Predictive Inferences for a Future Number of Failures Coming from Underlying Models under Parametric Uncertainty

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Abstract
In this paper, we present an accurate procedure to obtain prediction limits for the number of failures that will be observed in a future inspection of a sample of units, based only on the results of the first in-service inspection of the same sample. The failure-time of such units is modeled with a two-parameter Weibull distribution indexed by scale and shape parameters β and δ, respectively. It will be noted that in the literature only the case is considered when the scale parameter β is unknown, but the shape parameter δ is known. As a rule, in practice the Weibull shape parameter δ is not known. Instead it is estimated subjectively or from relevant data. Thus its value is uncertain. This δ uncertainty may contribute greater uncertainty to the construction of prediction limits for a future number of failures. In this paper, we consider the case when both parameters β and δ,are unknown. In literature, for this situation, usually a Bayesian approach is used. Bayesian methods are not considered here. We note, however, that although subjective Bayesian prediction has a clear personal probability interpretation, it is not generally clear how this should be applied to non-personal prediction or decisions. Objective Bayesian methods, on the other hand, do not have clear probability interpretations in finite samples. The technique proposed here for constructing prediction limits emphasizes pivotal quantities relevant for obtaining ancillary statistics. and represents a special case of the method of invariant embedding of sample statistics into a performance index. Two versions of prediction limits for a future number of failures are given.

Keywords Weibull distribution, parametric uncertainty, future number of failures, prediction limits

1 Introduction

This paper extends the results of Nelson [1]. Nelson's prediction limits were motivated by the following application. Nuclear power plants contain large heat exchangers that transfer energy from the reactor to steam turbines. Such exchangers typically have 10,000 to 20,000 stainless steel tubes that conduct the flow of steam. Due to stress and corrosion, the tubes develop cracks over time. Cracks are detected during planned inspections. The cracked tubes are subsequently plugged to remove them from service. To develop efficient inspection and plugging strategies, plant management can use a prediction of the added number of tubes that will need plugging by a specified future time.

Nelson presents simple prediction limits for the number of failures that will be observed in a future inspection of a sample of units. The past data consist of the cumulative number of failures in a previous inspection of the same sample of units. Life of such units is modeled with a Weibull distribution with a given shape parameter value.

Prediction of an unobserved random variable is a fundamental problem in statistics. Hahn and Nelson [2], Patel [3], and Hahn and Meeker [4] provided surveys of methods for statistical prediction for a variety of situations on this topic. In the areas of reliability and life-testing, this problem translates to obtaining prediction intervals for lifetime distributions. Nordman and Meeker [5] compared probability ratio, simplified probability ratio and likelihood ratio methods proposed by Nelson [1], assuming known the Weibull shape parameter δ.

In this paper, we use a frequentist procedure, which is called 'within-sample prediction of future order statistics', when the time-to-failure follows the two-parameter Weibull distribution indexed by scale and shape parameters β and δ. We consider the case when both parameters β and δ are unknown. The technique proposed here for constructing prediction limits emphasizes pivotal quantities relevant for obtaining ancillary statistics and represent a special case of the method of invariant embedding of sample statistics into a performance index applicable whenever the statistical problem is invariant under a group of transformations, which acts transitively on the parameter space (Nechval et al. [6-7]).

Conceptually, it is useful to distinguish between "new-sample" prediction, "within-sample" prediction, and "new-within-sample" prediction. Some mathematical preliminaries for the within-sample prediction are given below.

2 Mathematical Preliminaries for Within-Sample Prediction

Theorem 1 *Let $X_1 \leq \ldots \leq X_k$ be the first k ordered observations (order statistics) in a sample of size m from a continuous distribution with some probability density functio $f_\theta(x)$ and distribution function $F_\theta(x)$, where θ is a parameter (in general, vector). Then the joint probability density function of $X_1 \leq \ldots \leq X_k$*

and the lth order statistics $X_l (1 < k < l < m)$ *is given by*

$$g_\theta(x_1, \ldots, x_k, x_l) = g_\theta(x_1, \ldots, x_k) g_\theta(x_l|x_k), \tag{1}$$

where

$$g_\theta(x_1, \ldots, x_k) = \frac{m!}{(m-k)!} \Pi_{i=1}^{k} f_\theta(x_i)[1 - F_\theta(x_k)]^{m-k}, \tag{2}$$

$$\begin{aligned}
& g_\theta(x_l|x_k) \\
=& \frac{(m-k)!}{(l-k-1)!(m-l)!}[\frac{F_\theta(x_l) - F_\theta(x_k)}{1 - F_\theta(x_k)}]^{l-k-1}[1 - \frac{F_\theta(x_l) - F_\theta(x_k)}{1 - F_\theta(x_k)}]^{m-l} \frac{f_\theta(x_l)}{1 - F_\theta(x_k)} \\
=& \frac{(m-k)!}{(l-k-1)!(m-l)!} \sum_{j=0}^{l-k-1} \begin{pmatrix} l-k-1 \\ j \end{pmatrix} (-1)^j [\frac{1 - F_\theta(x_l)}{1 - F_\theta(x_k)}]^{m-l+j} \frac{f_\theta(x_l)}{1 - F_\theta(x_k)} \quad (3) \\
=& \frac{(m-k)!}{(l-k-1)!(m-l)!} \sum_{j=0}^{m-l} \begin{pmatrix} m-l \\ j \end{pmatrix} (-1)^j [\frac{F_\theta(x_l) - F_\theta(x_k)}{1 - F_\theta(x_k)}]^{l-k-1+j} \frac{f_\theta(x_l)}{1 - F_\theta(x_k)}
\end{aligned}$$

represents the conditional probability density function of X_l *given* $X_k = x_k$*.*

