# An Improvement on Kim's Inequality for Pricing Asian Option

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**Abstract**
In his study of a simple one-dimensional parabolic PDE, Kim[11] obtained an inequality that the price of Asian options satisfies. This paper aims to improve this particular Kim's inequality. Our inequality has an advantage over Kim's, for our estimation has an upper bound as R vanishes. Therefore, our newly established estimation can help us to determine the solution error of a second order partial differential equation with variable coefficients.
**Keywords** Asian option, parabolic PDE, error estimation

## 1 Introduction

As a new financial product, Asian options, also known as average options, can be seen as innovative European options. The commonality with the European options is that investors are only allowed to exercise their option contracts on the maturity dates, while the difference is that investors of Asian options decide whether or not to exercise their option contracts based on the price level of the average share price during the contract term. Because the value of European options on the maturity date has nothing to do with the price path and depends only on the maturity date of the stock price, it is difficult to prevent speculators from manipulating the maturity price and consequently from arbitrage. On the other hand, because Asian options are associated with the price path, they can be applied to ease market behaviors. They represent the exotic options most actively traded in financial derivatives market today. Their difference with the usual-sense stock options is the implementation of price limits so that the exercise price is the average price of the stock price of the secondary market over a period of six months prior to exercise. Their difference with the standard options is that the options' maturity-date benefits are determined not by the prevailing market price of the underlying asset, but by the average price of the underlying asset over some time period during the option contract period. This time period is known as an average period, over which either the arithmetic or geometric mean is applied. Asian options can be well designed to avoid stock price manipulation and damage of company interests caused by insider trading. So, they are welcomed by both investors and issuers. Additionally, compared to the standard options, Asian options also possess such advantages as lower prices and can be applied to hedge the risk of a specified time period. Along with baskets, spreads, as well as other strategies, Asian options have been extensively studied by many scholars. Even so, few of published works deal with the problem of pricing general Asian options.

In terms of the pricing of Asian options, [1-4] represent some of the most recent results, while Dewynne *et al.* provide a simplified means of pricing Asian options using parabolic PDE[5]. A progressive solution is established by using Laplace transforms and power series expansions. Equations of Kolmogorov type have also turned out to be relevant in option pricing in the setting of certain models for stochastic volatility and in the pricing of Asian options. Frentz *et al.* numerically solve the Cauchy problem of a general class of second order degenerate parabolic differential operators of Kolmogorov type with variable coefficients by using posteriori error estimates and an algorithm developed for adaptive weak approximation of stochastic differential equations[6]. On the basis of these works, we show how to apply the relevant results in the context of mathematical finance and option pricing. The approach outlined in this paper circumvents many of the difficulties confronted by any deterministic approach based on, for example, a finite-difference discretization of the partial differential equation. Meanwhile, Frentz[7] also analyze the second order PDE operators arising in the pricing of Asian options, while proving the optimal interior regularity. Min Dai presents a lattice algorithm for pricing both European- and American-style moving average barrier options (MABOs)[8]. They develop a finite-dimensional partial differential equation (PDE) model for discretely monitoring MABOs and solve it numerically by using a forward shooting grid method. However, their modeling PDE for continuously monitored MABOs is of infinite dimensions and cannot be solved directly by using existing numerical methods. Recently, Bayraktar *et al.* construct a sequence of functions based on the value of Asian option[9]. As a result, each term of the sequence is the unique classical solution of a parabolic PDE so that they provide the relevant numerical approximation.

Deelstra *et al.* obtain an approximation formula by using comonotonic bounds[10], leading to four different approximations: the upper, the improved upper, the lower, and the intermediary bounds. In this way, they improve the traditional hybrid moment matching method. Their methods have the advantage that they can be applied in other frameworks, e.g., in Lévy settings, as well. These results can be adapted to deal with options written in a foreign currency, such as compo and quanto options. Kim studies a simple one-dimensional parabolic PDE that the price of Asian options satisfies[11]. The result indicates that the generalized solution is a classical solution and cannot be obtained. Additionally, Kim's inequality is shown to be unbounded as the parameter R vanishes. By improving Kim's inequality, our estimation has an upper bound as R vanishes.

