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Mathematical Model of the Infection Spread in Transport Based on the Theory of Porous Medium

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Abstract: Mathematical model of the spread of a virus in trains which considers transmission of infection from two types of sick passengers (clearly infected and latently infected) was built and named SEI (Susceptible-Exposed-Infected) model. Ventilation and movements of the passengers inside the train are considered through the porous medium equation. An algorithm for filling empty seats in the train is developed. A route of ten hours duration with two stop stations was simulated. The results of numerical calculations allow to conclude that asymptomatic infected passengers are the most dangerous group in the spread of infection.

Keywords: mathematical model, SEI model, train, spread of infection, porous medium equation

1. INTRODUCTION

The spread of coronavirus disease COVID-19 caused serious problems for the economies and health systems of all the world. Most of countries took the necessary precautions to slow down the spread of the infection, but these steps do not allow to live without limits. The results of mathematical modeling of disease dynamics can be used in emergency situations [21].

Transport is a very dangerous channel which has huge influence on disease dynamics. Bo Xu et al. (2019) constructed the national highway network between 333 cities in China to show that road transport has significant impacts on Pandemic Influenza A (H1N1) in Mainland China [19]. Colizza et al. (2006) used worldwide air travel infrastructure to forecast global epidemics [6]. Balcan et al. (2003) used airline traffic and small-scale commuting ows to analyze their role in the global epidemics [2]. Therefore, after the outbreak of COVID-19 many countries limited transport mobility. Linka et al. (2020) suggested that an unconstrained mobility would have accelerated the spreading of the virus, especially in such countries as Spain, and France [14]. Analysis of the situation in Italy confirms that, in addition with other factors (socio-economic, territorial and pollutant variables), the availability of the transport system had the main impact on Covid-19 crisis [5]. Cartenì et al. (2021) in further researches showed that the certified COVID-19 cases certified are directly related to the number of public transport trips [4].

Rail transport is a dangerous source of infection. Zou et al. (2021) analyzed the influence of train-induced wind on the COVID-19 pandemic and their results confirmed significance of rail transport to the transmission of the virus [22]. An outbreak of COVID-19 started in Wuhan, Hubei. Wuhan is located in the center of China and has a convenient transportation system and Hubei is identified as a hub of traveling and transport in the country which played a big role in the spreading of COVID-19 [15]. The transmission risk of COVID-19 on high-speed trains in China in the period from 19 December 2019 through 6 March 2020 is quantified in [9]. This analysis helps to understand more clearly the risk of COVID-19

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transmission among train passengers. Viruses are transmitted through surfaces [18] and via airborne droplets, especially quickly indoors [12].

Most of the studies connected with the modeling of COVID-19 spread do not considered the influence of transport [10,16,17,20]. Though there are various works about modeling of COVID-19 spread via air travel (for instance, [7,13,14]), a few attention is paid to the disease transmission through rail travel passengers.

The aim of this study is to investigate the infection spread connected with passengers transportation. To achieve it, a modified SEI model is built. The infection spread within a train is described by the porous medium equation. A random process of filling free seats is considered through the developed algorithm and a pseudo-random number generator is used.

2. MATHEMATICAL MODEL

Modified SEI model (Susceptible-Exposed-Infected) with constant population N divided into three groups: susceptible (S), exposed (E) and infected (I) passengers. As for the rail travels the group of recovered is not so important because we study only the dynamics of the virus spreading, the infection itself is instantaneous. The consequences stretch over the time. Isolation can be discontinued between 5 or 10 days after symptom onset. One of the longest train journeys is the Trans–Siberian Railway, and one of the routes on it from Moscow to Vladivostok is about 7 days. These periods are very similar to each over, that is why it is assumed that during rail travel most of passengers spend less time than is required for full recovery. Also, in the case of COVID-19 the impact of asymptomatic patients is significant [8]. Thus, modified SEI model is the R group excluded and included the group of exposed (asymptomatic) passengers E, which also can infect susceptible passengers S as symptomatic infected individuals I. The compartmental model is formulated by the following set of ordinary differential equations:

$$\frac{\partial S}{\partial t} = -\frac{\alpha SI}{N} - \frac{\beta SE}{N} \tag{2.1}$$

$$\frac{\partial E}{\partial t} = \frac{\alpha_1 SI}{N} + \frac{\beta SE}{N} \tag{2.2}$$

$$\frac{\partial I}{\partial t} = \frac{\alpha_2 SI}{N} \tag{2.3}$$

$$S(t) + E(t) + I(t) = N \tag{2.4}$$

When symptomatic infected passengers I infect susceptible passengers S with the transmission rate α, individuals from group S can become symptomatic infected with the rate α_2 or asymptomatic infected E with the rate α_1 at $\alpha = \alpha_1 + \alpha_2$. If asymptomatic infected passengers E infect susceptible passengers S with the transmission rate β, the number of asymptomatic individuals increases at the rate β.

