b], \ \ i = \overline{1,n},$$
$$f_i\big(t, v_0(t), \dot{v}_0(t), \dot{v}_{0i}(t)\big) \cap \mathbb{R}_+ \ne \emptyset \ \textit{for almost all} \ t \in [a,b], \ \ i = \overline{1,n},$$

be given. Define a set-valued mapping $B : [a,b] \to \mathrm{C}^{\mathrm{C}}(\mathbb{R}^n)$ as follows:

$$B(t) \doteq \big[\dot{v}_0(t), \dot{u}_0(t)\big]_{\mathbb{R}^n}, \ \ t \in [a,b]. \tag{4.26}$$

Assume that for any $i = \overline{1,n}$, for almost all $t \in [a,b]$ and all $u \in B(t)$, the mapping $f_i(t,\cdot,\cdot,u) : \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{K}^{\mathrm{C}}(\mathbb{R})$ is antitone on the set $\mathbb{R}^n \times B(t)$. Then there exists a solution $x \in AC(B)$ to the problem (4.25), (4.12), (4.13)*. Moreover, the set of solutions to the problem* (4.25), (4.12), (4.13) *contains a solution with the least derivative.*

Remark 4.1:
By the virtue of the definition of the mapping B by (4.26)*, the fact that a solution x belongs to $AC(B)$ means that this absolutely continuous function satisfies the inequalities $\dot{u}_0(t) \le \dot{x}(t) \le \dot{v}_0(t)$ for almost all $t \in [a,b]$.*

Proof
We will denote the restrictions of mappings by the same symbols as the initial mappings

provided that leads to no ambiguity. We denote the components of the set-valued mapping $B : [a,b] \to \mathrm{C^C}(\mathbb{R}^n)$ as $B_i : [a,b] \to \mathrm{C^C}(\mathbb{R})$, i.e. $B_i(t) \doteq \big[\dot{v}_{0i}(t), \dot{u}_{0i}(t)\big]$, $t \in [a,b]$. We define the set $T_i \subset [a,b]$ of such $t \in [a,b]$ that for all $x \in \mathbb{R}^n$, $v \in B(t)$, the mapping $f_i(t,x,v,\cdot) : B_i(t) \to \mathrm{K^C}(\mathbb{R})$ is continuous, and $-\infty < \dot{v}_0(t), \dot{u}_0(t) < +\infty$. The Lebesgue measure of this set equals to $b-a$, and for all $t \in T_i$, the set $B_i(t)$ is a segment of the real line.

Let us verify the condition $(\mathrm{b}1)$ of Theorem 4.1. Let for any $i = \overline{1,n}$, for some $t \in T_i$, there exist $x \in \mathbb{R}^n$, $v \in B(t)$ such that the mapping $f_i(t,x,v,\cdot) : B_i(t) \to \mathrm{K^C}(\mathbb{R})$ does not order cover the set $\{0\} \subset \mathbb{R}$. Then there exists $z_0 \in \mathbb{R}$ such that the set $f_i(t,x,v,z_0)$ contains some positive number, and for any $z \in [\dot{v}_{0i}(t), z_0]$, zero does not belong to the set $f_i(t,x,v,z)$. Let us demonstrate that there exists $\delta > 0$ such that for any $z \in [\dot{v}_{0i}(t), z_0]$, it holds true that

$$f_i(t,x,v,z) \cap (-\delta,\delta) = \emptyset. \tag{4.27}$$

Otherwise, there exists a sequence $\{z_k\}_{k=1}^{\infty} \subset [\dot{v}_{0i}(t), z_0]$ whose elements satisfy the inequality $|z_k| < 2^{-k}$. This sequence is compact, so it contains a subsequence converging to some $\overline{z} \in [\dot{v}_{0i}(t), z_0]$. Due to continuity of the mapping $f_i(t,x,v,\cdot)$ at $\overline{z}$, the inclusion $0 \in f_i(t,x,v,\overline{z})$ holds true, which contradicts with the assumptions made.

