A Universal Nonlinear Control Law for the Synchronization of Arbitrary 3-D Continuous-time Quadratic Systems

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Abstract
In this letter we present a universal nonlinear control law for the synchronization of arbitrary 3-D continuous-time quadratic systems. This control law does not require any type of conditions on the considered systems.
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1 Introduction

Several methods have been successfully applied to chaos synchronization. For example, in [1] a method is introduced to synchronize two identical chaotic systems with different initial conditions. An adaptive control approach is presented in [2], a backstepping design was presented in [3], an active control method is presented in [4-6], and a nonlinear control scheme was given in [7-9]. Consequently, there are many applications of chaos synchronization in physical, chemical, and ecological systems, and in secure communications as shown in [1-2,10-13].

In this letter, we apply nonlinear control theory to synchronize two arbitrary 3-D continuous-time quadratic systems. The proposed control law does not need any conditions on the considered systems, and hence it is a universal synchronization approach for general 3-D continuous-time quadratic systems. In other words, the present letter is concerned with synchronization of nonlinear systems in the framework of nonlinear observers. The investigation is restricted to a pair of quadratic three dimensional systems, for which a control feedback can be chosen in such a way that global asymptotic stability of the error system can be established in the framework of classical Lyapunov theory. This restriction is justified by the importance of this type of systems in real applications [14] which is certainly a useful result.

2 Synchronization using a universal nonlinear control law

In this section, we consider two arbitrary 3-D continuous-time quadratic systems. The one with variables x_1, y_1, and z_1 will be controlled to be the new system given by

$$\begin{cases} x_1' = a_0 + a_1x_1 + a_2y_1 + a_3z_1 + f_1(x_1, y_1, z_1) \\ y_1' = b_0 + b_1x_1 + b_2y_1 + b_3z_1 + f_2(x_1, y_1, z_1) \\ z_1' = c_0 + c_1x_1 + c_2y_1 + c_3z_1 + f_3(x_1, y_1, z_1) \end{cases} \tag{1}$$

where

$$\begin{cases} f_1(x_1, y_1, z_1) = a_4x_1^2 + a_5y_1^2 + a_6z_1^2 + a_7x_1y_1 + a_8x_1z_1 + a_9y_1z_1 \\ f_2(x_1, y_1, z_1) = b_4x_1^2 + b_5y_1^2 + b_6z_1^2 + b_7x_1y_1 + b_8x_1z_1 + b_9y_1z_1 \\ f_3(x_1, y_1, z_1) = c_4x_1^2 + c_5y_1^2 + c_6z_1^2 + c_7x_1y_1 + c_8x_1z_1 + c_9y_1z_1 \end{cases} \tag{2}$$

and the one with variables x_2, y_2, and z_2 as the response system

$$\begin{cases} x_2' = d_0 + d_1x_2 + d_2y_2 + d_3z_2 + g_1(x_2, y_2, z_2) + u_1(t) \\ y_2' = r_0 + r_1x_2 + r_2y_2 + r_3z_2 + g_2(x_2, y_2, z_2) + u_2(t) \\ z_2' = s_0 + s_1x_2 + s_2y_2 + s_3z_2 + g_3(x_2, y_2, z_2) + u_3(t) \end{cases} \tag{3}$$

where

$$\begin{cases} g_1(x_2, y_2, z_2) = d_4x_2^2 + d_5y_2^2 + d_6z_2^2 + d_7x_2y_2 + d_8x_2z_2 + d_9y_2z_2 \\ g_2(x_2, y_2, z_2) = r_4x_2^2 + r_5y_2^2 + r_6z_2^2 + r_7x_2y_2 + r_8x_2z_2 + r_9y_2z_2 \\ g_3(x_2, y_2, z_2) = s_4x_2^2 + s_5y_2^2 + s_6z_2^2 + s_7x_2y_2 + s_8x_2z_2 + s_9y_2z_2 \end{cases} \tag{4}$$

Here $(a_i, b_i, c_i)_{0 \le i \le 9} \subset \mathbb{R}^{30}$ and $(d_i, r_i, s_i)_{0 \le i \le 9} \subset \mathbb{R}^{30}$ are bifurcation parameters, and $u_1(t), u_2(t), u_3(t)$ are the unknown (to be determined) nonlinear controller such that two systems (1) and (3) can be synchronized.