Movements of susceptible, symptomatic infected and asymptomatic infected along the train mutually influence on each other. To consider it parabolic partial differential equations are used to enter variables responsible for motion. The train can be represented as a rod with a length l, which is insulated at the boundaries. Characteristics of the spread $(u(x,t))$ of infection for groups inside the train (road) are functions of space (x) and time (t). They are presented as a porous medium equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^m \frac{\partial u}{\partial x}\right) \tag{2.5}$$

where m is a constant which describes porosity. It is equal to the ratio of the volume occupied by pores V_n to the total volume V [3, pp. 7-9]:

$$m = {}^{V_n}\!/_{V} \tag{2.6}$$

The effect of porosity considers that the train has a general ventilation, through which the infection can be transmitted, and movements of the passengers inside the train and the restaurant car. The solutions or the porous medium equation converge to solutions of the heat equation when $m < 1$ [1].

Variables which describes each group of SEI model and space (x) are normalized to one. The passengers from all the groups have influence on susceptible S and exposed E individuals. Dynamics of symptomatic infected passengers I depends on the groups S and I. Thus, considering the equations (2.1)- (2.3) as the source terms and porosity (2.5) the model is formulated by the following set of equations:

$$\frac{\partial S}{\partial t} = -\alpha SI - \beta SE + d_{11}\frac{1}{l^2}\frac{\partial}{\partial x}\left(S^m\frac{\partial S}{\partial x}\right) + d_{12}\frac{1}{l^2}\frac{\partial}{\partial x}\left(S^m\frac{\partial E}{\partial x}\right) + d_{13}\frac{1}{l^2}\frac{\partial}{\partial x}(S^m\frac{\partial I}{\partial x}) \tag{2.7}$$

$$\frac{\partial E}{\partial t} = \alpha_1 SI + \beta SE + d_{21}\frac{1}{l^2}\frac{\partial}{\partial x}\left(E^m\frac{\partial S}{\partial x}\right) + d_{22}\frac{1}{l^2}\frac{\partial}{\partial x}\left(E^m\frac{\partial E}{\partial x}\right) + d_{23}\frac{1}{l^2}\frac{\partial}{\partial x}(E^m\frac{\partial I}{\partial x}) \tag{2.8}$$

$$\frac{\partial I}{\partial t} = \alpha_2 SI + d_{31}\frac{1}{l^2}\frac{\partial}{\partial x}\left(I^m\frac{\partial S}{\partial x}\right) + d_{33}\frac{1}{l^2}\frac{\partial}{\partial x}(I^m\frac{\partial I}{\partial x}) \tag{2.9}$$

$$S(t) + E(t) + I(t) = 1 \tag{2.10}$$

where $D = \left(d_{ij}\right)$ is a positive definite matrix that characterized the intensity of infections.

From station to station the train is isolated at the ends and the actual state of the system cannot be known. That is why we used second-type boundary condition, as known as Neumann boundary condition, which specifies the value of the normal derivative of the function. This means that for an interval $0 < x < 1$ the boundary conditions for each group are:

$$\frac{\partial S}{\partial x}\Big|_{x=0} = \frac{\partial S}{\partial x}\Big|_{x=1} = 0 \tag{2.11}$$

$$\frac{\partial E}{\partial x}\Big|_{x=0} = \frac{\partial E}{\partial x}\Big|_{x=1} = 0 \tag{2.12}$$

$$\frac{\partial I}{\partial x}\Big|_{x=0} = \frac{\partial I}{\partial x}\Big|_{x=1} = 0 \tag{2.13}$$

Each passenger has his own place and at every point of the train we can calculate to which group the passenger belongs to. We consider that asymptomatic E and symptomatic I passengers are formulated as functions, where parameters ξ_e, ξ_i specify the proportion of the sick passengers at each place of the train, the points x_e, x_i determine the area of infection, and variance σ measures how sick passengers impact on susceptible neighbors. The initial conditions for the groups E and I are described by the following equations:

$$E(x,0) = \xi_e e^{\frac{-(x-x_e)^2}{4\sigma^2}} \tag{2.14}$$

$$I(x,0) = \xi_i e^{\frac{-(x-x_i)^2}{4\sigma^2}} \tag{2.15}$$

Parameters (ξ_e, x_e) and (ξ_i, x_i) refers to the groups E and I, respectively.

