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Equilibrium in Market Models with Known Elasticities

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Abstract: Closed and open market models, in which the supply and demand functions are restored by their price elasticities, are studied. For the closed market model criteria on the existence of equilibrium is obtained as the corollary of existence theorems for the solutions to systems of linear equations and inequalities. The results of the covering maps theory, namely existence theorems of a coincidence point, are applied to obtain sufficient conditions on the existence of an equilibrium in open market model and to develop search algorithm of an equilibrium in this model. Numerical experiments illustrating the obtained results are conducted.

Keywords: demand, supply, elasticity, market model, economical equilibrium

1. INTRODUCTION

The concept of equilibrium plays an important role in the research of market models. Basically it is a situation on the market in which none of the participants is interested in the change of their state.

Let us explain the meaning of equilibrium. Assume that the market consists of the producers and the consumers. The producers tend to maximize their own income by manufacturing some goods. The consumers spend their budget to purchase these goods satisfying their needs. The total volume of goods produced is called the supply. The total volume of goods needed is called the demand.

It is obvious that insufficient amount of goods on the market can lead to unfavorable consequences such as hunger, epidemic or even death depending on the market described. Therefore, there must be enough goods on the market to satisfy the needs of the consumers. On the other hand, if some goods produced is not sold to the consumers, the producers suffer an obvious income loss. Hence, they must produce enough goods to satisfy the needs of the consumers, but not more than this required amount. From these considerations we obtain that the amount of goods produced must be equal to the amount of goods needed. In other words, the value of the supply must be equal to the value of the demand. This situation on the market is called an equilibrium.

The concept of equilibrium allows us to determine the best prices for the producers and the consumers to set in the modeled region. Indeed, with these prices (which are called equilibrium prices) producers obtain maximal profit since they sell all the goods produced, and consumers obtain the goods needed in the full amount. However, the mathematical approach to this concept remained poor due to the state of the progress of mathematics at the time.

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The advancement in the theory of covering mappings and coincidence points (see, e.g., [1], [2]) makes the deep research into the equilibrium theory possible. This mathematical field allows us to make a research on the equilibrium even in the nonlinear market models. For example, in [5]– [7] theorems on the existence of coincidence points for mappings of metric spaces are applied to obtain sufficient conditions on the existence of an equilibrium in the market models. In [3], [4], [7] the mappings of supply and demand are determined as solutions of extremal problems.

The supply function can also be obtained as the realization of some production model (see, e.g., [5], [13]). Here we restore the supply and demand function using the concept of elasticitiy. Elasticities are the values that connect supply and demand volumes with parameters obtained from the statistical data (e.g., prices, income, transport costs etc.). That allows us to determine the supply and demand functions that describe real market of some region. Using equilibrium theory, we can improve the economical situation in the region.

In this paper, we investigate the existence of equilibrium in open and closed market models. Here, we used price elasticity coefficients of goods presented on the market to obtain the explicit forms of the supply and demand functions. For these functions, we obtained a criteria for the existence of equilibrium in the closed market model. Also for the open market model we obtained sufficient conditions using corollaries of coincidence points theorems. We also developed a search algorithm of an equilibrium for the open market model and conducted a numerical experiment. This algorithm is based on pattern search method and illustrate the effectiveness of the obtained sufficient conditions. This method is chosen due to its simplicity and visibility. The obtained results show that equilibrium is found in the case when the model satisfies the sufficient conditions with moderate precision.

2. MARKET MODELS

This section provides sample equations, figures, and tables.

2.1. Closed Market Model

Consider a market of n goods with uniform prices set:

$$p = (p_1, \dots, p_n), p_i \in [c_{1i}, c_{2i}],$$

where vectors c_1, c_2 set the natural restrictions on these prices.

Suppose that we have $n \times n$ matrices $\mathcal{E} = (E_{ij})_{i,j=\overline{1,n}}, \tilde{\mathcal{E}} = (\tilde{E}_{ij})_{i,j=\overline{1,n}}$ ($E_{ij}, \tilde{E}_{ij} \in \mathbb{R}$ for all $i, j = \overline{1, n}$, vectors $\bar{c}_1 = (c_{11}, ..., c_{1n})$, $\bar{c}_2 = (c_{21}, ..., c_{2n}) \in \mathbb{R}^n_+$ with $0 < c_{1i} < c_{2i}$ for all $i = \overline{1, n}$, a vector $\bar{p}^* = (p^*_1, ..., p^*_n) \in \mathbb{R}^n_+$ such that $c_{1i} \le p^*_i \le c_{2i}$ for all $i = \overline{1, n}$, and vectors $\bar{D}^* = (D^*_1, ..., D^*_n), \bar{S}^* = (S^*_1, ..., S^*_n) \in \mathbb{R}^n_+$.

We call

$$\sigma_c = (\mathcal{E}, \tilde{\mathcal{E}}, \bar{D}^*, \bar{S}^*, \bar{p}^*, \bar{c}_1, \bar{c}_2) \tag{2.1}$$

a closed market model (the goods presented on the market are manufactured by the producers only). The set of all σ_c is denoted by Σ_c.

The parameters of the model have the following sense. Elements of $\mathcal{E}$ are the price elasticities of demand:

$$E_{ij} = \frac{\partial D_i}{\partial p_j} \frac{p_j}{D_i}, i, j = \overline{1, n}. \tag{2.2}$$

Here p_i is the price of the ith good, $D_i = D_i(p_1, ..., p_n)$ is a total demand on the ith good, E_{ij} is the elasticity of demand on the ith good for the price of the jth good, $i, j = \overline{1, n}$.

Elements of $\tilde{\mathcal{E}}$ are the price elasticities of supply:

$$\tilde{E}_{ij} = \frac{\partial S_i}{\partial p_j}\frac{p_j}{S_i}, i, j = \overline{1,n}. \tag{2.3}$$

Here $S_i = (S_i(p_1, ..., p_n)$ is a total supply of the ith good, $\tilde{E}_{ij}$ is the elasticity of supply on the ith good for the price of the jth good, $i, j = \overline{1,n}$.

Components of the vectors $\bar{c}_1$ and $\bar{c}_2$ generate natural conditions on the prices of the goods:

$$c_{1i} \leq p_i \leq c_{2i}, i = \overline{1,n}.$$

Component of the vector $\bar{D}^* = \bar{D}(p_1^*, ..., p_n^*)$ is the demand on the corresponding good and component of the vector $\bar{S}^* = \bar{S}(p_1^*, ..., p_n^*)$ is the supply of the corresponding good for the prices set $\bar{p}^* = (p_1^*, ..., p_n^*)$.