*Proof.*The joint density of $X_1 \leq \ldots \leq X_k$ and X_l is given by

$$g_\theta(x_1, \ldots, x_k, x_l) = \frac{(m)!}{(l-k-1)!(m-l)!} \prod_{i=1}^{k} f_\theta(x_i)[F_\theta(x_l) - F_\theta(x_k)]^{l-k-1} f_\theta(x_l)$$

$$[1 - F_\theta(x_l)]^{m-l} = g_\theta(x_1, \ldots, x_k) g_\theta(x_l|x_k). \tag{4}$$

It follows from (4) that

$$g_\theta(x_l|x_1, \ldots, x_k) = \frac{g_\theta(x_1, \ldots, x_k, x_l)}{g_\theta(x_1, \ldots, x_k)} = g_\theta(x_l|x_k), \tag{5}$$

i.e., the conditional distribution of X_l l, given $X_i = x_i$ for all $i = 1, \ldots, k$, is the same as the conditional distribution of X_l , given only $X_k = x_k$,which is given by (3). This ends the proof.

Corollary 1.1. The conditional probability distribution function of X_l given $X_k = x_k$ is

$$\begin{aligned}
& P_\theta(X_l \leq x_l | X_k = x_k) \\
=& 1 - \frac{(m-k)!}{(l-k-1)!(m-l)!} \sum_{j=0}^{l-k-1} \begin{pmatrix} l-k-1 \\ j \end{pmatrix} \frac{(-1)^j}{m-l+1+j} \Big[\frac{1 - F_\theta(x_l)}{1 - F_\theta(x_k)}\Big]^{m-l+1+j} \quad (6) \\
=& \frac{(m-k)!}{(l-k-1)!(m-l)!} \sum_{j=0}^{m-l} \begin{pmatrix} m-l \\ j \end{pmatrix} \frac{(-1)^j}{l-k+j} \Big[\frac{F_\theta(x_l) - F_\theta(x_k)}{1 - F_\theta(x_k)}\Big]^{l-k+j}.
\end{aligned}$$

Corollary 1.2. Let $X_1 \leq \ldots \leq X_k$ be the first k order statistics in a sample of size m from the two-parameter Weibull distribution with the probability density function

$$f_\theta(x) = \frac{\delta}{\beta}(\frac{x}{\beta})^{\delta-1}\exp[-(\frac{x}{\beta})^\delta]\ (x > 0), \tag{7}$$

where $\theta = (\beta, \sigma)$,$\beta > 0$ and $\sigma > 0$ are the scale and shape parameters, respectively. Then the conditional probability distribution function of X_l given $X_k = x_k$ is

$$\begin{aligned} P_\theta\{X_l \leq x_l | X_k = x_k\} = 1 - \frac{(m-k)!}{(l-k-1)!(m-l)!}\sum_{j=0}^{l-k-1}\left(\begin{array}{c} l-k-1 \\ j \end{array}\right) \\ \frac{(-1)^j}{m-l+1+j}\Big[\exp\Big(-\frac{x_l^\delta - x_k^\delta}{\beta^\delta}\Big)\Big]^{m-l+1+j}. \end{aligned} \tag{8}$$

Theorem 2 *If in (8) the scale parameter is unknown, then the predictive probability distribution function of X_l based on (x_k, δ) is given by*

$$\begin{aligned} P_\delta\Big\{\Big(\frac{X_l}{X_k}\Big)^\delta \leq \Big(\frac{x_l}{x_k}\Big)^\delta\} = 1 - \frac{m!}{(l-k-1)!(m-l)!} \times \left(\begin{array}{c} l-k-1 \\ j \end{array}\right) \\ \frac{(-1)^j}{m-l+1+j}\Big(\Pi_{s=0}^{k-1}\Big[\Big(\frac{x_l}{x_k}\Big)^\delta - 1)(m-l+1+j) + (m-k+1+s)\Big]\Big)^{-1}. \end{aligned} \tag{9}$$

*Proof.*We reduce (8) to

$$\begin{aligned} & P_\theta\Big\{\Big(\frac{X_l}{X_k}\Big)^\delta \leq \Big(\frac{X_l}{X_k}\Big)^\delta | \Big(\frac{X_k}{\beta}\Big)^\delta = \Big(\frac{X_k}{\beta}\Big)^\delta\Big\} \\ =& 1 - \frac{(m-k)!}{(l-k-1)!(m-l)!}\sum_{j=0}^{l-k-1}\left(\begin{array}{c} l-k-1 \\ j \end{array}\right)\frac{(-1)^j}{m-l+1+j} \\ & \Big[\exp(-\omega[\nu^\delta - 1])\Big]^{m-l+1+j} \\ =& P_\delta\{V^\delta \leq \nu^\delta | W = \omega\}, \end{aligned} \tag{10}$$