First, let us define the following quantities depending on the above two systems in which we can proceed with our proposed method:

$$\begin{cases} \xi_1 = a_1 + d_1 + a_4(x_1 + x_2) + d_4(x_1 + x_2) + a_7y_1 + a_8z_1 + d_7y_2 + d_8z_2 \\ \xi_2 = a_2 + d_2 + a_5(y_1 + y_2) + d_5(y_1 + y_2) + a_9z_1 + d_9z_2 \\ \xi_3 = a_3 + d_3 + a_6(z_1 + z_2) + d_6(z_1 + z_2) \\ \xi_4 = \eta_1 + \eta_2 + \eta_3 \\ \xi_5 = b_1 + r_1 + b_4(x_1 + x_2) + r_4(x_1 + x_2) + b_7y_1 + b_8z_1 + r_7y_2 + r_8z_2 \end{cases} \tag{5}$$

and

$$\begin{cases} \xi_6 = b_2 + r_2 + b_5(y_1 + y_2) + r_5(y_1 + y_2) + b_9 z_1 + r_9 z_2 \\ \xi_7 = b_3 + r_3 + b_6(z_1 + z_2) \\ \xi_8 = \eta_4 + \eta_5 + \eta_6 \\ \xi_9 = c_1 + s_1 + c_4(x_1 + x_2) + s_4(x_1 + x_2) + c_7 y_1 + c_8 z_1 + s_7 y_2 + s_8 z_2 \\ \xi_{10} = c_2 + s_2 + c_5(y_1 + y_2) + s_5(y_1 + y_2) + c_9 z_1 + s_9 z_2 \\ \xi_{11} = c_3 + s_3 + c_6(z_1 + z_2) + s_6(z_1 + z_2) \\ \xi_{12} = \eta_7 + \eta_8 + \eta_9 \end{cases} \tag{6}$$

where

$$\begin{cases} \eta_1 = d_4 x_1^2 + d_7 x_1 y_2 + d_8 x_1 z_2 + d_1 x_1 - a_4 x_2 - a_7 x_2 y_1 - a_8 x_2 z_1 \\ \eta_2 = -a_1 x_2 + d_5 y_1^2 + d_9 y_1 z_2 + d_2 y_1 - a_5 y_2^2 - a_9 y_2 z_1 - a_2 y_2 \\ \eta_3 = d_6 z_1^2 + d_3 z_1 - a_6 z_2^2 - a_3 z_2 - a_0 + d_0 \\ \eta_4 = r_4 x_1^2 + r_7 x_1 y_2 + r_8 x_1 z_2 + r_1 x_1 - b_4 x_2^2 - b_7 x_2 y_1 - b_8 x_2 z_1 \\ \eta_5 = -b_1 x_2 + r_5 y_2^2 + r_9 y_1 z_2 + r_2 y_1 - b_5 y_2^2 - b_9 y_2 z_1 - b_2 y_2 \\ \eta_6 = r_6 z_1^2 + r_3 z_1 - b_6 z_2^2 - b_3 z_2 - b_0 + r_0 \\ \eta_7 = s_4 x_1^2 + s_7 x_1 y_2 + s_8 x_1 z_2 + s_1 x_1 - c_4 x_2^2 - c_7 x_2 y_1 - c_8 x_2 z_1 \\ \eta_8 = -c_1 x_2 + s_5 y_1^2 + s_9 y_1 z_2 + s_2 y_1 - c_5 y_2^2 - c_9 y_2 z_1 - c_2 y_2 \\ \eta_9 = s_6 z_1^2 + s_3 z_1 - c_6 z_2^2 - c_3 z_2 - c_0 + s_0 \end{cases} \tag{7}$$

The above quantities comes from the formulation of the problem as the system in (8) below. Now let the error states be $e_1 = x_2 - x_1, e_2 = y_2 - y_1$, and $e_3 = z_2 - z_1$. Then the error system is given by

$$\begin{cases} e_1' = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + \xi_4 + u_1(t) \\ e_2' = \xi_5 e_1 + \xi_6 e_2 + \xi_7 e_3 + \xi_8 + u_2(t) \\ e_3' = \xi_9 e_1 + \xi_{10} e_2 + \xi_{11} e_3 + \xi_{12} + u_3(t) \end{cases} \tag{8}$$