## 2 Main Results

Kim (2009)[11] considers the following parabolic type equation

$$u_t + \frac{1}{2}[x - e^{-\int_0^t dv(s)} q(t)]^2 \sigma^2 u_{xx} = 0 \tag{1}$$

along with the boundary condition

$$u(T, x) = max(x - K_1, 0) \tag{2}$$

where $v(t)$ stands for dividend yield, $\sigma$ the volatility of the underlying asset, and $q(t)$ the trading strategy.

First let us introduce a lemma about Gauss estimation. Suppose that $g(x)$ is a continuous function in the interval $[-R, R]$ satisfying

$$\frac{1}{2} \leq g(x) \leq \frac{3}{2}, x \in [-R, R]$$

Denote

$$Q := \{(t, x) \in R^2 : 0 < t < 2, |x| < R\},$$
$$\Omega := \{(t, x) \in Q : t > g(x)\}, \Sigma := \{(t, x) \in Q : t = g(x)\}.$$

Lemma1[11] Let $\Omega$ and $\Sigma$ be defined as above and $a(t, x)$ a function satisfying

$$\begin{cases} Lu := u_t - a(t, x)u_{xx} = 0, (t, x) \in \Omega \\ u = 0, (t, x) \in \Sigma \end{cases}$$

Assume that $u \in C^{1,2}_{loc}(\Omega) \bigcap C(\bar{\Omega})$ and $a(t, x)$ satisfy

$$0 \leq a(t, x) \leq 1, \forall (t, x) \in \Omega$$

Then, we have the following estimate:

$$|u|_{0;\Omega'} \leq (\frac{16}{\sqrt{2\pi}})R^{-1}e^{-\frac{R^2}{32}} |u|_{0;\Omega}, \Omega' := \{(t, x) \in \Omega : |x| < \frac{R}{2}\} \tag{3}$$

Proposition 1 If $|x| < R$, then

$$\int_E \phi(2, x - y)dy < 2\int_0^{2R} \phi(2, y)dy + \frac{1}{\sqrt{2\pi}R} e^{-2R^2} \tag{4}$$

where $E = \bigcup_{j \in Z} ((4j + 1)R, (4j + 3)R)$.

Proof: First we prove that the following equality holds true:

$$\int_E \phi(2, x - y)dy = \sum_{j=1}^{\infty} \int_{(4j-3)R+x}^{(4j-1)R+x} \phi(2, y)dy + \sum_{j=0}^{\infty} \int_{(4j+1)R-x}^{(4j+3)R-x} \phi(2, y)dy$$

In fact,

$$
\begin{aligned}
&\int_E \phi(2,x-y)dy \\
&= \int_{\bigcup\limits_{j=1}^{\infty}((-4j+1)R,(-4j+3)R)} \phi(2,x-y)dy + \int_{\bigcup\limits_{j=0}^{\infty}((4j+1)R,(4j+3)R)} \phi(2,x-y)dy \\
&= \sum_{j=1}^{\infty}\int_{(-4j+1)R}^{(-4j+3)R} \phi(2,x-y)dy + \sum_{j=0}^{\infty}\int_{(4j+1)R}^{(4j+3)R} \phi(2,x-y)dy \\
&= \sum_{j=1}^{\infty}\int_{x-(-4j+1)R}^{x-(-4j+3)R} -\phi(2,y)dy + \sum_{j=0}^{\infty}\int_{x-(4j+1)R}^{x-(4j+3)R} \phi(2,y)dy
\end{aligned}
$$

$\because \quad \phi(2,y) = \frac{1}{\sqrt{8\pi}}e^{-\frac{y^2}{8}}$ is an even function,

$$
\begin{aligned}
\therefore \quad \int_E \phi(2,x-y)dy &= \sum_{j=1}^{\infty}\int_{(-4j+1)R-x}^{(-4j+3)R-x} \phi(2,y)dy + \sum_{j=0}^{\infty}\int_{(4j+1)R-x}^{(4j+3)R-x} \phi(2,y)dy \\
&= \sum_{j=1}^{\infty}\int_{(4j-3)R+x}^{(4j-1)R+x} -\phi(2,y)dy + \sum_{j=0}^{\infty}\int_{(4j+1)R-x}^{(4j+3)R-x} \phi(2,y)dy
\end{aligned}
$$

Secondly, we prove that when $-R < X < 0$, the following inequality holds true

$$
\sum_{j=1}^{\infty}\int_{(4j-3)R+x}^{(4j-1)R+x} \phi(2,y)dy \le \int_0^{2R} \phi(2,y)dy + \frac{1}{2\sqrt{2\pi}R}e^{-2R^2} \tag{5}
$$