The relation (4.27), due to the connectedness of the values $f_i(t,x,v,z)$, implies that the segment $U \doteq [\dot{v}_{0i}(t), z_0]$ is a union of the following two sets:

$$U_+ \doteq \big\{z \in U : \ f_i(t,x,v,z) \subset [\delta,+\infty)\big\}, \ \ U_- \doteq \big\{z \in U : \ f_i(t,x,v,z) \subset (-\infty,\delta]\big\}.$$

Both these sets should be closed as the mapping $f_i(t,x,v,\cdot)$ is continuous. However, this is not possible by the virtue of the connectedness of the segment U. Thus, it is proved that for any $t \in T_i$, for all $x \in \mathbb{R}^n$ and $v \in B(t)$, the mapping $f_i(t,x,v,\cdot) : B_i(t) \to \mathrm{K^C}(\mathbb{R})$ order covers the set $\{0\} \subset \mathbb{R}$. Considering the fact that the Lebesgue measure of the set T_i equals to $b-a$, the condition $(\mathrm{b}1)$ is satisfied.

The validity of the rest of the conditions of Theorem 4.1 under the assumptions of the statement being proved is obvious. Thus, there exists a solution $x \in AC(B)$ of the problem (4.25), (4.12), (4.13).

We now prove the existence of a solution with the least derivative. Let us remind the reader (see the proof of Theorem 4.1) that the derivatives of the solutions of the problem (4.25), (4.12), (4.13) are the solutions of the following system of inclusions

$$f_i\big(t, \gamma + \int_a^t u(s)ds, u(t), u_i(t)\big) \ni 0, \ \ t \in [a,b], \ \ i = \overline{1,n}. \tag{4.28}$$

Thus, it remains to prove that the set of solutions of the system (4.28) possesses the least element.

Denote by $\mathcal{R}$ the set (in the space $L(B)$) of solutions of the system (4.28). According to Theorem 2.1, the set $\mathcal{R}$ possesses a minimal element $\widehat{u} \in L(B)$. Assume that this element is not the least in $\mathcal{R}$. Then there exists a solution $z \in \mathcal{R}$ of (4.28) such that $z \ngeq \widehat{u}$. For each $i = \overline{1,n}$, we define the sets

$$E_{i+} = \big\{t \in [0,1] : \ \widehat{u}_i(t) \le z_i(t)\big\}, \ \ E_{i-} = \big\{t \in [0,1] : \ \widehat{u}_i(t) > z_i(t)\big\}.$$

Define a measurable function $\widehat{z}$ with the components equal to

$$\widehat{z}_i(t) = \min\big\{z_i(t), \widehat{u}_i(t)\big\} = \begin{cases} \widehat{u}_i(t) & \text{for } t \in E_{i+}, \\ z_i(t) & \text{for } t \in E_{i-}, \end{cases} \quad i = \overline{1,n}.$$

Thus, the inequality $\widehat{z} < \widehat{u}$ is fulfilled. Hence, for almost all $t \in E_{i+}$, the inclusion

$$0 \in f_i\big(t, \gamma + \int_a^t \widehat{u}(s)ds, \widehat{u}(t), \widehat{u}_i(t)\big), \ \ i = \overline{1,n},$$

due to antitonicity of the mapping $f_i(t,\cdot,\cdot,u)$ on $\mathbb{R}^n \times B(t)$, implies that for any $i=\overline{1,n}$, the set

$$f_i\big(t,\gamma+\int_a^t \widehat{z}(s)ds,\widehat{z}(t),\widehat{z}_i(t)\big)=f_i\big(t,\gamma+\int_a^t \widehat{z}(s)ds,\widehat{z}(t),\widehat{u}_i(t)\big)$$

contains a non-negative number. Analogously, for any $i=\overline{1,n}$, we get that for almost all $t\in E_{i\,-}$, some non-negative number belongs to the set

$$f_i\big(t,\gamma+\int_a^t \widehat{z}(s)ds,\widehat{z}(t),\widehat{z}_i(t)\big).$$

By Theorem 2.1, there exists a solution $\xi\in\mathcal{R}$ of the system (4.28) such that $\xi\le\widehat{z}<\widehat{u}$. The latter, however, contradicts with the minimality of $\widehat{u}$ in $\mathcal{R}$. □

Finally, we note that the equation (4.18) considered in Example 3.1 under the restriction (4.20) satisfies not only the assumptions of Theorem 4.1, but also the assumptions of Theorem 4.2 if one takes

$$v_0(t)=\gamma+\int_a^b q_1(s)ds,\ \ u_0(t)=\gamma+\int_a^b \big(q_1(s)-r\big)ds,\ \ t\in[a,b].$$

Therefore, the set of solutions of the problem (4.18), (4.19), (4.13) such that

$$q_1(t)-r\le\dot{x}(t)\le q_1(t),\ \ t\in[a,b], \tag{4.29}$$

contains a solution with the least derivative.