We propose the following universal control law for the system (3):

$$\begin{cases} u_1 = -(\xi_1 + 1)e_1 - (\xi_2 + \xi_5)e_2 - \xi_4 \\ u_2 = -(\xi_6 + 1)e_2 - (\xi_7 + \xi_{10})e_3 - \xi_8 \\ u_3 = -(\xi_3 + \xi_9)e_1 - (\xi_{11} + 1)e_3 - \xi_{12} \end{cases} \tag{9}$$

Then the two 3-D continuous-time quadratic systems (1) and (3) approach synchronization for any initial condition. Indeed, the error system (8) becomes

$$\begin{cases} e_1' = -e_1 - \xi_5 e_2 + \xi_3 e_3 \\ e_2' = \xi_5 e_1 - e_2 - \xi_{10} e_3 \\ e_3' = -\xi_3 e_1 + \xi_{10} e_2 - e_3 \end{cases} \tag{10}$$

and if we consider the Lyapunov function $V = \frac{e_1^2+e_2^2+e_3^2}{2}$, then it is easy to verify the asymptotic stability of the error system (10) by Lyapunov stability theory since we have $\frac{dV}{dt} = -e_1^2 - e_2^2 - e_3^2 < 0$ for all $(a_i, b_i, c_i)_{0\leq i\leq 9} \subset R^{30}$, $(d_i, r_i, s_i)_{0\leq i\leq 9} \subset R^{30}$ and for all initial conditions. In particular, if the two systems (1) and (3) are chaotic, then the control law (9) guarantees also their synchronization for any initial condition. A practical example of this situation can be found in [8]. On the other hand, any 3-D continuoustime quadratic chaotic system can be stabilized (controlled) to a stable 3-D continuous-time quadratic system that converges to an equilibrium point (to a 3-D continuous-time quadratic system that converges to a periodic solution). Furthermore, any 3-D continuous-time quadratic system can be chaotified to a chaotic 3-D continuous-time quadratic system.

3 Example

The most known example of 3-D quadratic systems, is the original Lorenz system given by:

$$\begin{cases} x_1' = a_1 x_1 - a_1 y_1 \\ y_1' = b_1 x_1 - y_1 - x_1 z_1 \\ z_1' = -c_3 z_1 + x_1 y_1 \end{cases} \tag{11}$$

To apply the above method, we consider the one with variables x_2, y_2, and z_2 as the response system

$$\begin{cases} x_2' = d_1 x_2 - d_1 y_2 \\ y_2' = r_1 x_1 - y_2 - x_2 z_2 + u_2(t) \\ z_2' = -s_3 z_2 + x_1 y_2 + u_3(t) \end{cases} \tag{12}$$

Thus, the universal control law for the system (12) is given by:

$$\begin{cases} u_1(t) = -(\xi_1 + 1)e_1 - (\xi_2 + \xi_5)e_2 - \xi_4 \\ u_2(t) = e_2 - \xi_8 \\ u_3(t) = -\xi_9 e_1 - (\xi_{11} + 1)e_3 - \xi_{12} \end{cases} \tag{13}$$

where

$$\begin{cases} \xi_1 = a_1 + d_1, \xi_2 = -a_1 - d_1, \xi_4 = d_1x_1 - a_1x_2 - d_1y_1 + a_1y_2 \\ \xi_5 = -z_1 - z_2, \xi_6 = -2, \xi_8 = -x_1z_2 + z_1x_2 - y_1 + y_2 \\ \xi_9 = y_1 + y_2, \xi_{11} = -c_3 - s_3, \xi_{12} = x_1y_2 - x_2y_1 - s_3z_1 + c_3z_2 \end{cases} \tag{14}$$

In particular, if the two systems (11) and (12) are chaotic, then the control law (13) guarantees their synchronization for any initial condition.

4 Conclusion

We have presented a universal nonlinear control law (without any conditions) for the synchronization of arbitrary 3-D continuous-time quadratic systems. This universal law (9) can be considered either as a stabilization, or a control, or as a chaotification approach for the system under consideration.

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