The number of susceptible is determined from the (2.10):

$$S(x,0) = 1 - E(x,0) - I(x,0) \tag{2.16}$$

Determination of seats that sick passengers can buy is a random process. That is why, if the train is empty, values x_e, x_i are determinated, using the pseudo-random number generator. Equations (2.14) – (2.15) are calculated given number of times. The results obtained at each point are summarized. The arithmetic mean is taken.

The algorithm to set the initial conditions at interchange points is very similar. It consists of the following steps:

1. Divide the length of the train by the number of cars.
2. Select the car that people leave using a pseudo-random number generator
3. Choose how many percent leaves the car using a pseudo-random number generator.
4. Divide this percentage evenly between each of the groups
5. Make steps 2-4 until the train is free for a chosen percent
6. Fill selected cars according to the statistics of each group depending on the city.
7. Make steps 2-6 a given number of times to average the results.

The transmission rate α between susceptible passengers S with symptomatic infected passengers I is assumed to be α =0,42. Since passengers are indoors and the disease may not occur clearly immediately, probability to become asymptomatic infected E is higher than probability to become symptomatic infected I. Therefore, the rate α_1 between susceptible passengers S with asymptomatic infected passengers E is equal to $\alpha_1 = 0{,}34$. The rate α_2 between susceptible passengers S with symptomatic infected passengers I is equal to $\alpha_2 = 0{,}08$. According to [11], the transmission rate β is even higher than α, and within this study of the model is assumed to be β = 0,5. The matrix D has positive definite eigenvalues, which ensures the non-negativity of the solutions, and is presented in the such form in this study:

$$D = \begin{pmatrix} 0{,}13 & 0{,}12 & 0{,}01 \\ 0{,}12 & 0{,}13 & 0{,}01 \\ 0{,}01 & 0 & 0{,}009 \end{pmatrix}$$

3. RESULTS

The total travel time is restricted by ten hours. There are two stop stations at which structure of passengers changed. The travel time between stations no greater than five hours. The train consists of sixteen cars, each with nine compartments. Considering staff, the number of the passengers is $N = 600$. We set the porosity $m = ¼$.

Model (2.7) - (2.9) with boundary and initial conditions (2.11) - (2.16) are implemented with explicit method on Python. The random module was used to generate pseudo-random numbers.

Fig. 3.1 shows the initial conditions for each group (S, E, I) at first station. Panels (b)–(c) from Fig.1 shows that the asymptomatic infected E and symptomatic infected I passengers are unevenly distributed throughout the train. The share of each group is $S\,(0) = 0{,}9809$; $E(0) = 0{,}0103$; $I\,(0) = 0{,}0088$.