Parameters $(\mathcal{E}, \tilde{\mathcal{E}}, \bar{D}^*, \bar{S}^*, \bar{p}^*, \bar{c}_1, \bar{c}_2)$ uniquely define the function of demand

$$\begin{gathered} D : [c_{11}; c_{21}] \times ... \times [c_{1n}; c_{2n}] \to \mathbb{R}^n_+, \\ D(p_1, ..., p_n) = (D_1(p_1, ..., p_n), ..., D_n(p_1, ..., p_n)) \end{gathered} \tag{2.4}$$

and the function of supply

$$\begin{gathered} S : [c_{11}; c_{21}] \times ... \times [c_{1n}; c_{2n}] \to \mathbb{R}^n_+, \\ S(p_1, ..., p_n) = (S_1(p_1, ..., p_n), ..., S_n(p_1, ..., p_n)) \end{gathered} \tag{2.5}$$

Solving the system of partial differential equations (2.2) we obtain the explicit form of the demand function:

$$D_i(p_1, ..., p_n) = D_i^* \prod_{j=1}^{n} (p_j^*)^{-E_{ij}} p_j^{E_{ij}}. \tag{2.6}$$

Similarly, solving the system of partial differential equations (2.3) we get the explicit form of the supply function:

$$S_i(p_1, ..., p_n) = S_i^* \prod_{j=1}^{n} (p_j^*)^{-\tilde{E}_{ij}} p_j^{\tilde{E}_{ij}}. \tag{2.7}$$

Definition 2.1:
A vector $\bar{p} \in [c_{11}, c_{21}] \times ... \times [c_{1n}, c_{2n}]$ is called an equilibrium price vector (an equilibrium), in the model σ_c if $D(\bar{p}) = S(\bar{p})$.

2.2. Open Market Model

Suppose that we have $n \times n$ matrices $\mathcal{E} = (E_{ij})_{i,j=\overline{1,n}}, \tilde{\mathcal{E}} = (\tilde{E}_{ij})_{i,j=\overline{1,n}}$ $(E_{ij}, \tilde{E}_{ij} \in \mathbb{R}$ for all $i, j = \overline{1,n}$, vectors $\bar{c}_1 = (c_{11}, ..., c_{1n})$, $\bar{c}_2 = (c_{21}, ..., c_{2n}) \in \mathbb{R}^n_+$ with $0 < c_{1i} < c_{2i}$ for all $i = \overline{1,n}$, a vector $\bar{p}^* = (p_1^*, ..., p_n^*) \in \mathbb{R}^n_+$ such that $c_{1i} \leq p_i^* \leq c_{2i}$ for all $i = \overline{1,n}$, and vectors $\bar{D}^* = (D_1^*, ..., D_n^*), \bar{S}^* = (S_1^*, ..., S_n^*) \in \mathbb{R}^n_+$. Moreover, we suppose that a vector $\bar{a} = (a_1, ..., a_n) \in \mathbb{R}^n_+$, such that there exists at least one number $i = \overline{1,n} : a_i > 0$, is given.

We call

$$\sigma_o = (\mathcal{E}, \tilde{\mathcal{E}}, \bar{D}^*, \bar{S}^*, \bar{p}^*, \bar{c}_1, \bar{c}_2, \bar{a}) \tag{2.8}$$

an open market model (at least one kind of the goods is imported externally). The set of all σ_o is denoted by Σ_o.

The parameters $\mathcal{E}, \tilde{\mathcal{E}}, \bar{D}^*, \bar{S}^*, \bar{p}^*, \bar{c}_1, \bar{c}_2$ of the model (2.8) have the same sense as the ones for the model (2.1). The component of vector $\bar{a}$ with the number $i = \overline{1,n}$ equals to the volume of imported ith good.

The parameters $(\mathcal{E}, \tilde{\mathcal{E}}, \bar{D}^*, \bar{S}^*, \bar{p}^*, \bar{c}_1, \bar{c}_2, \bar{a})$ uniquely define the demand function (2.4) and supply function (2.5). These functions are defined by (2.6) and (2.7) respectively.

Definition 2.2:
A vector $\bar{p} \in [c_{11}, c_{21}] \times ... \times [c_{1n}, c_{2n}]$ is called an equilibrium price vector (an equilibrium), in the model σ_o if $D(\bar{p}) = S(\bar{p}) + \bar{a}$.

3. EQUILIBRIUM IN MARKET MODELS

The following theorems (criteria on the existense of an equilibrium) are extenstions of the results obtained in [8] for the closed market model.

Theorem 3.1:
In closed market model $\sigma_C \in \Sigma_C$ there exists an equilibrium vector iff for any vectors $\bar{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n_+$ and $\bar{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ the following conditions hold:

$$\sum_{i,j=1}^{n} \lambda_i a_{ij} x_j + \sum_{i=1}^{n} \lambda_i b_i + \sum_{i=1}^{n} \lambda_i x_i = 0,$$

$$\sum_{i=1}^{n} \lambda_i \ln \frac{c_{1i}}{c_{2i}} \le 0$$

with

$$a_{ij} = \tilde{E}_{ij} - E_{ij},\ i, j = \overline{1, n},$$

$$b_i = \ln \frac{D_i^*}{S_i^*} + \sum_{k=1}^{n} (\tilde{E}_{ik} - E_{ik}) \ln p_k^*,\ \ i = \overline{1, n}.$$

Proof
follows from Alexandrov theorem and Fan Ky theorem on the consistency of linear inequality system (see, e.g., [9]) and the fact that for closed market model $\sigma_c \in \Sigma_c$ the condition on the existence of an equilibrium is equivalent to the condition on the existence of the solution to the following system of equations (which is linear by $\ln p_j,\ j = \overline{1, n}$):

$$\sum_{j=1}^{n} (\tilde{E}_{ij} - E_{ij}) \ln p_j = \ln \frac{D_i^*}{S_i^*} + \sum_{j=1}^{n} (\tilde{E}_{ij} - E_{ij}) \ln p_j^*,\ \ i = \overline{1, n},$$

satisfying inequalities:

$$c_{1j} \le p_j \le c_{2j}\ \ \forall j = \overline{1, n}.$$

□

The following Theorem is another criteria on the existence of equilibrium. Its conditions (unlike Theorem 3.1) can be easily verified. To do that, one must simply calculate the finite number of determinants.