Analogously, for the system of inclusions (4.21), (4.19) with the initial condition (4.13), which was considered in Example 3.1, if the relations (4.22) take place, the set of solutions that satisfy the inequalities (4.29) contains a solution with the least derivative. The same statement is valid for the Cauchi problem (4.23), (4.19), (4.13). Namely, if the condition (4.24) takes place, the set of solutions satisfying the inequalities (4.29) possesses a solution with the least derivative.

5. EXISTENCE AND ESTIMATES OF EQUILIBRIUM PRICES IN DYNAMICAL CONTINUOUS SUPPLY-AND-DEMAND MODELS

In this section, Theorems 4.1 and 4.2 are applied to investigation of equilibrium in dynamical supply-and-demand models. The model of economic processes we consider has its roots in the classical works of G.C. Evans [11], P.A. Samuelson [12], and R. Allen [13]. A considerable advance in the investigation of models describing equilibrium processes in market models were the results of A.V. Arutyunov, N.G. Pavlova, S.E. Zhukovskiy, A.A. Shananin (see [14–17]) obtained using the results on covering mappings of metric spaces. In particular, in [17] the authors managed to investigate the models with the mappings of supply and demand that are dependent not only on the market prices, but also on the the prices change rates. In this section, the models under consideration contain set-valued mappings of supply and demand. Such generalisation is natural as the values of these mappings are defined in the economic problems as the set of solutions of constraint minimisation problems (see e.g. [23, §4.2.2]). In the present research, in contrast to the cited works that base on the results on mappings of metric spaces, methods of analysis of mappings of partially ordered spaces are employed.

Let us formulate the problem.

Let us have n types of goods, whose prices at any time $t\in[a,b]$ we denote by $p_i(t)$, $i=\overline{1,n}$. Let these prices be known at the initial moment of time

$$p_i(a)=\gamma_i,\ \ i=\overline{1,n}. \tag{5.30}$$

We define the vector $p(t) = \big(p_1(t), \dots, p_n(t)\big) \in \mathbb{R}^n_+$ of prices. Assume that there are some restrictions on the change rates of the prices, i.e. there is a measurable set-valued mapping $B : [a, b] \to \mathrm{K}(\mathbb{R}^n)$ and the inclusion

$$\dot{p}(t) = \big(\dot{p}_1(t), \dots, \dot{p}_n(t)\big) \in B(t) \text{ for almost all } t \in [a, b] \tag{5.31}$$

takes place.

Next, let the set-valued mappings $D : [a, b] \times \mathbb{R}^n_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{K}(\mathbb{R}^n_+)$ and $S : [a, b] \times \mathbb{R}^n_+ \times \mathbb{R}^n \to \mathrm{K}(\mathbb{R}^n_+)$ be given. We assume that for any $x \in \mathbb{R}^n_+$ and any $v, u \in \mathbb{R}^n$, the functions $D(\cdot, x, v, u), S(\cdot, x, v) : [a, b] \to \mathrm{K}(\mathbb{R}^n_+)$ are measurable; for almost all $t \in [a, b]$ and any $v, u \in \mathbb{R}^n$, the functions $D(t, \cdot, v, u), S(t, \cdot, v) : \mathbb{R}^n_+ \to \mathrm{K}(\mathbb{R}^n_+)$ are right continuous in each of the arguments $x_1, \dots, x_n$; for almost all $t \in [a, b]$ and any $x \in \mathbb{R}^n_+$, $u \in \mathbb{R}^n$, the functions $D(t, x, \cdot, u), S(t, x, \cdot) : \mathbb{R}^n \to \mathrm{K}(\mathbb{R}^n_+)$ are right continuous in each of the arguments $v_1, \dots, v_n$; for almost all $t \in [a, b]$ and any $x \in \mathbb{R}^n_+$, $v \in \mathbb{R}^n$, the function $D(t, x, v, \cdot) : \mathbb{R}^n \to \mathrm{K}(\mathbb{R}^n_+)$ is continuous. We assume that the aggregate consumer demand at time t is defined by the formula $D\big(t, p(t), \dot{p}(t), \dot{p}(t)\big)$ and aggregate manufacturer's supply — by the formula $S\big(t, p(t), \dot{p}(t)\big)$.