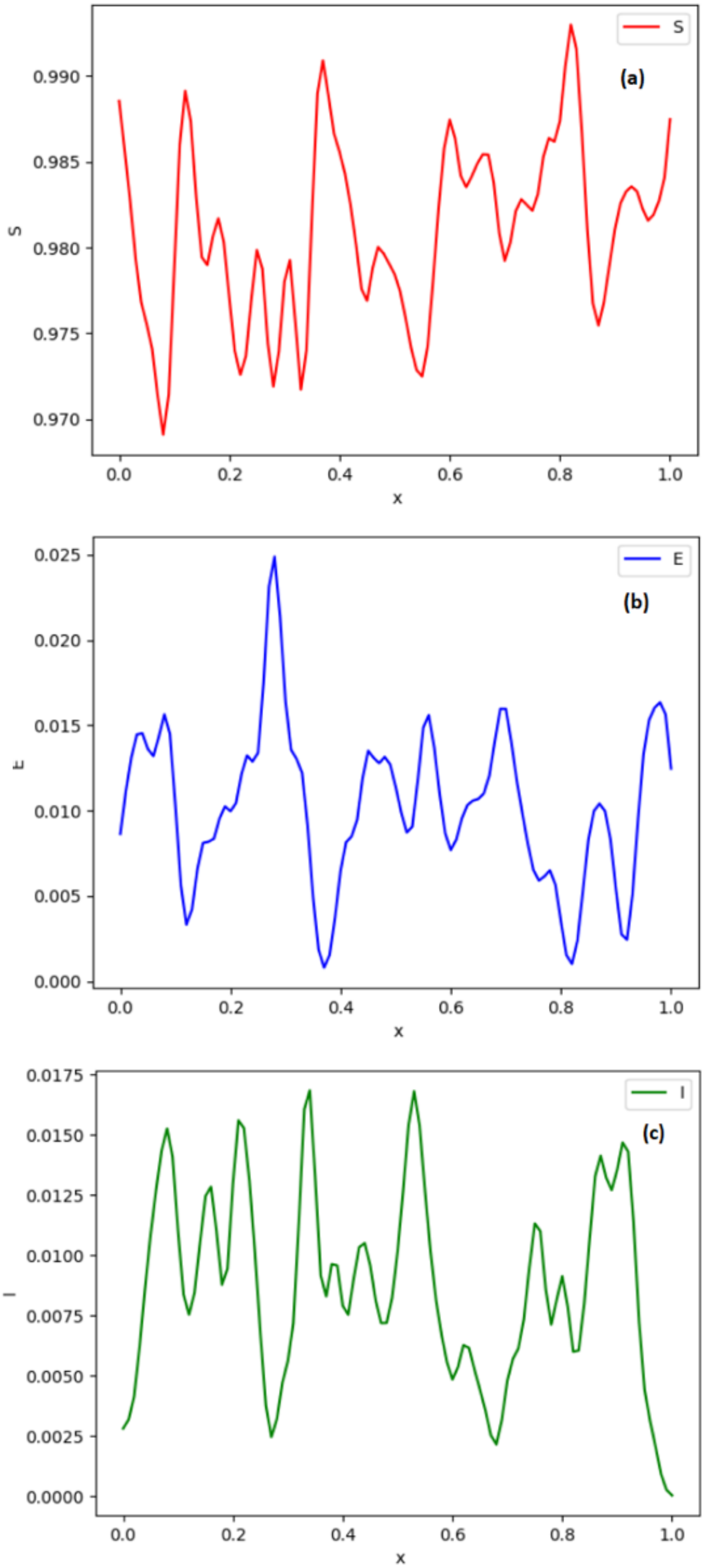

Fig. 3.1. The initial conditions for susceptible S (a) asymptomatic infected E (b) and symptomatic infected I (c) passengers at station one. The ordinate axis shows the share of each group (S, E, I) at the point x, the abscissa axis presents the location in the train (x)

Fig. 3.2 shows the dynamics over the time and space for each group (S, E, I) through the trip from first to second station. It can be seen from the surfaces that the susceptible group (Fig. 3.2 (a)) tends to fall, while the number of asymptomatic infected (Fig. 3.2 (b)) passengers grows. The asymptomatic infected people almost do not change due to their inactivity.