In fact, when $-R < X < 0$,

$$
\begin{aligned}
\sum_{j=1}^{\infty}\int_{(4j-3)R+x}^{(4j-1)R+x} \phi(2,y)dy &< \sum_{j=0}^{\infty}\int_{4jR}^{(4j+2)R} \phi(2,y)dy \\
&= \int_0^{2R} \phi(2,y)dy + \sum_{j=1}^{\infty}\int_{4jR}^{(4j+2)R} \phi(2,y)dy
\end{aligned}
$$

$$
\begin{aligned}
\because \quad \sum_{j=1}^{\infty}\int_{4jR}^{(4j+2)R} \phi(2,y)dy &< \int_{4R}^{\infty} \phi(2,y)dy = \int_{4R}^{\infty} \frac{1}{\sqrt{8\pi}}e^{-\frac{y^2}{8}}dy \\
&< \frac{1}{\sqrt{8\pi}}\int_{4R}^{\infty} \frac{y}{4R}e^{-\frac{y^2}{8}}dy = \frac{1}{2\sqrt{2\pi}R}e^{-2R^2}
\end{aligned}
$$

$$\therefore \quad \sum_{j=1}^{\infty} \int_{(4j-3)R+x}^{(4j-1)R+x} \phi(2,y)dy \leq \int_{0}^{2R} \phi(2,y)dy + \frac{1}{2\sqrt{2\pi}R} e^{-2R^2}$$

Thirdly, we prove that as $0 < X < R$, we have

$$\sum_{j=0}^{\infty} \int_{(4j+1)R-x}^{(4j+3)R-x} \phi(2,y)dy \leq \int_{0}^{2R} \phi(2,y)dy + \frac{1}{2\sqrt{2\pi}R} e^{-2R^2} \tag{6}$$

In practice, as $0 < X < R$,

$$\begin{aligned}
\sum_{j=0}^{\infty} \int_{(4j+1)R-x}^{(4j+3)R-x} \phi(2,y)dy &\leq \sum_{j=0}^{\infty} \int_{4jR}^{(4j+2)R} \phi(2,y)dy \\
&= \int_{0}^{2R} \phi(2,y)dy + \sum_{j=1}^{\infty} \int_{4jR}^{(4j+2)R} \phi(2,y)dy \\
&\leq \int_{0}^{2R} \phi(2,y)dy + \frac{1}{2\sqrt{2\pi}R} e^{-2R^2}
\end{aligned}$$

Put (5) and (6) together, the proof is completed. QED.

Proposition 2, If $|x| < R$, then

$$\int_{E} \phi(2, x-y)dy < \sqrt{1-e^{-R^2}} + \frac{1}{\sqrt{2\pi}R} e^{-2R^2} \tag{7}$$

Proof: For arbitrary $R > 0$, we have inequality $\int_0^{2R} \phi(2,y)dy < \frac{1}{2}\sqrt{1-e^{-R^2}}$. In fact,

$$\begin{aligned}
\left(\int_{0}^{2R} \phi(2,y)dy\right)^2 &= \int_{0}^{2R} \int_{0}^{2R} \frac{1}{8\pi} e^{-\frac{x^2+y^2}{8}} dxdy \\
&\leq \iint_{D} \frac{1}{8\pi} e^{-\frac{x^2+y^2}{8}} dxdy \\
&= \frac{1}{4}(1-e^{-R^2})
\end{aligned} \tag{8}$$

where $D = \{(x,y)|x^2+y^2 \leq 8R^2, x \geq 0, y \geq 0\}$. Hence $\int_0^{2R} \phi(2,y)dy < \frac{1}{2}\sqrt{1-e^{-R^2}}$.

By substituting the above inequality in proposition 1, inequality (7) is obtained at once. QED.

Proposition 3 If $k > 1$, $R^2 < \frac{2ln2}{16k-9}$, then

$$\int_{4jR}^{(4j+2)R} \phi(2,y)dy \leq \int_{(8j-4)R}^{8jR} \phi(2,\sqrt{k}y)dy \tag{9}$$

Proof: After changing the variable $t=\frac{y+4R}{2}$, we have

$$\int_{(8j-4)R}^{8jR}\phi(2,\sqrt{k}y)dy=\frac{1}{\sqrt{8\pi}}\int_{(8j-4)R}^{8jR}e^{-\frac{ky^2}{8}}dy=\frac{1}{\sqrt{8\pi}}\int_{4jR}^{(4j+2)R}e^{ln2-\frac{k(y-2R)^2}{2}}dy$$

As $R^2<\frac{2ln2}{16k-9}$, $4jR<y<(4j+2)R$, $j=1,2,\cdots$, the following inequality holds true:$ln2-\frac{k(y-2R)^2}{2}+\frac{y^2}{8}\geq 0$.