where $V = X_l/X_k$ is the ancillary statistic whose distribution does not depend on the parameter β. Since X_k does not depend on V, $W = (X_k/\beta)^\delta$ is the pivotal quantity, whose distribution is known and does not depend on the parameters β and δ, we eliminate the parameter from the problem as

$$P_\delta\{X_l \leq x_l\} = \int_0^\infty P_\theta\{X_l \leq x_l | X_k = x_k\} g_\theta(x_k) dx_k, \tag{11}$$

where

$$g_\theta(x_k) = \frac{m!}{(k-1)!(m-k)!}F_\theta^{k-1}(x_k)\Big[1 - F_\theta(x_k)\Big]^{m-k} f_\theta(x_k), x_k \in (0, \infty), \tag{12}$$

represents the probability density function of the kth order statistic X_k k. Indeed, it follows from (12) that

$$\begin{aligned} g_\theta(x_k)dx_k =&\frac{m!}{(k-1)!(m-k)!}\Big[1-\exp\Big(-\big(\frac{x_k}{\beta}\big)^{\delta}\Big)\Big]^{k-1}\exp\Big(-\big(\frac{x_k}{\beta}\big)^{\delta(m-k)}\Big) \\ &\exp\Big(-\big(\frac{x}{\beta}\big)^{\delta}\Big)d\big(\frac{x}{\beta}\big)^{\delta} \\ =&\frac{m!}{(k-1)!(m-k)!}[1-e^{-\omega}]^{k-1}e^{-\omega(m-k+1)}d\omega = g(\omega)d\omega. \end{aligned} \tag{13}$$

It follows from (10) and (13) that

$$\begin{aligned} &P_\delta\{V^\delta \le \nu^\delta\} = \int_0^\infty P_\delta\{V^\delta \le \nu^\delta | W=\omega\}g(\omega)d\omega \\ =&1-\frac{(m)!}{(l-k-1)!(m-l)!}\sum_{j=0}^{l-k-1}\left(\begin{array}{c} l-k-1 \\ j \end{array}\right)\frac{(-1)^j}{m-l+1+j}\Big(\Pi_{s=0}^{k-1}\Big[(\nu^\delta \\ &-1)(m-l+1+j)+(m-k+1+s)\Big]\Big)^{-1}. \end{aligned} \tag{14}$$

Now (9) follows from (14). This ends the proof.

Corollary 2.1. If the parameter $\delta = 1$, i.e. we deal with the exponential distribution, then the predictive probability distribution function of Xl based on x_k is given by

$$\begin{aligned} &P\Big\{\Big(\frac{X_l}{X_k}\Big) \le \Big(\frac{x_l}{x_k}\Big)\Big\} = 1-\frac{m!}{(l-k-1)!(m-l)!}\times\sum_{j=0}^{l-k-1}\left(\begin{array}{c} l-k-1 \\ j \end{array}\right) \\ &\frac{(-1)^j}{m-l+1+j}\Big(\Pi_{s=0}^{k-1}\Big[\Big(\frac{x_l}{x_k}-1\Big)(m-l+1+j)+(m-k+1+s)\Big]\Big)^{-1}. \end{aligned} \tag{15}$$

Theorem 3 *Let $X_1 \le \ldots, \le X_k$ be the first k ordered observations from a sample of size m from the two-parameter Weibull distribution (7). Then the joint probability density function of the pivotal quantities*

$$W_2 = \frac{\delta}{\widehat{\delta}}, \quad W_3 = \Big(\frac{\widehat{\beta}}{\beta}\Big)^{\hat{\delta}}, \tag{16}$$

conditional on fixed $z^k = (z_i, \ldots, z_k)$,where $Z_i = (X_i/\widehat{\beta})^{\hat{\delta}},\ \ i = 1, \ldots, k$,are ancillary statistics, any $k-2$ of which form a functionally independent set,$\widehat{\beta}$ and $\widehat{\delta}$ are the estimators of β and δ,based on the first k ordered observations $(X_1 \le \ldots \le X_k)$ from a sample of size m from the two-parameter Weibull distribution (7), such that W_2 and W_3 are the pivotal quantities (in particular, the maximum likelihood estimators of β and δ,

$$\widehat{\beta} = \Big(\Big[\sum_{i=1}^{k} x_i^{\hat{\delta}} + (m-k)x_k^{\hat{\delta}}\Big]/k\Big)^{1/\hat{\delta}} \tag{17}$$

and

$$\widehat{\delta} = \Big[\Big(\sum_{i=1}^{k} x_i^{\hat{\delta}} \ln x_i + (m-k)x_k^{\hat{\delta}} \ln x_k\Big)\Big(\sum_{i=1}^{k} x_i^{\hat{\delta}} + (m-k)x_k^{\hat{\delta}}\Big)^{-1} - \frac{1}{k}\sum_{i=1}^{k} \ln x_i\Big]^{-1} \tag{18}$$