Theorem 3.2:
For closed market model $\sigma_C \in \Sigma_C$ there exists an equilibrium vector iff there exist such natural numbers $i_k \le n$, $j_l \le n$, $k, l = \overline{1, r}$ with $r = rang\left(\tilde{\mathcal{E}} - \mathcal{E}\right)$ and square matrix $A = \{a_{i_k j_l}\}$, $a_{i_k j_l} = \tilde{E}_{i_k j_l} - E_{i_k j_l}$ with rang r such that the following conditions hold:

$$detA \neq 0,$$

$$det\left(\begin{array}{ccc|c} & A & & \begin{matrix} b_{i_1} \\ \vdots \\ b_{i_r}\end{matrix} \\ \hline a_{ij_1} & \cdots & a_{ij_r} & b_i \end{array}\right) = 0 \ \ \forall i = \overline{1,n},$$

$$det\left(\begin{array}{ccc|c} & A & & \begin{matrix} b_{i_1} \\ \vdots \\ b_{i_r}\end{matrix} \\ \hline \delta_{ij_1} & \cdots & \delta_{ij_r} & \ln c_{1i} \end{array}\right) \leq 0, \ \ det\left(\begin{array}{ccc|c} & A & & \begin{matrix} b_{i_1} \\ \vdots \\ b_{i_r}\end{matrix} \\ \hline \delta_{ij_1} & \cdots & \delta_{ij_r} & \ln c_{2i} \end{array}\right) \geq 0 \ \ \forall i = \overline{1,n}$$

with

$$b_i = \ln \frac{D_i^*}{S_i^*} + \sum_{k=1}^{n} (\tilde{E}_{ik} - E_{ik}) \ln p_k^*, \ \ i = \overline{1,n}.$$

and δ_{ij} is Kronecker delta.

Proof
The vector $\bar{p} = (p_1, ..., p_n)$ is an equilibrium prices vector in model σ_c iff it is the solution to the following system of equations and inequalities

$$\begin{cases} \sum\limits_{j=1}^{n} (\tilde{E}_{ij} - E_{ij}) \ln p_j = \ln \frac{D_i^*}{S_i^*} + \sum\limits_{j=1}^{n} (\tilde{E}_{ij} - E_{ij}) \ln p_j^*, \\ p_i \geq c_{1i}, \\ p_i \leq c_{2i}, i = \overline{1,n}. \end{cases} \tag{3.9}$$

The system (3.9) is consistent iff the following system is consistent

$$\begin{cases} \sum\limits_{j=1}^{n} a_{ij} x_i = b_i, \\ x_i \geq C_{1i}, \\ x_i \leq C_{2i}, \end{cases} \quad i = \overline{1,n}. \tag{3.10}$$

Here

$$C_{1i} = \ln c_{1i}, i = \overline{1,n}; C_{2i} = \ln c_{2i}, i = \overline{1,n}.$$

Applying Theorem 1.3 from [10] to the system of linear equations and inequalities (3.10) we obtain the conditions of the Theorem.

□

3.1. Equilibrium in Open Market Models

Let us introduce the following notation:

$$\bar{\alpha}(\sigma) = \left[\max_{i=\overline{1,n}} \left(\left(S_i^* \prod_{j=1}^{n} (p_j^*)^{-\tilde{E}_{ij}} \min\left\{ c_{1j}^{|\tilde{E}_{ij}|}, c_{2j}^{-|\tilde{E}_{ij}|} \right\} \right)^{-1} \times \sum_{k=1}^{n} \frac{c_{2k} - c_{1k}}{2} c_{2k} |\tilde{E}_{ki}^{-1}| \right) \right]^{-1},$$

$$\bar{\beta}(\sigma) = \max_{i=\overline{1,n}} \left(\left(D_i^* \prod_{j=1}^{n} (p_j^*)^{-E_{ij}} \max\left\{ c_{2j}^{|E_{ij}|}, c_{1j}^{-|E_{ij}|} \right\} \right) \times \sum_{k=1}^{n} \frac{c_{k2} - c_{1k}}{2c_{1k}} |E_{ik}| \right),$$

$$\bar{\gamma}(\sigma) = \max_{i=\overline{1,n}} |S_i(\tilde{c}) + a_i - D_i(\tilde{c})| ,$$

where E_{ij}^{-1} is the element of the inverse matrix E^{-1} to $\tilde{\mathcal{E}}$, $\tilde{c} = \frac{c_1+c_2}{2}$.

Theorem 3.3:
Let the parameters of the open market model $\sigma_o \in \Sigma_o$ satisfy the conditions:

- $\bar{\beta}(\sigma) < \bar{\alpha}(\sigma)$;
- $\bar{\gamma}(\sigma) < \bar{\alpha}(\sigma) - \bar{\beta}(\sigma)$.

Then there exists an equilibrium prices vector $\bar{p} \in \mathbb{R}^n_+$ with $\bar{c}_1 \leq \bar{p} \leq \bar{c}_2$.

To prove this Theorem, we need the following definitions and results from the theory of covering maps. Let us formulate them. Let (X, ρ_X) and (Y, ρ_Y) be the metric spaces and Ψ and Φ be the maps from X to Y. By $B_X(x, r)$ denote a closed ball with the center at point x and radius r in the space X. Analogously we define $B_Y(x, r)$.

Definition 3.1:
(see [11]). A map $\Psi : X \to Y$ is called α-covering if

$$\Psi(B_X(x, r)) \supseteq B_Y(\Psi(x), \alpha r) \forall x \in X, \forall r > 0.$$

Definition 3.2:
(see [11]). A map Ψ is called metrically κ-regular if $\forall x_0 \in X, y \in Y \exists \in X : \Psi(x) = y$ and

$$\rho_X(x, x_0) \leq \kappa \rho_Y(y, \Psi(x_0)).$$

Proposition 3.1:
A map Ψ is α-covering iff Ψ is $1/\alpha$-regular.

Note that the maps Ψ and Φ are, obviously, surjective. It is easy to show that from the properties of metric regularity we can obtain the following Proposition.

Proposition 3.2:
Let $\sigma_o \in \Sigma_o$, $S, D : \mathbb{R}^n_+ \to \mathbb{R}^n$ be such that S is α-covering and D is Lipschitz continuous with Lipschitz constant β, $\exists p^*, p^{**} \in \mathbb{R}^n_+ : S(p^*) = D(p^*), S(p^{**}) = D(p^*)$. If $S(p^*) = S(p^{**})$, then $p^* = p^{**}$.