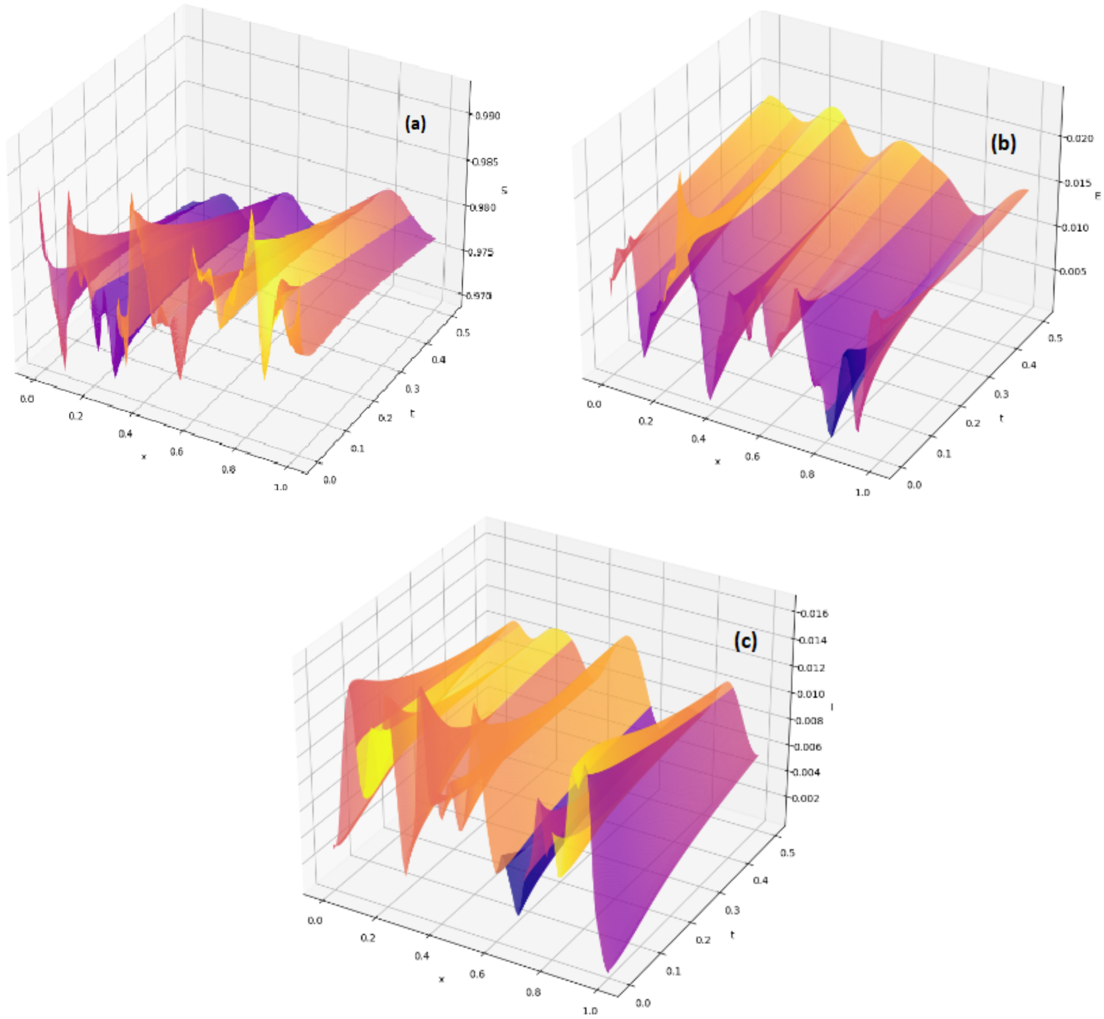

Fig. 3.2. The surfaces for susceptible S (a) asymptomatic infected E (b) and symptomatic infected I (c) passengers during the trip. The abscissa axis presents the location in the train (x), on the ordinate axis is time (t), on the applicate axis is the share of each of each group (S, E, I)

Consider the simulation results on arrival at station two of each group.

Fig. 3.3 shows the share at each point. The share of each group at last step is $S\,(0) = 0{,}976$; $E\,(0) = 0,0145$; $I\,(0) = 0{,}009$. During a five-hour trip the number of susceptible passengers decreased by 0,5 percent, number of asymptomatic infected passengers increased by 0,4 percent. It is clearly seen that location of symptomatic infected passengers (Fig. 3.3 (c)) is very similar to initial conditions (Fig.3.1 (c)). By comparing the dynamics over the time of asymptomatic infected passengers from departure (Fig. 3.1 (b)) to arrival (Fig. 3.3 (b)) it can be seen at the beginning most of the sick passengers are in the first half of the train. Due to the effect of porosity, this difference is more noticeable at the end. The more asymptomatic infected passengers are in the area, the more possibility to get infected.