respectively, lead to the pivotal quantities W_2 and W_3)is given by

$$\begin{aligned} & f(\omega_2, \omega_3 | z^{(k)}) \\ =& \vartheta^{\bullet}(z^{(k)})\omega_2^{k-1} \prod_{i=1}^{k} z_i^{\omega_2} \omega_3^{k\omega_2 - 1} \exp\Big(-\omega_3^{\omega_2}\Big[\sum_{i=1}^{k} z_i^{\omega_2} + (m-k)z_k^{\omega_2}\Big]\Big) \\ =& \vartheta^{\bullet}(z^{(k)})\omega_2^{k-2} \prod_{i=1}^{k} z_i^{\omega_2} \omega_3^{\omega_2(k-1)} \exp\Big(-\omega_3^{\omega_2}\Big[\sum_{i=1}^{k} z_i^{\omega_2} + (m-k)z_k^{\omega_2}\Big]\Big)\omega_2\omega_3^{\omega_2-1} \\ =& f(\omega_2|z^{(k)}) f(\omega_3|\omega_2, z^{(k)}), \quad \omega_3 \in (0,\infty), \end{aligned} \tag{19}$$

where

$$\vartheta^{\bullet}(z^{(k)}) = \Big[\int_0^{\infty} \Gamma(k)\omega_2^{k-2} \prod_{i=1}^{k} z_i^{\omega_2}\Big(\sum_{i=1}^{k} z_i^{\omega_2} + (m-k)z_k^{\omega_2}\Big)^{-k} d\omega_2\Big]^{-1} \tag{20}$$

is the normalizing constant,

$$f(\omega_2|z^{(k)}) = \vartheta(z^{(k)})\omega_2^{k-2} \prod_{i=1}^{k} z_i^{\omega_2}\Big(\sum_{i=1}^{k} z_i^{\omega_2} + (m-k)z_k^{\omega_2}\Big)^{-k}, \omega_2 \in (0,\infty), \tag{21}$$

$$\vartheta(z^{(k)}) = \Big[\int_0^{\infty} \omega_2^{k-2} \prod_{i=1}^{k} z_i^{\omega_2}\Big(\sum_{i=1}^{k} z_i^{\omega_2} + (m-k)z_k^{\omega_2}\Big)^{-k} d\omega_2\Big]^{-1}, \tag{22}$$

$$\begin{aligned} f(\omega_3, \omega_2|z^{(k)}) &= \frac{\Big[\sum_{i=1}^{k} z_i^{\omega_2} + (m-k)z_k^{\omega_2}\Big]^k}{\Gamma(k)}\omega_3^{w_2(k-1)} \\ &\times \exp\Big(-\omega_3^{w_2}\Big[(\sum_{i=1}^{k} z_i^{\omega_2} + (m-k)z_k^{\omega_2}\Big]\Big)\omega_2\omega_3^{\omega_2-1}, \omega_3 \in (0,\infty). \end{aligned} \tag{23}$$

*Proof.*The joint density $X_1 \le \ldots \le X_k$ is given by

$$f_\theta(x_1,\ldots,x_k) = \frac{m!}{(m-k)!}\prod_{i=1}^{k} \frac{\delta}{\beta}(\frac{x_i}{\beta})^{\delta-1} \exp(-(\frac{x_i}{\beta})^{\delta}) \exp(-(m-k)((\frac{x_k}{\beta})^{\delta}). \tag{24}$$

Using $\widehat{\beta}$ and $\widehat{\delta}$ (the maximum likelihood estimators of β and δ obtained from solution of (17) and (18)) and the invariant embedding technique [8-14], we transform (24) as follows:

$$
\begin{aligned}
& f_\theta(x_1,\dots,x_k)d\widehat{\beta}d\widehat{\delta}\\
=&\frac{m!}{(m-k)!}\prod_{i=1}^{k} x_i^{-1}\delta^k\prod_{i=1}^{k}\big(\frac{x_i}{\beta}\big)^{\delta}\exp\Big(-\sum_{i=1}^{k}\big(\frac{x_i}{\beta}\big)^{\delta}-(m-k)\big(\frac{x_k}{\beta}\big)^{\delta}\Big)d\widehat{\beta}d\widehat{\delta}\\
=&-\frac{m!}{(m-k)!}\widehat{\beta}\widehat{\delta}^k\prod_{i=1}^{k} x_i^{-1}\big(\frac{\delta}{\widehat{\delta}}\big)^{k-2}\prod_{i=1}^{k}\big(\frac{x_i}{\widehat{\beta}}\big)^{\hat{\delta}(\frac{\delta}{\hat{\delta}})}\big(\frac{\widehat{\beta}}{\beta}\big)^{\hat{\delta}(\frac{\delta}{\hat{\delta}})(k-1)}\times\exp\Big(-\big(\frac{\widehat{\beta}}{\beta}\big)^{\hat{\delta}(\frac{\delta}{\hat{\delta}})}\\
&\big[\sum_{i=1}^{k}\big(\frac{x_i}{\widehat{\beta}}\big)^{\hat{\delta}(\frac{\delta}{\hat{\delta}})}+(m-k)\big(\frac{x_k}{\widehat{\beta}}\big)^{\hat{\delta}(\frac{\delta}{\hat{\delta}})}\big]\Big)\Big(\frac{\widehat{\delta}(\frac{\delta}{\widehat{\delta}})}{\beta}\big(\frac{\widehat{\beta}}{\beta}\big)^{\hat{\delta}(\frac{\delta}{\hat{\delta}})-1}d\widehat{\beta}\Big)\Big(-\frac{\delta}{\widehat{\delta}^2}d\widehat{\delta}\Big)\\
=&-\frac{m!}{(m-k)!}\widehat{\beta}\widehat{\delta}^k\prod_{i=1}^{k} x_i^{-1}\omega_2^{k-2}\prod_{i=1}^{k} z_i^{\omega_2}\omega_3^{\omega_2(r-1)}\exp\Big(-\omega_3^{\omega_2}\big[\sum_{i=1}^{k} z_i^{\omega_2}+\\
&(m-k)z_k^{\omega_2}\big]\Big)d(\omega_3^{\omega_2})d\omega_2\\
=&-\frac{m!}{(m-k)!}\widehat{\beta}\widehat{\delta}^k\prod_{i=1}^{k} x_i^{-1}\omega_2^{k-2}\prod_{i=1}^{k} z_i^{\omega_2}\omega_3^{\omega_2(k-1)}\exp\Big(-\omega_3^{\omega_2}\big[\sum_{i=1}^{k} z_i^{\omega_2}\\
&+(m-k)z_k^{\omega_2}\big]\Big)\omega_2\omega_3^{\omega_2-1}d\omega_2 d\omega_3.
\end{aligned}
\tag{25}
$$