Proof
Indeed, if S is α-covering, then S is $1/\alpha$-regular and

$$\rho_X(p^*, p^{**}) \leq \frac{1}{\alpha} \rho_Y(S(p^*), S(p^{**})) = 0,$$

which leads to $p^* = p^{**}$. □

Theorem 3.4:
(see Theorem 1 from [11]) Let the space X be complete, $x_0 \in X, \alpha > 0, R > 0$. Let the map $\Psi : X \to Y$ be closed and α-covering on $B_X(x_0, R)$. Then for any nonnegative $\beta < \alpha$ and any map $\Phi : B_X(x_0, R) \to Y$ satisfying Lipschitz condition with the constant β and such that

$$\rho_Y(\Psi(x_0), \Phi(x_0)) \leq (\alpha - \beta)R,$$

there exists a coincidence point $\xi \in X$ for the maps Ψ, Φ, i.e., $\Psi(\xi) = \Phi(\xi)$, such that

$$\rho_Y(x_0, \xi) \leq \frac{\rho_Y(\Psi(x_0), \Phi(x_0))}{\alpha - \beta}.$$

Proof
Proof of Theorem 3.3. Introduce the following notation. By cov$(S|M)$ denote the supremum of all $\alpha > 0$ such that S is α-covering on M. By lip$(D|M)$ denote the infimum of all $\beta \geq 0$ such that D satisfies Lipschitz condition with a constant β. Then

$$\text{lip}(D|M) = \sup_{p \in \text{int}M} \left\| \frac{\partial D}{\partial p}(p) \right\|.$$

In spaces $\mathbb{R}^n_+, \mathbb{R}^n$ define the norms by

$$\|x\|_1 = 2 \max_{j=\overline{1,n}} \frac{|x_j|}{c_{2j} - c_{1j}} \forall x = (x_1, ..., x_n) \in \mathbb{R}^n,$$

$$\|y\|_2 = \max_{j=\overline{1,n}} |y_j| \forall y = (y_1, ..., y_n) \in \mathbb{R}^n.$$

Let $X = \mathbb{R}^n_+, Y = \mathbb{R}^n$. Put $M = B_X(\tilde{c}, 1)$. It is obvious that $M = [c_{11}, c_{21}] \times ... \times [c_{1n}, c_{2n}]$. Consider the metric spaces (X, ρ_Y) and (Y, ρ_Y) with the metrics ρ_X, ρ_Y defined by the norms $\| \cdot \|_1, \| \cdot \|_2$ correspondingly. It is obvious that $\mathbb{R}^n_+$ is not complete, but we only need that $B_X(\tilde{c}, 1)$ is complete.

Let us estimate lip$(D|M)$. To do that, estimate $\left\| \frac{\partial D}{\partial p}(p) \right\|$ first. From (2.2) it follows that

$$\left\| \frac{\partial D}{\partial p}(p) \right\| = \frac{D_i^* E_{ik}}{p_k} \prod_{j=1}^n (p_j^*)^{-E_{ij}} p_j^{E_{ij}}$$

Therefore,

$$\left\| \frac{dD}{dp} \right\| = \max_{\|x\|=1} \left\| \frac{dD}{dp} x \right\| = \max_{\|x\|=1} \max_{i=\overline{1,n}} \sum_{k=1}^n \left| \frac{\partial D_i}{\partial p_k} x_k \right| \leq$$

$$\leq \max_{\|x\|=1} \max_{i=\overline{1,n}} \sum_{k=1}^n \frac{D_i^* |E_{ik}|}{p_k} |x_k| \prod_{j=1}^n (p_j^*)^{-E_{ij}} p_j^{E_{ij}} \leq$$

$$\leq \max_{i=\overline{1,n}} \left(\prod_{j=1}^n (p_j^*)^{-E_{ij}} p_j^{E_{ij}} D_i^* \right) \sum_{k=1}^n \frac{c_{2k} - c_{1k}}{2p_k} |E_{ik}| \leq$$

$$\leq \max_{i=\overline{1,n}} D_i^* \left(\prod_{j=1}^n (p_j^*)^{-E_{ij}} \max \left\{ c_{2j}^{E_{ij}}, c_{1j}^{E_{ij}} \right\} \right) \times$$

$$\times \sum_{k=1}^n \frac{c_{k2} - c_{1k}}{2c_{1k}} |E_{ik}| = \bar{\beta}(\sigma).$$

Now we estimate cov$(S|M)$. According to Proposition 3.1, if the map S is α-covering, the inverse map S^{-1} is $1/\alpha$-Lipschitz continuous. We obtain the estimate using this proposition. Firstly we find $(\partial S/\partial p)^{-1}$. By (2.3) we have

$$\frac{\partial S_i}{\partial p_k}(p) = \frac{\tilde{E}_{ik} S_i^*}{p_k} \prod_{j=1}^n (p_j^*)^{-\tilde{E}_{ij}} p_j^{\tilde{E}_{ij}}$$

Therefore,

$$\det \frac{\partial S(p)}{\partial p} = \prod_{i=1}^{n} \left(S_i^* p_i^{-1} \prod_{j=1}^{n} (p_j^*)^{-\tilde{E}_{ij}} p_j^{\tilde{E}_{ij}} \right) \det \tilde{\mathcal{E}}.$$

By $\mathcal{S}_{ik}, \tilde{\mathcal{E}}_{ik}$ denote a cofactor to element $\partial S/\partial p_k, \tilde{E}_{ik}$ of $\partial S/\partial p, \tilde{\mathcal{E}}$ correspondingly. Thus:

$$\mathcal{S}_{ik} = \left(\prod_{\substack{m=1 \\ m \neq k}}^{n} p_m^{-1} \right) \prod_{\substack{l=1 \\ l \neq i}}^{n} \left(S_l^* \prod_{j=1}^{n} (p_j^*)^{-\tilde{E}_{lj}} p_j^{\tilde{E}_{lj}} \right) \tilde{\mathcal{E}}_{ik}.$$

Hence, the element of inverse matrix $(\partial S/\partial p)^{-1}$:

$$\left(\frac{\partial S(p)}{\partial p} \right)^{-1}_{ki} = \frac{\tilde{E}_{ki}^{-1}}{p_k^{-1} S_i^* \prod\limits_{j=1}^{n} (p_j^*)^{-\tilde{E}_{ij}} p_j^{\tilde{E}_{ij}}},$$

where $\tilde{E}_{ki}^{-1}$ is the element of inverse matrix to $\tilde{\mathcal{E}}$.