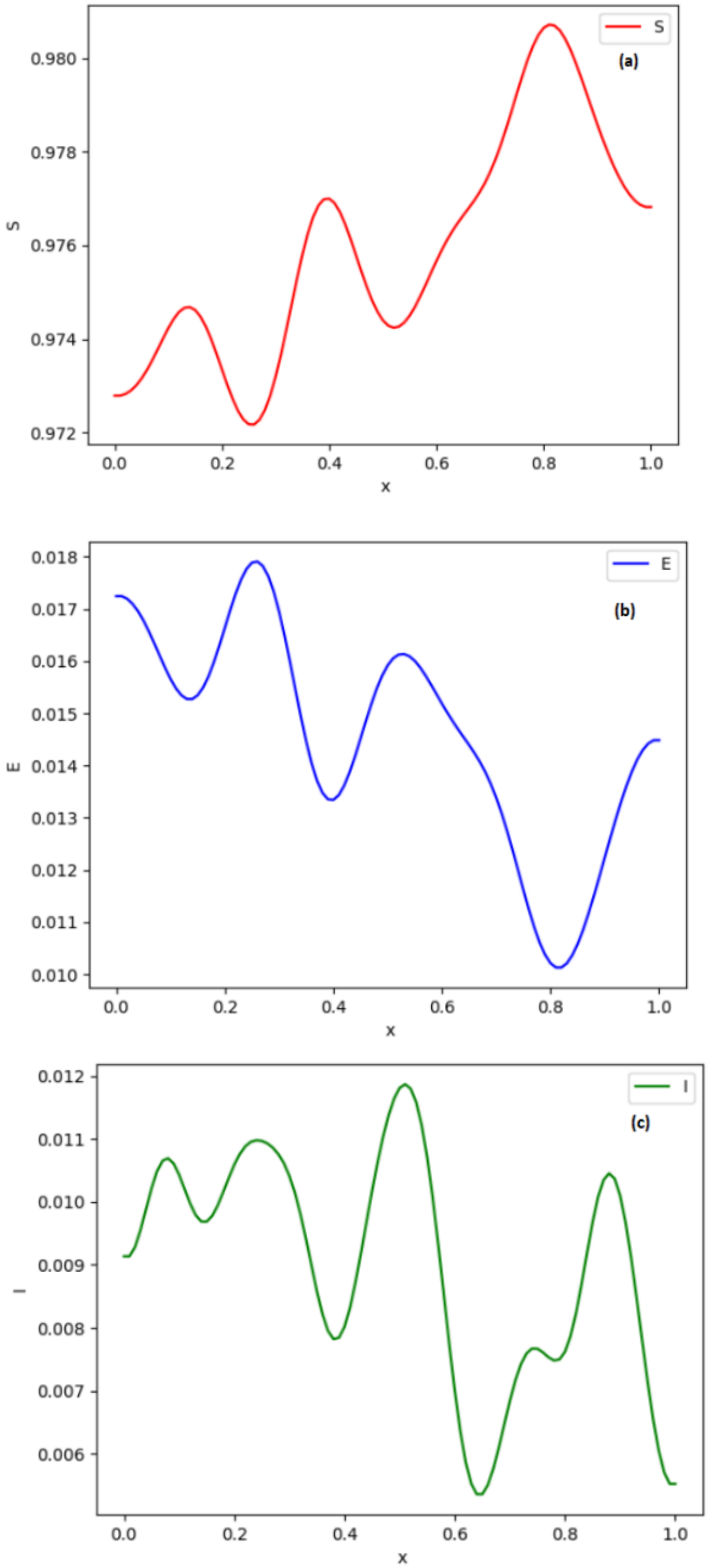

Fig. 3.3. The share for susceptible S (a) asymptomatic infected E (b) and symptomatic infected I (c) passengers on arrival at station two. The ordinate axis shows the share of each group (S, E, I) at the point x, the abscissa axis presents the location in the train (x)

Fig. 3.4 shows the initial conditions for each group *(S, E, I)* at second station. Based on the algorithm at the interchange points there are 7 percent who leaves the train and the free cars are filled by 98 percent susceptible and asymptomatic infected passengers respectively. It is assumed that the probability of taking seats by symptomatic infected passengers is very small with such filling.

Despite the small change in the initial conditions, there is a small difference between at arrival at second station (Fig. 3.3) and departure from this station (Fig. 3.4). The share of each group is $S(0) = 0{,}976;\ E(0) = 0{,}0149;\ I(0) = 0{,}0084$.