Normalizing (25), we obtain (19). This ends the proof.

It will be noted that more general case of distributions indexed by location and scale parameters has been considered in [15].

Theorem 4 *If in (8) both parameters β and δ are unknown, then the predictive probability distribution function of X_l based on $(x_k,\widehat{\delta})$ and conditional on fixed $z^{(k)}$ is given by*

$$
\begin{aligned}
& P\Big\{\big(\frac{X_l}{X_k}\big)^{\hat{\delta}}\le\big(\frac{x_l}{x_k}\big)^{\hat{\delta}}|z^{(k)}\Big\}\\
=&1-\frac{m!}{(l-k-1)!(m-l)!}\times\int_0^\infty\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}\frac{(-1)^j}{m-l+1+j}\\
&\Big(\prod_{s=0}^{k-1}\Big[\Big(\big(\big[\frac{x_l}{x_k}\big]^{\hat{\delta}}\big)^{\omega_2}-1\Big)(m-l+1+j)+(m-k+1+s)\Big]\Big)^{-1}f\big(\omega_2|z^{(k)}\big)d\omega_2.
\end{aligned}
\tag{26}
$$

Proof. We reduce (9) to

$$
\begin{aligned}
&P_\delta\Big\{\Big(\frac{X_l}{X_k}\Big)^{\hat{\delta}(\frac{\delta}{\hat{\delta}})} \le \Big(\frac{x_l}{x_k}\Big)^{\hat{\delta}(\frac{\delta}{\hat{\delta}})}\Big\} = 1 - \frac{m!}{(l-k-1)!(m-l)!} \times \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \\
&\frac{(-1)^j}{m-l+1+j}\Big(\prod_{s=0}^{k-1}\Big[\Big(\Big(\frac{x_l}{x_k}\Big)^{\hat{\delta}(\frac{\delta}{\hat{\delta}})}-1\Big)(m-l+1+j)+(m-k+1+s)\Big]\Big)^{-1} \\
=&1 - \frac{m!}{(l-k-1)!(m-l)!} \times \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} \qquad (27)\\
&\Big(\prod_{s=0}^{k-1}\Big[(\nu_2^{\omega_2}-1)(m-l+1+j)+(m-k+1+s)\Big]\Big)^{-1} \\
=&P\Big\{V_2^{W_2} \le \nu_2^{\omega_2}\Big\},
\end{aligned}
$$

where $V_2 = (X_l/X_k)^{\hat{\delta}}$ is the ancillary statistic whose distribution does not depend on the parameters β and δ. Since the pivotal quantity W_2, whose distribution is given by (21), does not depend on V_2, it follows from (21) and (27) that

$$
P\Big\{V_2 \le \nu_2|z^{(k)}\Big\} = \int_0^\infty P\Big\{V_2^{W_2} \le \nu_2^{\omega_2}\Big\} f\Big(\omega_2|z^{(k)}\Big) d\omega_2, \qquad (28)
$$

where the unknown parameters β and δ are eliminated from the problem. Now (26) follows from (28). This ends the proof.

3 Prediction Limits for a Future Number of Failures

Consider the situation in which m units start service at time 0 and are observed until a time t_c when the available Weibull failure data are to be analyzed. Failure times are recorded for the k units that fail in the interval $[0, t_c]$. Then the data consist of the k smallest-order statistics $X_1 \le \ldots \le X_k \le t_c$ and the information that the other $m?k$ units will have failed after t_c.With time (or Type I) censored data, t_c is prescribed and k is random. With failure (or Type II) censored data, k is prescribed and $t_c = X_k$ is random.

The problem of interest is to use the information obtained up to t_c to construct the Weibull within-sample prediction limits (lower and upper) for the number of units that will fail in the time interval $[t_c, t_\omega]$.For example, this t_ω could be the end of a warranty period.