Now we estimate the Lipschitz constant of $(\partial D/\partial p)^{-1}$:

$$\left\| \left(\frac{\partial S(p)}{\partial p} \right)^{-1} \right\| = \max_{\|x\|_1 = 1} \max_{i=\overline{1,n}} \sum_{k=1}^{n} \left\| \left(\frac{\partial S_i(p)}{\partial p_k} \right)^{-1} \right\| \leq$$

$$\leq \max_{\|x\|_1 = 1} \max_{i=\overline{1,n}} \sum_{k=1}^{n} \frac{p_k \left| \tilde{E}_{ki}^{-1} x_k \right|}{S_i^* \prod\limits_{j=1}^{n} (p_j^*)^{-\tilde{E}_{ij}} \min \left\{ c_{1j}^{\tilde{E}_{ij}}, c_{2j}^{\tilde{E}_{ij}} \right\}} \leq$$

$$\leq \max_{i=\overline{1,n}} \left(S_i^* \prod_{j=1}^{n} (p_j^*)^{-\tilde{E}_{ij}} \min \left\{ c_{1j}^{\tilde{E}_{ij}}, c_{2j}^{\tilde{E}_{ij}} \right\} \right)^{-1} \times$$

$$\times \sum_{k=1}^{n} \left| \tilde{E}_{ki}^{-1} \right| \frac{c_{2k} - c_{1k}}{2} c_{2k} = \frac{1}{\bar{\alpha}(\sigma)}.$$

By the conditions of Theorem and inequalities $\text{cov}(S|M) \geq \bar{\alpha}(\sigma)$, $\text{lip}(D|M) \leq \bar{\beta}(\sigma)$ there exist positive numbers α, β such that $\bar{\beta}(\sigma) < \beta < \alpha < \bar{\alpha}(\sigma)$, S is α-covering on M, D is β-Lipschitz continuous on M. Since $\rho_Y(S(\tilde{c} + a, D(\tilde{c})) = \bar{\gamma}(\sigma)$, from Condition 2 of the Theorem it follows that $\rho_Y(S(\tilde{c} + a, D(\tilde{c})) < \alpha - \beta$. Therefore, there exists a vector $\bar{p} \in X$ such that $D(\bar{p}) = S(\bar{p}) + a$ and

$$\rho_Y(\bar{p}, \tilde{c}) \leq \frac{\rho_Y(S(\tilde{c}) + a, D(\tilde{c}))}{\alpha - \beta}.$$

From the last inequality it follows that $p \in \text{int}M$, since $M = B_X(\tilde{c}, 1)$ and $\rho_y(S(\tilde{c}) + a, D(\tilde{c})) < \alpha - \beta$. Hence, $c_{1j} < p_j < c_{2j}, j = \overline{1, n}$. □

Theorem 3.5:
Let the following conditions be satisfied for the model $\sigma_o \in \Sigma_o$:

- $\bar{\alpha}(\sigma) > 2\bar{\beta}(\sigma)$;
- $\exists \tilde{p} = (\tilde{p}_1, \ldots, \tilde{p}_n) \in \mathbb{R}^n_+$: $S(\tilde{p}_1, \ldots, \tilde{p}_n) = D(p_1^*, \ldots, p_n^*)$, *where* $(p_1^*, \ldots, p_n^*)$ *is an equilibrium in model* σ_o.

Then in the model σ_o *there exists an equilibrium prices vector not equal* $(p_1^*, \ldots, p_n^*)$.

Proof
follows from Theorem 1 in [12]. □

The results similar to Theorems 3.3–3.5 for dynamic market models may be obtained, applying the results of [13]– [16] besides existence theorems for coincidence points. In such models the supply and demand functions depend not only on prices on the goods, but on price change rates. In turn, the question of the existence for an equilibrium can be considered as the question of existence for the solution of a system of differential equations.

4. ITERATION PROCESS

Define the norms in the spaces $\mathbb{R}^n_+$ and $\mathbb{R}^n$:

$$\|x\|_X = 2\max_{j=\overline{1,n}} \frac{|x_j|}{c_{2j} - c_{1j}} \ \ \forall x = (x_1, ..., x_n) \in \mathbb{R}^n,$$

$$\|y\|_Y = \max_{j=\overline{1,n}} |y_j| \ \ \forall y = (y_1, ..., y_n) \in \mathbb{R}^n,$$

and consider metric spaces $(\mathbb{R}^n_+, \rho_X), (\mathbb{R}^n, \rho_Y)$ with the metrics ρ_X and ρ_Y generated by the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ correspondingly. Fix an arbitrary $\delta > 0$. From Definition (3.1) we obtain that

$$\rho_Y(S(x'), D(x)) \leq \delta\rho_Y(S(x), D(x)), \tag{4.11}$$

$$\rho_X(x', x) \leq \alpha^{-1}\rho_Y(S(x), D(x)). \tag{4.12}$$

Based on these inequalities we can construct the following iteration process. Let $S, D : X \to Y$, S be α-covering and D be Lipschitz continuous with the constant β. Fix an arbitrary $p_0 \in X$ and a sequence of nonnegative numbers $\{\delta_i\}$:

$$\beta + \alpha \overline{\lim_{i\to\infty}} \delta_i < \alpha. \tag{4.13}$$

Then there exists a sequence $\{p_i\}, i = 0, 1, ...$ such that

$$\rho_Y(S(p_{i+1}), D(p_i)) \leq \delta_i\rho_Y(S(p_i), D(p_i)), \tag{4.14}$$

$$\rho_X(p_{i+1}, p_i) \leq \alpha^{-1}\rho_Y(S(p_i), D(p_i)). \tag{4.15}$$

Theorem 4.1 (A. Arutyunov, see [1])**:**
Let the space X *be complete,* $S : X \to Y$ *be* α*-covering, gph*$S = \{(x, y) \in X \times Y : y = S(x)\}$ *be closed and* $D : X \to Y$ *satisfy Lipschitz condition with a constant* $\beta < \alpha$ *and, moreover,*

$$\beta + \alpha \overline{\lim_{i\to\infty}} \delta_i < \alpha. \tag{4.16}$$

Then $\forall p_0 \in X$ *there exists a sequence* $\{p_i\}$ *satisfying* (4.14), (4.15) $\forall i$*, and any such sequence converges to some coincidence point* $\xi = \xi(x_0)$ *with*

$$\rho_X(\xi, p_0) \leq \alpha^{-1} \left(1 + \sum_{j=1}^{\infty} \prod_{j=1}^{i} \left(\delta_i + \frac{\beta}{\alpha}\right)\right) \rho_Y(S(p_0), D(p_0)).$$