For the defined above set-valued mappings of supply and demand, we consider the problem of existence of an *equilibrium* — a solution $p \in AC(B)$ (i.e. absolutely continuous function, whose derivative satisfies the restriction (5.31)) of the inclusion

$$D\big(t, p(t), \dot{p}(t), \dot{p}(t)\big) - S\big(t, p(t), \dot{p}(t)\big) \ni 0, \tag{5.32}$$

with the initial condition (5.30). Here the difference $D - S$ of sets in the space $\mathbb{R}^n$ is the set of all vectors $x - y$, where $x \in D$, $y \in S$.

We will denote the restrictions of mappings by the same symbols as the initial mappings indicating their domains and ranges. Applying Theorem 4.1 to the mapping $f : [a, b] \times \mathbb{R}^n_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{K}(\mathbb{R}^n_+)$ defined by the relation

$$f(t, x, v, u) \doteq D(t, x, v, u) - S(t, x, v) \text{ for almost all } t \in [a, b] \text{ and any } x \in \mathbb{R}^n_+,\ v, u \in \mathbb{R}^n_+,$$

we get the following statement.

Corollary 5.1:
Let a non-negative function $\overline{p} \in AC(B)$ such that $\overline{p}(a) \geq \gamma$ and

$$\Big(D\big(t, \overline{p}(t), \dot{\overline{p}}(t), \dot{\overline{p}}(t)\big) - S\big(t, \overline{p}(t), \dot{\overline{p}}(t)\big)\Big) \cap \mathbb{R}^n_+ \neq \emptyset \textit{ for almost all } t \in [a, b],$$

be given. Let the set of measurable selections of the set-valued mapping $B(\cdot) \cap \mathcal{O}_{\mathbb{R}^n}\big(\overline{p}(\cdot)\big) : [a, b] \to \mathrm{C}(\mathbb{R}^n)$ be integrally bounded from below (i.e. there exists a number C such that for any measurable function $u \in W^n$, satisfying the relations $u(t) \in B(t)$ and $u(t) \leq \dot{\overline{p}}(t)$ for almost all $t \in [a, b]$, it holds true that $\int_a^b u(t)dt \geq C$) and the following conditions take place:

(c1) *for almost all $t \in [a, b]$, any $x \in \mathbb{R}^n_+$ and $v \in B(t)$, the mapping $D(t, x, v, \cdot) : B(t) \to \mathrm{K}(\mathbb{R}^n_+)$ order covers the set $\{0\} \subset \mathbb{R}^n$;*
(c2) *for almost all $t \in [a, b]$ and any $u \in B(t)$, the mapping $D(t, \cdot, \cdot, u) : \mathbb{R}^n_+ \times B(t) \to \mathrm{K}(\mathbb{R}^n_+)$ is antitone, and the mapping $S(t, \cdot, \cdot) : \mathbb{R}^n_+ \to \mathrm{K}(\mathbb{R}^n_+)$ is isotone.*

Then there exists a solution $p \in AC(B)$ of the problem (5.32), (5.31), (5.30) *such that*

$$\dot{p}(t) \leq \dot{\overline{p}}(t) \textit{ for almost all } t \in [a, b]. \tag{5.33}$$

Note that the assumptions $(c2)$ have natural economic sense: a decrease of the prices and the growth rates of prices result in an increase of the demand and a decrease of supply.

The main distinctions of Corollary 5.1 from the results of [14–17] are: set-valued mappings of supply and demand, omission of the additional condition of essentially boundedness of the derivative $\dot{p}$ of the prices function, and the estimate of the equilibrium price derivative of the form (5.33). Corollary 5.1 has the type of the Chaplygin comparison theorem for the differential inclusion modelling the economic processes under consideration.

Finally, let us formulate a statement allowing to obtain two-sided estimates of the equilibrium prices and guarantee the existence of the equilibrium price having the least rate of change.