Consider the situation when $t_c = X_k$. Under conditions of Theorem 4, the lower prediction limit for the number of units that will fail in the time interval $[t_c, t_\omega]$ is given by

$$
L_{lower} = l_{max} - k, \qquad (29)
$$

where

$$l_{max} = \max_{k<l\leq m} \arg P\Big(\big\{X_l > t_\omega | z^k\big\} \leq \alpha\Big) \tag{30}$$

The upper prediction limit for the number of units that will fail in the time interval $[t_c, t_\omega]$ is given by

$$L_{upper} = l_{min} - k - 1, \tag{31}$$

where

$$l_{min} = \min_{k<l\leq m} \arg P\Big(\big\{X_l > t_\omega | z^k\big\} \geq 1-\alpha\Big) \tag{32}$$

In the above case, where both parameters β and δ are unknown, the prediction limits (lower and upper) for the number of units that will fail in the time interval $[t_c, t_\omega]$ are based on $(x_k, \widehat{\delta})$ and conditional on fixed $z^{(k)}$. If l, which satisfies (30), does not exist then $l_{max} = k$ and the lower prediction limit for the number of units that will fail in the time interval $[t_c, t_\omega]$ is given by

$$L_{lower} = l_{max} - k = 0. \tag{33}$$

If l, which satisfies (32), does not exist then $l_{min} = m+1$ and upper prediction limit for the number of units that will fail in the time interval $[t_c, t_\omega]$ is given by

$$L_{upper} = l_{min} - k - 1 = m - k, \tag{34}$$

4 Second Version of Prediction Limits for a Future Number of Failures

In this section, we wish to show how to obtain the second version of prediction limits for a future number of failures. The methodology is based on the following results.

Theorem 5 *Let $X_1 \leq \ldots \leq X_k$ be the first k ordered observations from a sample of size m from the two-parameter Weibull distribution (7). Then the joint probability density function of the pivotal quantities*

$$W_1 = \Big(\frac{\widehat{\beta}}{\beta}\Big)^{\delta}, \quad W_3 = \frac{\delta}{\widehat{\delta}}, \tag{35}$$

conditional on fixed $z^{(k)} = (z_i, \ldots, z_k)$, where $Z_i = (X_i/\widehat{\beta})^{\hat{\delta}}, i = 1, \ldots, k$ are ancillary statistics, any $k-2$ of which form a functionally independent set, $\widehat{\beta}$ and $\widehat{\delta}$ are, for instance, the maximum likelihood estimators for β and δ based on the first k ordered observations $(X_1 \leq \ldots \leq X_k)$ from a sample of size m from the *two-parameter Weibull distribution (7), which can be found from solution of (17) and (18), is given by*

$$f(\omega_1,\omega_2|z^{(k)}) = \vartheta^\bullet(z^{(k)})\omega_2^{k-2}\prod_{i=1}^{k} z_i^{\omega_2}\omega_1^{k-1}\exp(-\omega_1[\sum_{i=1}^{k} z_i^{\omega_2}+(m-k)z_k^{\omega_2}])$$
$$= f(\omega_2|z^{(k)})f(\omega_1|\omega_2,z^{(k)}), \quad \omega_1\in(0,\infty),\omega_2\in(0,\infty), \tag{36}$$

where

$$\vartheta^\bullet(z^{(k)}) = \Big[\int_0^\infty \Gamma(k)\omega_2^{k-2}\prod_{i=1}^{k} z_i^{\omega_2}\Big(\sum_{i=1}^{k} z_i^{\omega_2}+(m-k)z_k^{\omega_2}\Big)^{-k}d\omega_2\Big]^{-1} \tag{37}$$

is the normalizing constant, $f(\omega_2|z^{(k)})$ is given by (21),

$$f(\omega_1,\omega_2|z^{(k)}) = \frac{\big[\sum_{i=1}^{k} z_i^{\omega_2}+(m-k)z_k^{\omega_2}\big]^k}{\Gamma(k)}\omega_1^{k-1}\exp\Big(-\omega_1\big[\sum_{i=1}^{k} z_i^{\omega_2}$$
$$+(m-k)z_k^{\omega_2})\big]\Big)\omega_1\in(0,\infty), \tag{38}$$

Proof. The joint density of $X_1\le\ldots\le X_k$ is given by

$$f_\theta(x_1,\ldots,x_k) = \frac{m!}{(m-k)!}\prod_{i=1}^{k}\frac{\delta}{\beta}(\frac{x_i}{\beta})^{\delta-1}\exp(-(\frac{x_i}{\beta})^\delta)\exp(-(m-k)(\frac{x_k}{\beta})^\delta). \tag{39}$$