We construct search algorithm on the base of iteration process (4.14), (4.15) and Theorem 4.1. The search of a new point on every step of this iteration process is based on direct search method. Namely, we search the new point of the iteration process in the form:

$$p_{i+1} = p_i + h = (p_{1i} + h_1, ..., p_{ni} + h_n), \tag{4.17}$$

where $h = (h_1, ..., h_n)$ is the step that needs to be defined on every iteration.

The search radius, i.e., the maximal value of every coordinate of the step, can be found using (4.15). Indeed, from (4.15) we obtain that:

$$\max_{i=\overline{1,n}} \frac{|h_i|}{c_{i1} - c_{i2}} \leqslant \frac{\alpha^{-1}}{2} \max_{i=\overline{1,n}} |S_i(p) - D_i(p)|,$$

where $S(p) = (S_1(p), ..., S_n(p)), D(p) = (D_1(p), ..., D_n(p))$. Then:

$$\frac{|h_i|}{c_{i1} - c_{i2}} \leqslant \frac{\alpha^{-1}}{2} \max_{i=\overline{1,n}} |S_i(p) - D_i(p)|.$$

Hence we obtain the estimate for the search radius:

$$|h_i| \leqslant \frac{c_{i1} - c_{i2}}{2\alpha} \max_{i=\overline{1,n}} |S_i(p) - D_i(p)|. \tag{4.18}$$

Next we must define the elements of the sequence $\{\delta_i\}$. From (4.16) it follows that:

$$\overline{\lim_{i\to\infty}} \delta_i < 1 - \frac{\beta}{\alpha}. \tag{4.19}$$

Here we put:

$$\delta_i = \delta = 1 - \frac{\beta}{\alpha} - \varepsilon, \tag{4.20}$$

where $\varepsilon > 0$ is an arbitrary small positive number (process error). Constructed sequence obviously satisfies (4.19).

Now we describe the step of the iteration process. Let p_i be the point obtained on ith iteration. Define the search radius by considering an equality in (4.18) and put

$$\tilde{p}_{i+1} = (p_{1i} + h_1, p_{2i}, ..., p_{ni}),$$

i.e., take the initial point with ¡¡incremented¿¿ first coordinate. For this point we check the condition (4.14). If it is satisfied, we put $p_{i+1} = \tilde{p}_{i+1}$ but we continue the search process by decreasing h_1. If we find another point satisfying (4.14), we compare it with the previous one by the value at $\rho_2(S(\cdot), D(p_i))$ and take the one with the minimal value. Then we decrease h_1 and continue the search while $h_1 > \varepsilon$.

After we complete the search by the first coordinate in positive direction we put

$$\tilde{p}_{i+1} = (p_{1i} - h_1, p_{2i}, ..., p_{ni}),$$

and conduct this search for this point. Once we finish this search (i.e. $h_1 < \varepsilon$), we ¡¡increment¿¿ the second coordinate and conduct this search for it etc. When we complete the search for the last coordinate we finish the iteration and obtain the point p_{i+1}.

Numerical experiments show that on the first step of the iteration process δ_i chosen above can violate this process. Condition (4.19) allows us to redefine δ_i to satisfy (4.14), from which we obtain that

$$\max_{p\in U_h(p_i)} \rho_Y(S(p), D(p_i)) \leq \delta_i \rho_Y(S(p_i), D(p_i)),$$

where $U_h(p_i) = \{x \in \mathbb{R}^n_+ : |p_ij - x_j| \leq h_j, j = \overline{1, n}\}$. From the last inequality we get the lower estimate for δ_i:

$$\delta_i \geq (\rho_Y(S(p^k), D(p^k)))^{-1} \max_{p \in U_h(p^k)} \rho_Y(S(p), D(p^k)).$$

It is possible to define δ_i as shown above on each iteration but this can significantly decrease the iteration process.

5. CONCLUSION

The open and closed market models are studied. The existence of an equilibrium is investigated. For the closed market model we obtained necessary and sufficient conditions for the existence of an equilibrium price vector. To do that, we used corollaries of consistence theorems for the systems of linear equations and inequalities. These conditions can be easily verified numerically. For the open market model we obtained sufficient existence conditions. To do that, we used corollaries of the theorems on coincidence points for covering and Lipschitz continuous mappings. These conditions can be also verified numerically. Also we obtained the conditions for the uniqueness and nonuniqueness of equilibrium in the open and closed market models. They are easily obtained as corollaries from coincidence points theorems. All these results can be applied to investigate the power of the set of equilibrium price vectors and to find them numerically.

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APPENDIX. NUMERICAL EXPRERIMENT RESULTS

Table 5.1. Numerical results ($n = 1$).