Hence $\frac{1}{\sqrt{8\pi}}\int_{4jR}^{(4j+2)R}e^{-\frac{y^2}{8}}dy\leq\frac{1}{\sqrt{8\pi}}\int_{4jR}^{(4j+2)R}e^{ln2-\frac{k(y-2R)^2}{2}}dy$.

That is $\int_{4jR}^{(4j+2)R}\phi(2,y)dy\leq\int_{(8j-4)R}^{8jR}\phi(2,\sqrt{k}y)dy$. QED.

Proposition 4 If $k>1$, $R^2<\frac{2ln2}{16k-9}$, then

$$\int_E\phi(2,x-y)dy<2\int_0^{2R}\phi(2,y)dy+\frac{1}{\sqrt{2\pi}kR}e^{-2kR^2} \tag{10}$$

Proof: Combining the results of proposition 2 and proposition 3 leads to

$$\begin{aligned}\int_E\phi(2,x-y)dy&\leq 2\int_0^{2R}\phi(2,y)dy+2\sum_{j=1}^{\infty}\int_{4jR}^{(4j+2)R}\phi(2,y)dy\\&\leq 2\int_0^{2R}\phi(2,y)dy+2\sum_{j=1}^{\infty}\int_{(8j-4)R}^{8jR}\phi(2,\sqrt{k}y)dy\\&\leq 2\int_0^{2R}\phi(2,y)dy+2\int_{4R}^{\infty}\phi(2,\sqrt{k}y)dy\\&\leq 2\int_0^{2R}\phi(2,y)dy+\frac{1}{\sqrt{2\pi}}\int_{4R}^{\infty}\frac{y}{4R}e^{-\frac{ky^2}{8}}dy\\&=2\int_0^{2R}\phi(2,y)dy+\frac{1}{\sqrt{2\pi}kR}e^{-2kR^2}.QED.\end{aligned}$$

Proposition 5 If $k>1$, $R^2<\frac{2ln2}{16k-9}$, then $\int_E\phi(2,x-y)dy<\sqrt{1-e^{-R^2}}+\frac{1}{\sqrt{2\pi}kR}e^{-2kR^2}$.

Proof: By substituting inequality (8) in Proposition 4, one knows that inequality (10) is true. QED.

## 3 Conclusions

With the increasing globalization of investment in recent years, a variety of Asian options has obtained a wide range of applications and development. For example, Quanto option represents a contingent claim that the option's benefit depends on the prices of financial derivatives in a foreign currency, while the actual transaction of the option is paid in local currency. Quanto options, as a kind of option designed for avoiding different types of risks involved in international trades and investments, represent a financial derivative instrument. They can be used as a hedge in international trades against fluctuations in exchange rates. Therefore, with the rapid development of international trades and the globalization of the financial industry, this option will definitely attract more attention. The revenue function of Quanto option stands for a joint concern of foreign asset prices and exchange rates, creating many alternative investing and hedging opportunities. Additionally, other than Asian options, Power options represent yet another new class of options. They alter the pricing structure of assets and greatly increase the sensitivity of price over time. Power-Asian options are a unification of the two, making it possible for investors to better avoid risks. An example is the moving average call option. It has been often used to design a poison pill, a business strategy used to increase the likelihood of negative results over positive ones against a party that attempts a takeover. The moving average calls, issued to existing shareholders, would be triggered by the event of a hostile takeover. The French investment bank Compagnie Financiere Indosuez and the French construction company Bouygues have successfully issued such options/warrants to protect themselves against potentially unfriendly investors.

Kim's inequality is very useful for our work[12]. However, Kim's estimation is possibly unbounded as R vanishes. Our inequality, developed in this paper, has an advantage over Kims inequality, because our estimation is bounded on the top as R vanishes. Therefore, our work in this paper may help us to look for the solution error of a second order partial differential equation with variable coefficients.

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