Using the invariant embedding technique [8-14], we transform (39) to

$$\begin{aligned}
&f_\theta(x_1,\ldots,x_k)d\widehat{\beta}d\widehat{\delta}\\
=&\frac{m!}{(m-k)!}\prod_{i=1}^{k}x_i^{-1}\delta^k\prod_{i=1}^{k}\Big(\frac{x_i}{\beta}\Big)^\delta\exp\Big(-\sum_{i=1}^{k}\Big(\frac{x_i}{\beta}\Big)^\delta-(m-k)\Big(\frac{x_k}{\beta}\Big)^\delta\Big)d\widehat{\beta}d\widehat{\delta}\\
=&-\frac{m!}{(m-k)!}\widehat{\beta}\widehat{\delta}^k\prod_{i=1}^{k}x_i^{-1}\Big(\frac{\delta}{\widehat{\delta}}\Big)^{k-2}\prod_{i=1}^{k}\Big(\frac{x_i}{\widehat{\delta}}\Big)^{\hat{\delta}(\frac{\delta}{\hat{\delta}})}\Big(\frac{\widehat{\beta}}{\beta}\Big)^{\delta(k-1)}\times\exp\Big(-\Big(\frac{\widehat{\beta}}{\beta}\Big)^\delta\\
&\Big[\sum_{i=1}^{k}(\frac{x_i}{\widehat{\delta}})^{\hat{\delta}(\frac{\delta}{\hat{\delta}})}+(m-k)\Big(\frac{x_k}{\widehat{\delta}}\Big)^{\hat{\delta}(\frac{\delta}{\hat{\delta}})}\Big]\Big)\Big(\frac{\delta}{\beta}\Big(\frac{\widehat{\beta}}{\beta}\Big)^{\delta(k-1)}d\widehat{\beta}\Big)(-\frac{\delta}{\widehat{\delta}^2})d\widehat{\delta}^2\\
=&-\frac{m!}{(m-k)!}\widehat{\beta}\widehat{\delta}^k\prod_{i=1}^{k}x_i^{-1}\omega_2^{k-2}\prod_{i=1}^{k}x_i^{-1}z_i^{\omega_2}\omega_1^{k-1}\exp\Big(-\omega_1\Big[\sum_{i=1}^{k}z_i^{\omega_2}+\\
&(m-k)z_k^{\omega_2}\Big]\Big)d\omega_1\omega_2.
\end{aligned} \tag{40}$$

Normalizing (40), we obtain (36). This ends the proof.

Corollary 5.1. If the parameter δ is known then

$$W_1\sim f(\omega_1)=\frac{k^k}{\Gamma(k)}\omega_1^{k-1}\exp(-\omega_1 k),\ \omega_1\in(0,\infty). \tag{41}$$

Theorem 6 *If in (8) the scale parameter β is unknown, then the predictive probability distribution function of X_l based on $(\widehat{\beta}, \delta)$ and conditional on fixed x_k is given by*

$$P_\delta\{X_l \le x_l | X_k = x_k\} = 1 - \frac{(m-k)!}{(l-k-1)!(m-l)!} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} \Big[1 - (m-l+1+j) \frac{x_l^\delta - x_k^\delta}{k\widehat{\beta}^\delta}\Big]^{-k} \tag{42}$$

Proof. We reduce (8) to

$$\begin{aligned} & P_\theta\{X_l \le x_l | X_k = x_k\} \\ = & 1 - \frac{(m-k)!}{(l-k-1)!(m-l)!} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} \Big[\exp\Big(-\big[\frac{\widehat{\beta}}{\beta}\big]^\delta \frac{x_l^\delta - x_k^\delta}{\widehat{\beta}^\delta}\Big)\Big]^{m-l+1+j} \\ = & 1 - \frac{(m-k)!}{(l-k-1)!(m-l)!} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} \Big[\exp\Big(-\omega_1 \frac{x_l^\delta - x_k^\delta}{\widehat{\beta}^\delta}\Big)\Big]^{m-l+1+j}. \end{aligned} \tag{43}$$

Now, we eliminate the unknown parameter β from the problem and find (42) as

$$P_\delta\{X_l \le x_l | X_k = x_k\} = \int_0^\infty P_\theta\{X_l \le x_l | X_k = x_k\} f(\omega_1) d\omega_1. \tag{44}$$

This ends the proof.
Corollary 6.1. If the parameter $\delta = 1$, i.e. we deal with the exponential distribution, then the predictive probability distribution function of X_l based on $\widehat{\beta}$ and conditional on fixed xk is given by x_k

$$P\{X_l \le x_l | X_k = x_k\} = 1 - \frac{1}{B(l-k, m-l+1)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} \Big[1 + (m-l+1+j) \frac{x_l - x_k}{k\widehat{\beta}}\Big]^{-k}, \tag{45}$$

where

$$k\widehat{\beta} = \sum_{i=1}^k x_i + (m-k)x_k. \tag{46}$$

Theorem 7 *If in (8) both parameters β and δ are unknown, then the predictive probability distribution function of Xl based on $(wideparen\beta, wideparen\delta)$ and* *conditional on fixed x_k and $z^{(k)}$ is given by*

$$P_\theta\{X_l \le x_l | X_k = x_k; z^{(k)}\} = 1 - \frac{(m-k)!}{(l-k-1)!(m-l)!} \times \int_0^\infty \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} \Big[1 + (m-l+1+j)\Big(\big(\frac{x_l}{\widehat{\beta}}\big)^{\hat{\delta}\omega_2} - \big(\frac{x_k}{\widehat{\beta}}\big)^{\hat{\delta}\omega_2}\Big)\Big(\sum_{i=1}^{k} z_i^{\omega_2} + (m-k) z_i^{\omega_2}\Big)^{-1}\Big]^{-k} \times f(\omega_2 | z^{(k)}) d\omega_2. \tag{47}$$