a	$\mathcal{E}$	$\tilde{\mathcal{E}}$	c_1	c_2	D^*	S^*	p^*	Exact solution	Approximate solution	Difference
12,04967257	0,49849705	0,31751280	3,42513070	3,90654450	52,32966270	39,78066120	3,66583760	3,53063718	3,53071553	0,00007835
23,94610559	0,37996909	0,87865800	1,08465500	2,26080760	62,55706230	42,41923650	1,67273130	1,10991876	1,11005981	0,00014105
75,39041169	0,61294372	0,94264690	2,00816940	2,23830410	82,05652560	7,06134190	2,12323675	2,14250395	2,14248499	0,00001896
3,46479208	0,84918128	0,98566300	1,17153660	1,86127260	14,18057210	10,31352890	1,51640460	1,21939422	1,21963361	0,00023939
12,04967254	0,49849705	1,31751280	3,42513070	3,90654450	52,32966270	39,78066120	3,66583760	3,73464490	3,73464999	0,00000509
45,75960188	0,21808695	1,36433650	1,68366000	3,10197130	83,76817070	36,67268590	2,39281565	2,49140247	2,49136671	0,00003576
57,92058658	0,29802699	1,39381170	5,47845230	6,84553920	98,16366610	37,78317680	6,16199575	6,76017829	6,76013104	0,00004725
52,01725023	1,78504326	0,74302160	1,31061930	1,41118560	83,42842960	27,72930280	4,00613680	1,32130417	1,32138381	0,00001928
45,82652174	1,48409565	0,97468880	3,82624060	4,18603300	61,03724680	15,94939750	2,94413005	4,04545893	4,04543965	0,00004024
25,40951761	1,11906593	0,25285110	2,84528020	3,04297990	97,14177200	74,41436640	4,27683430	3,03158898	3,03154874	0,00001191
24,66958655	1,35448468	0,81663360	3,81639170	4,73727690	80,88850350	61,82636120	7,17070685	4,66852786	4,66851595	0,00037521
8,24245960	1,74156854	1,94871380	1,51486620	1,92540030	10,94697020	2,62021260	6,90775325	1,70972480	1,70973503	0,00000348
59,38903653	1,97225893	1,92996670	6,59084110	7,22466540	76,13983210	20,30144890	5,10374760	7,12536138	7,12535790	0,00035818
25,98130490	1,06158955	1,96279730	4,74293100	5,46456420	45,88108570	20,36532230	3,88474745	5,41417342	5,41381524	0,00027041
4,17417959	1,18727726	1,12764460	3,24925490	4,52024000	86,81987700	82,28327370	0,75434715	3,74615923	3,74642964	0,00067130
4,86894508	0,45991870	-0,66585270	7,67719070	8,79748820	9,24301545	4,38458911	8,23733945	8,24943189	8,24943135	0,00000054
8,49601314	1,80795421	-1,48582480	1,25348280	1,37778650	8,12000301	0,12086598	1,31563465	1,35907473	1,35907484	0,00000011
0,35324002	2,73432405	-2,70023780	0,27421480	8,02557370	0,60356316	4,57846568	4,14989425	6,26317387	6,26317374	0,00000013
4,29331268	3,94833192	-3,49545330	6,40132230	7,26925600	9,98524981	4,81045034	6,83528915	6,72759315	6,72759330	0,00000015

Table 5.2. Numerical results ($n = 2$).

a	$\mathcal{E}$	$\tilde{\mathcal{E}}$	c_1	c_2	D^*	S^*	p^*	Exact solution	Approximate solution	Difference (10^{-4})
5,44924634 1,50269678	0.1890 0.8820 0.6993 0.9698	0.7892 0.2051 0.0357 0.8606	6,15663800 3,73111020	11,82780960 13,40970390	8,12078380 7,41039350	0,97495070 4,25438820	8,99222380 8,57040705	7,45510945 6,24363197	7,45509020 6,24363197	0,00001925
6,08640569 5,29118933	2.2765 1.3760 4.5251 7.3793	0.8156 0.2402 0.0305 0.9765	0,77748450 5,38254820	0,93917660 5,45608770	6,40652017 3,79437135	6,25215623 8,70644523	0,85833055 5,41931795	0,93917282 5,45607010	0,93898026 5,45587926	0,00027111
7,84596691 2,77771560	6.1338 8.3264 9.2562 6.3013	0.9173 0.3166 0.6689 0.6981	7,42234360 2,18748460	7,50771710 2,34259840	8,47626315 2,37613970	6,92245370 5,24030145	7,46503035 2,26504150	7,50771074 2,34258083	7,50770123 2,34256957	0,00001474
0,93754308 3,03700992	6.2204 4.7321 8.4400 5.6222	0.9847 0.1637 0.3153 0.7035	7,34771970 4,77912010	7,45629940 4,87652930	7,56454929 5,34618234	9,52376609 9,61665866	7,40200955 4,82782470	7,45627236 4,87648342	7,45611410 4,87636307	0,00019882
32,64472260 1,97423317	89.9777 22.7800 43.3595 23.2559	9.6736 6.4568 6.9741 6.6399	87,44205800 4,94665380	87,48379640 5,03385410	36,55847468 51,91810778	81,65630919 80,30565520	87,46292720 4,99025395	87,48379045 5,03385305	87,48367331 5,03359694	0,00028163
6,62858147 75,82674314	83.4105 37.7344 40.7935 73.9584	5.9630 0.3356 0.9847 8.3960	94,76327610 44,18489150	94,87513010 44,31103040	40,59597359 99,78815744	81,50984576 93,82997677	94,81920310 44,24796095	94,87507150 44,31100340	94,87504711 44,31093681	0,00007092
0,93754308 3,03700992	6.2204 4.7321 8.4400 5.6222	0.9847 0.1637 0.3153 0.7035	7,34771970 4,77912010	7,45629940 4,87652930	7,56454929 5,34618234	9,52376609 9,61665866	7,40200955 4,82782470	7,45627236 4,87648342	7,45606759 4,87632134	0,00026115
2,60770848 5,41952209	4.0164 9.4726 3.8716 1.4276	0.3168 0.4243 0.7969 0.6316	7,55571660 5,03938100	7,56824890 5,06077320	6,04567702 5,25349915	9,75816992 8,17917163	7,56198275 5,05007710	7,56824533 5,06076588	7,56821205 5,06070798	0,00006678
7,58087811 0,60925099	2.8164 1.4588 4.1622 6.0416	0.7191 0.2219 0.2973 0.6845	3,12702660 9,16040530	3,32658470 9,23570010	9,14185023 7,33798284	9,46440126 8,85951959	3,22680565 9,19805270	3,32656708 9,23568740	3,32658467 9,23562828	0,00006168
0,37634516 0,37634516	0.9667 0.5779 0.4516 0.2216	-0.5492 -0.1576 -0.8893 -0.6166	5,03020750 5,03020750	5,30800240 5,30800240	9,06978203 9,06978203	8,77302014 8,77302014	5,16910495 5,16910495	5,09265685 5,20897841	5,08568269 5,30173641	0,00697416
0,37634516 0,37634516	0.9667 0.5779 0.4516 0.2216	-0.5492 -0.1576 -0.8893 -0.6166	5,03020750 5,03020750	5,30800240 5,30800240	9,06978203 9,06978203	8,77302014 8,77302014	5,16910495 5,16910495	5,08548870 5,30189090	5,08568269 5,30173641	0,00019399
5,34465776 5,34465776	0.0806 0.9113 0.4183 0.2131	-0.2613 -0.7458 -0.7018 -0.2888	5,89700700 5,89700700	6,01063950 6,01063950	9,77384569 9,77384569	4,51118083 4,51118083	5,95382325 5,95382325	5,98007240 6,01063950	5,98016342 6,01063948	0,00009102
4,43204423 4,43204423	0.3271 0.3822 0.0578 0.6583	-0.9388 -0.0428 -0.0556 -0.6059	7,05488490 7,05488490	7,14020900 7,14020900	5,94099236 5,94099236	1,49789301 1,49789301	7,09754695 7,09754695	7,07485070 7,09225690	7,07485227 7,09227383	0,00000157