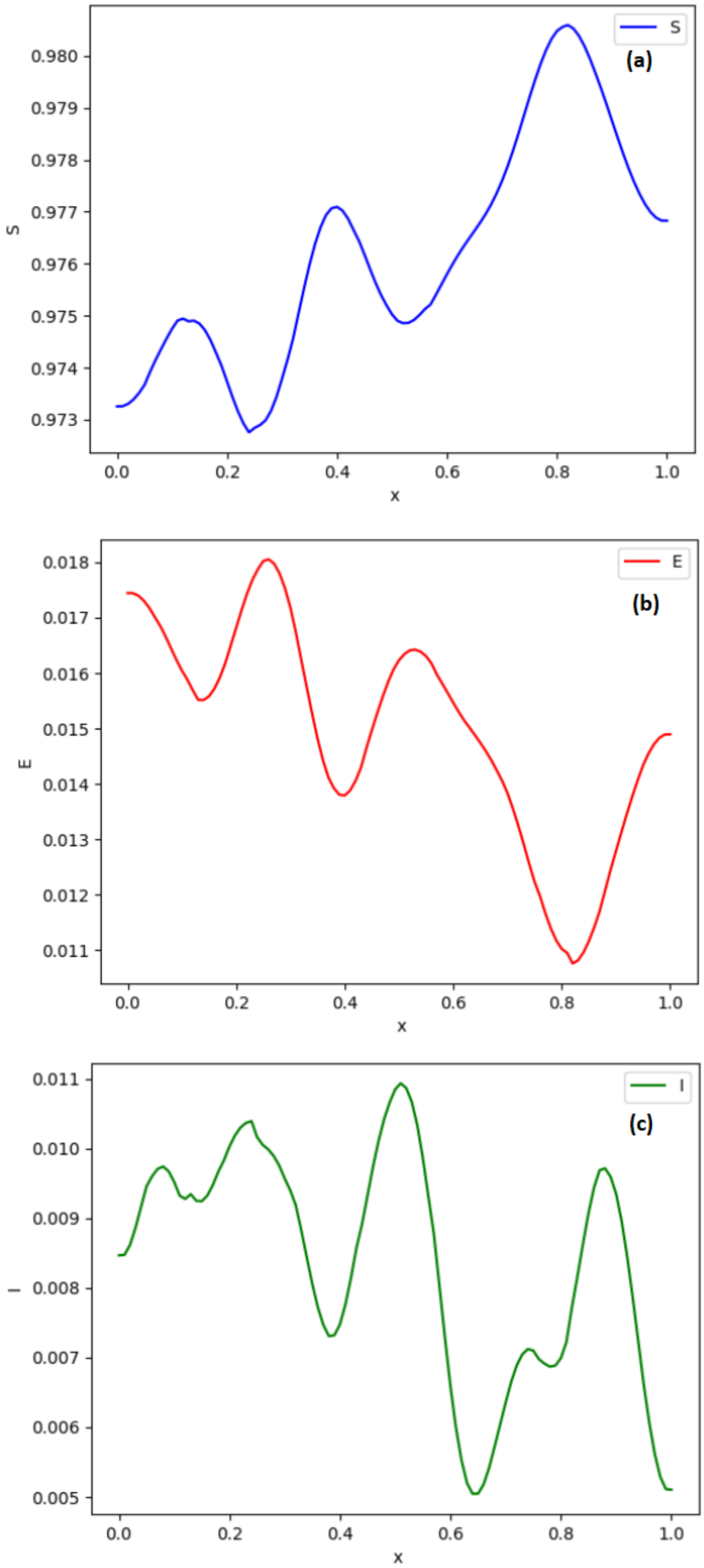

Fig. 3.4. The initial conditions for susceptible S (a) asymptomatic infected E (b) and symptomatic infected I (c) passengers at station two. The ordinate axis shows the share of each group (S, E, I) at the point x, the abscissa axis presents the location in the train (x)

The share of each group at last step is:

$S(0) = 0{,}97;\ E(0) = 0{,}021;\ I(0) = 0{,}0087.$

The number of asymptomatic infected passengers E has already increased by 0,6 percent which is one and a half bigger than the results of simulation from first to second station. Higher asymptomatic infected people lead to a faster spread od virus. The number symptomatic infected passengers, on the other hand, was almost unchanged. Despite their danger during the trip, they are inactive and with a small change of number of passengers there is a small probability that free seats will be filled by people from this group.

5. CONCLUSION

Modified SEI model in the porous medium was built. The probability of infecting from clearly infected and latently infected groups, passenger movement and ventilation were considered. The numerical calculations showed that the effect of a porous medium affects the breadth of the spread of infection. An algorithm to determine the initial conditions in the case of filling a free train and at interchange stations was developed. This algorithm considers factor of random filling of seats of passengers of different groups. Within the model problem, a small change in the composition of the train at the stop station led to increase of the virus spread. The numerical calculations made it possible to conclude that asymptomatic infected passengers have the greatest influence on the spread of infection than symptomatic infected. The increase in the number of asymptomatic infected passengers has led to a rapid increase spreading of disease. For a more detailed analysis, it is necessary to obtain reliable data on passenger traffic.

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