Proof. We reduce (8) to

$$\begin{aligned} &P_\theta\{X_l \le x_l | X_k \\ &= x_k\} = 1 - \frac{(m-k)!}{(l-k-1)!(m-l)!} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \\ &\quad \frac{(-1)^j}{m-l+1+j} \Big[\exp\Big(-\big[\frac{\widehat{\beta}}{\beta}\big]^\delta \Big[\big(\frac{x_l}{\widehat{\beta}}\big)^{\hat{\delta}(\frac{\delta}{\hat{\delta}})} - \big(\frac{x_k}{\widehat{\beta}}\big)^{\hat{\delta}(\frac{\delta}{\hat{\delta}})}\Big]\Big)\Big]^{m-l+1+j} \\ &= 1 - \frac{(m-k)!}{(l-k-1)!(m-l)!} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} \\ &\quad \exp\Big[\Big(-\omega_1\Big[\big(\frac{x_l}{\widehat{\beta}}\big)^{\hat{\delta}\omega_2} - \big(\frac{x_k}{\widehat{\beta}}\big)^{\hat{\delta}\omega_2}\Big]\Big)\Big]^{m-l+1+j}. \end{aligned} \tag{48}$$

Now, we eliminate the unknown parameters β and δ from the problem and find (47) as

$$\begin{aligned} &P_\theta\{X_l \le x_l | X_k = x_k; z^{(k)}\} \\ &= \int_0^\infty \int_0^\infty P_\theta\{X_l \le x_l | X_k = x_k\} f(\omega_1, \omega_2 | z^{(k)}) d\omega_1 d\omega_2 \\ &= \int_0^\infty \int_0^\infty P_\theta\{X_l \le x_l | X_k = x_k\} f(\omega_1 | \omega_2, z^{(k)}) f(\omega_2 | z^{(k)}) d\omega_1 d\omega_2. \end{aligned} \tag{49}$$

This ends the proof.

Under conditions of Theorem 7, the lower prediction limit for the number of units that will fail in the time interval $[t_c, t_\omega]$ is given by

$$L_{lower} = l_{max} - k, \tag{50}$$

where

$$l_{max} = \max_{k<l\le m} \arg\Big(P\{X_l > t_\omega | X_k = x_k; z^{(k)})\} \le \alpha\Big), \tag{51}$$

The upper prediction limit for the number of units that will fail in the time interval $[t_c, t_\omega]$ is given by

$$L_{upper} = l_{min} - k - 1, \tag{52}$$

$$l_{min} = \min_{k<l\leq m} \arg\Big(P\{X_l > t_\omega | X_k = x_k; z^{(k)})\} \geq 1-\alpha\Big), \tag{53}$$

In the above case, when both parameters β and δ are unknown, the prediction limits (lower and upper) for the number of units that will fail in the time interval $[t_c, t_\omega]$ are based on $(\widehat{\beta}, \widehat{\delta})$ and conditional on fixed x_k,$z^{(k)}$.

If l, which satisfies (51), does not exist then $l_{max} = k$ and the lower prediction limit for the number of units that will fail in the time interval $[t_c, t_\omega]$ is given by $L_{lower} = 0$. If l, which satisfies (53), does not exist then $l_{min} = m + 1$ and upper prediction limit for the number of units that will fail in the time interval $[t_c, t_\omega]$ is given by $L_{upper} = m - k$.

5 Numerical Example

For the sake of simplicity, but without loss of generality, we consider (for illustration) the special case of Theorem 2 where $m = 40$ items simultaneously tested have life times, which follow the Weibull distribution with $\delta = 1$. In other words, we deal with the exponential distribution. Two items have failed by the inspection at times, $X_1 = 45$ and $X_2 = 100$ hours. Let us assume that the situation takes place when $t_c = X_k = 100$ hours, where $k = 2$. Suppose, say, $t_\omega = 450$ hours. Taking into account (15), we find the lower prediction limit for the number of units that will fail in the time interval $[t_c, t_\omega]$ as

$$L_{lower} = l_{max} - k = 3 - 2 = 1, \tag{54}$$

$$l_{max} = \max_{k<l\leq m} \arg\Big(P\{X_l > t_\omega\} \leq \alpha\Big) = 3,\ \alpha = 0.05 \tag{55}$$

$$\begin{aligned} P\{X_l > t_\omega\} &= \frac{m!}{(l-k-1)!(m-l)!} \sum_{j=0}^{l-k-1} \left(\begin{array}{c} l-k-1 \\ j \end{array} \right) \frac{(-1)^j}{m-l+1+j} \\ &\left(\prod_{s=0}^{k-1} \Big[\Big(\frac{t_\omega}{x_k} - 1\Big)(m-l+1+j) + (m-k+1+s)\Big]\right)^{-1}, \end{aligned} \tag{56}$$

The upper prediction limit for the number of units that will fail in the time interval $[t_c, t_\omega]$ is given by

$$L_{upper} = l_{min} - k - 1 = 17 - 2 - 1 = 14, \tag{57}$$

$$l_{min} = \min_{k<l\leq m} \arg\Big(P\{X_l > t_\omega | X_k = x_k; z^{(k)})\} \geq 1-\alpha\Big) = 17. \tag{58}$$

It will be noted that when both parameters β and δ are unknown, the lower and upper prediction limits for the number of units that will fail in the time interval $[t_c, t_\omega]$ can be found either from (29) and (31), which are based on $(x_k, \widehat{\delta})$,or from (50) and (52), which are based on $(\widehat{\beta}, \widehat{\delta})$.

Conclusion and Future Work

The methodology described here can be extended in several different directions to handle various problems that arise in practice.

We have illustrated the prediction method for log-location-scale distributions (such as the Weibull or exponential distributions). Application to other distributions could follow directly.

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