Table 5.3. Numerical results ($n = 3$).

a	$\mathcal{E}$	$\tilde{\mathcal{E}}$	c_1	c_2	D^*	S^*	p^*	Exact solution	Approximate solution	Difference
0,00067164	-0,5231 0,2173 0,9689	0,5578 -0,7804 0,5495	0,00090800	0,00158780	0,00081357	0,00050346	0,00124790	0,00128320	0,00140421	0,00012101
0,00011286	0,7210 -0,1726 0,9837	0,6900 -0,3330 0,2800	0,00050400	0,00090860	0,00086791	0,00087520	0,00070630	0,00090860	0,00076905	
0,00027498	0,0000 0,6294 -0,9423	-0,6848 -0,4306 0,2699	0,00020500	0,00097810	0,00065843	0,00051156	0,00059155	0,00088850	0,00083523	
0,00010750	0,2274 0,8585 -0,3710	0,0715 0,8418 0,6096	0,00002280	0,00052410	0,00093567	0,00039017	0,00027345	0,00016020	0,00045987	0,00029967
0,00095715	-0,7547 -0,5598 -0,2748	0,6020 0,7708 0,9675	0,00004870	0,00040810	0,00092320	0,00086616	0,00022840	0,00013530	0,00014492	
0,00053950	0,0000 -0,2463 0,0460	0,7082 0,9049 -0,7075	0,00006260	0,00078740	0,00077132	0,00013670	0,00042500	0,00062040	0,00074918	
0,00003963	-0,5026 0,4313 0,7671	0,5729 -0,9934 0,3843	0,00002070	0,00004580	0,00005873	0,00000503	0,00003325	0,00004580	0,00004580	0,00000000
0,00007197	0,1215 0,1276 -0,0768	-0,8328 0,6271 0,8835	0,00001550	0,00011190	0,00005827	0,00008790	0,00006370	0,00001560	0,00001550	
0,00006047	0,0000 -0,8044 -0,7787	-0,1941 -0,7383 0,1019	0,00007900	0,00012810	0,00003524	0,00008354	0,00010355	0,00008680	0,00007900	
0,00008432	0,6210 0,8308 0,7780	-0,2949 -0,1874 -0,2297	0,00084450	0,00103460	0,00009384	0,00038078	0,00093955	0,00103420	0,00103460	0,00000040
0,00020863	0,1216 0,0103 0,6153	0,1970 -0,2776 0,9637	0,00025450	0,00056260	0,00030163	0,00017909	0,00040855	0,00031830	0,00028144	
0,00032173	0,0000 -0,8867 0,3545	-0,1950 0,5971 0,4874	0,00002730	0,00041180	0,00068158	0,00014939	0,00021955	0,00045820	0,00039854	
0,00005837	-0,5243 -0,1608 -0,0334	-0,0845 0,5885 -0,9718	0,00005280	0,00013200	0,00007452	0,00005578	0,00009240	0,00013200	0,00012900	0,00000300
0,00001550	-0,3629 -0,4256 -0,9427	0,2073 -0,5534 0,9400	0,00009480	0,00016860	0,00009964	0,00003764	0,00013170	0,00009490	0,00009495	
0,00009541	0,0000 0,0858 0,4931	0,5091 -0,8264 -0,7454	0,00008170	0,00017370	0,00005752	0,00004115	0,00012770	0,00011700	0,00012971	
0,00003587	0,8321 -0,8926 0,1500	0,7844 -0,2131 -0,8089	0,00008500	0,00012410	0,00004221	0,00008118	0,00010455	0,00008500	0,00008500	0,00000000
0,00008547	0,8150 0,7389 -0,3049	0,6097 0,6885 -0,2391	0,00003130	0,00003170	0,00008874	0,00001068	0,00003150	0,00003130	0,00003130	
0,00005685	0,0000 -0,7913 -0,3917	-0,2206 -0,8458 -0,7431	0,00005280	0,00012790	0,00007893	0,00004632	0,00009035	0,00009920	0,00012790	
0,00009064	-0,4217 -0,6801 0,2354	-0,7936 0,2152 0,6938	0,00003370	0,00004470	0,00005739	0,00008024	0,00003920	0,00004470	0,00004467	0,00000003
0,00009087	0,7460 -0,4548 -0,6924	0,8062 0,0382 0,2716	0,00005140	0,00008490	0,00003060	0,00002684	0,00006815	0,00005140	0,00005140	
0,00006311	0,0000 0,0796 -0,2799	0,9650 -0,5817 0,3558	0,00000050	0,00005690	0,00002820	0,00005324	0,00002870	0,00011230	0,00002213	
0,00006769	-0,6240 -0,6595 0,2290	-0,5791 -0,9208 -0,1698	0,00002790	0,00010820	0,00007469	0,00009147	0,00006805	0,00010820	0,00010820	0,00000000
0,00005905	0,9422 -0,0283 -0,2845	0,2006 -0,7921 -0,6689	0,00002230	0,00002860	0,00001285	0,00001899	0,00002545	0,00002860	0,00002860	
0,00002881	0,0000 -0,6351 0,1713	-0,9941 0,0697 -0,6779	0,00004570	0,00008370	0,00001968	0,00003436	0,00006470	0,00005400	0,00008122	
0,00007705	0,9151 0,7877 0,1382	0,9113 -0,4715 0,5701	0,00003990	0,00013470	0,00004845	0,00002284	0,00008730	0,00013470	0,00013450	0,00000020
0,00003230	-0,0208 0,7708 0,1809	-0,7565 -0,2716 -0,4085	0,00007680	0,00013410	0,00000904	0,00005978	0,00010545	0,00013410	0,00013410	
0,00008744	0,0000 -0,8708 0,8600	-0,4455 0,6463 0,5418	0,00005160	0,00012380	0,00005810	0,00004879	0,00008770	0,00015160	0,00011073	