Let the components D_i, S_i of the mappings D, S be the functions $D_i : [a,b] \times \mathbb{R}_+^n \times \mathbb{R}^n \times \mathbb{R} \to \mathrm{K}^{\mathrm{C}}(\mathbb{R})$, $S_i : [a,b] \times \mathbb{R}_+^n \times \mathbb{R}^n \to \mathrm{K}^{\mathrm{C}}(\mathbb{R})$, $i = \overline{1,n}$ (having compact connected values). We assume that for any $x \in \mathbb{R}_+^n$, $v \in \mathbb{R}^n$, $z \in \mathbb{R}$, the functions $D_i(\cdot, x, v, z) : [a,b] \to \mathrm{K}^{\mathrm{C}}(\mathbb{R})$, $S_i(\cdot, x, v) : [a,b] \to \mathrm{K}^{\mathrm{C}}(\mathbb{R})$ are measurable; for almost all $t \in [a,b]$, any $v \in \mathbb{R}^n$ and $z \in \mathbb{R}$, the functions $D_i(t, \cdot, v, z) : \mathbb{R}_+^n \to \mathrm{K}^{\mathrm{C}}(\mathbb{R})$, $S_i(t, \cdot, v) : \mathbb{R}_+^n \to \mathrm{K}^{\mathrm{C}}(\mathbb{R})$ are right continuous in each of the arguments $x_1, \ldots, x_n$; for almost all $t \in [a,b]$, any $x \in \mathbb{R}_+^n$ and $z \in \mathbb{R}$, the functions $D_i(t, x, \cdot, z) : \mathbb{R}^n \to \mathrm{K}^{\mathrm{C}}(\mathbb{R})$, $S_i(t, x, \cdot) : \mathbb{R}^n \to \mathrm{K}^{\mathrm{C}}(\mathbb{R})$ are right continuous in each of the arguments $v_1, \ldots, v_n$; for almost all $t \in [a,b]$ and any $x \in \mathbb{R}_+^n$, $v \in \mathbb{R}^n$, the function $D_i(t, x, v, \cdot) : \mathbb{R} \to \mathrm{K}^{\mathrm{C}}(\mathbb{R})$ is continuous. Consider a particular case of the inclusion (5.32) — the system

$$D_i(t, x, \dot{x}, \dot{x}_i) - S_i(t, x, \dot{x}) \ni 0, \;\; t \in [a,b], \;\; i = \overline{1,n}. \tag{5.34}$$

Theorem 4.2 implies the following statement.

Corollary 5.2:
Let functions $u_0, v_0 \in AC^n$ *such that* $u_0(a) \le \gamma \le v_0(a)$,

$$\Big(S_i\big(t, u_0(t), \dot{u}_0(t)\big) - D_i\big(t, u_0(t), \dot{u}_0(t), \dot{u}_{0i}(t)\big)\Big) \cap \mathbb{R}_+ \neq \emptyset \text{ for almost all } t \in [a,b], \; i = \overline{1,n},$$

$$\Big(D_i\big(t, v_0(t), \dot{v}_0(t), \dot{v}_{0i}(t)\big) - S_i\big(t, v_0(t), \dot{v}_0(t)\big)\Big) \cap \mathbb{R}_+ \neq \emptyset \text{ for almost all } t \in [a,b], \; i = \overline{1,n},$$

be given. Define a set-valued mapping $B : [a,b] \to \mathrm{C}^{\mathrm{C}}(\mathbb{R}^n)$ *by the relation* (4.26)*. Assume that for any* $i = \overline{1,n}$*, for almost all* $t \in [a,b]$ *and all* $u \in B(t)$*, the mapping* $D_i(t, \cdot, \cdot, u) : \mathbb{R}_+^n \times \mathbb{R}^n \to \mathrm{K}^{\mathrm{C}}(\mathbb{R})$ *is antitone on the set* $\mathbb{R}^n \times B(t)$ *and the mapping* $S_i(t, \cdot, \cdot) : \mathbb{R}_+^n \times \mathbb{R}^n \to \mathrm{K}^{\mathrm{C}}(\mathbb{R})$ *is isotone on the set* $\mathbb{R}^n \times B(t)$*. Then there exists a solution* $p \in AC(B)$ *of the problem* (5.34), (5.31), (5.30)*. Moreover, the set of solutions of the problem* (5.34), (5.31), (5.30) *contains a solution with the least derivative.*

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