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On Solvability of Equations Defined by Continuous and Smooth Regular Mappings

Sergey E. Zhukovskiy[1*]

[1] *V. A. Trapeznikov Institute of Control Sciences of RAS, Moscow, Russia*

Abstract: We consider equations defined by continuous mappings acting between finite-dimensional real vector spaces. It is assumed that the mappings are differentiable in the first variable. A regularity condition for this type of equations is obtained. It is shown that the regularity assumption implies the existence of solutions to the considered equations. Systems of two equations defined by continuous mappings acting between finite-dimensional real vector spaces are considered. It is assumed that the first mapping is differentiable in the first variable and the second mapping is differentiable in the second variable. A regularity condition for this type of systems is obtained. It is shown that the regularity assumption implies the existence of solutions to the considered system. The proofs of the main results of the paper are based on Brouwer's fixed point theorem and global implicit function theorem.

Keywords: nonlinear equations, regularity, covering, fixed point

1. INTRODUCTION

Given positive integers $n,\ k$ and a continuous mapping $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$, consider the following equation

$$f(x, x) = 0 \tag{1.1}$$

with unknown $x \in \mathbb{R}^n$. Our goal is to obtain sufficient conditions for the existence of a solution to this equations.

One of the standard approaches to this problem is based on the application of the covering mappings theory (see, for example, [1, 2]). The corresponding results guarantee that if f is covering in the first variable and is Lipschitz continuous in the second variable with a sufficiently small Lipschitz constant, then there exists a solution to equation (1.1). In the most general settings, these assertions provide sufficient solvability conditions for analogous equations defined by mappings acting between metric spaces. In this paper, we consider a specific case of finite-dimensional real linear spaces and differential mappings. We show that in this specific case, the assumption of Lipschitz continuity is redundant.

The proof of our main result is based on two assertions. One of them is the well-known Brouwer's fixed-point theorem (see, for example, Chapter II, §5.7 in [3]). The second is a global implicit function theorem (see Theorem 2 in [4]). In the second section of this paper, we recall this implicit function theorem as well as the related concepts and assertions. In the third section, we present solvability conditions for equation (1.1) and provide a proof of this result. The last section is devoted to a development of the main result to systems of equations.

*Corresponding author: s-e-zhuk@yandex.ru

2. PRELIMINARIES

Let us recall the concept of covering constant of a linear operator. Denote by $\mathcal{L}_{n\times k}$ the space of linear operators $A:\mathbb{R}^n\to\mathbb{R}^k$, denote by $\mathcal{SL}_{n\times k}$ the set of all surjective operators $A\in\mathcal{L}_{n\times k}$. Denote by $B_n(r)$ the closed ball in the space $\mathbb{R}^n$ centered at a point $x\in\mathbb{R}^n$ with a radius $r\ge 0$. Here and below we assume that $\mathbb{R}^n$ and $\mathbb{R}^k$ are equipped with norms which we denote by $|\cdot|$, and the space $\mathcal{L}_{n\times k}$ is equipped with the corresponding operator norm.

For a linear operator $A\in\mathcal{L}_{n\times k}$, put

$$\mathrm{cov}A := \sup\{\alpha\ge 0:\ B_k(\alpha)\subset AB_n(1)\}.$$

It is a straightforward task to ensure that $\mathrm{cov}A>0$ if and only if $A\in\mathcal{SL}_{n\times k}$.

Let us recall the global implicit function theorem from [4]. Given a topological space Σ and a mapping $f:\mathbb{R}^n\times\Sigma\to\mathbb{R}^k$, assume that for every $\sigma\in\Sigma$ the mapping $f(\cdot,\sigma):\mathbb{R}^n\to\mathbb{R}^k$ is differentiable. For $t\ge 0$, put

$$\alpha(t) := \inf\left\{\mathrm{cov}\frac{\partial f}{\partial x}(x,\sigma):\ x\in B_n(t),\quad \sigma\in\Sigma\right\}.$$

Theorem 2.1:
(see Theorem 2 in [4]) Assume that

(A1) *the mapping $f(\cdot,\cdot)$ is continuous on $\mathbb{R}^n\times\Sigma$, for every $\sigma\in\Sigma$ the mapping $f(\cdot,\sigma):\mathbb{R}^n\to\mathbb{R}^k$ is differentiable on $\mathbb{R}^n$, the mapping $\frac{\partial f}{\partial x}(\cdot,\cdot)$ is continuous on $\mathbb{R}^n\times\Sigma$.*

If

$$\int_0^{+\infty}\alpha(t)\,dt=+\infty \quad or \quad \sup_{\sigma\in\Sigma}|f(0,\sigma)|<\int_0^{+\infty}\alpha(t)\,dt,$$

then for every $\varepsilon>0$ there exists a continuous mapping $g:\Sigma\to\mathbb{R}^n$ such that

$$f(g(\sigma),\sigma)=0\quad\forall\,\sigma\in\Sigma,$$

$$\int_0^{|g(\sigma)|}\alpha(t)\,dt\le(1+\varepsilon)|f(0,\sigma)|\quad\forall\,\sigma\in\Sigma.$$

Below we also use the following corollary of Theorem 2.1.

Corollary 2.1:
Let f satisfies the assumption **(A1)**. *If there exists $\bar r>0$ such that*

$$\sup_{\sigma\in\Sigma}|f(0,\sigma)|<\alpha(\bar r)\bar r,$$

then for every $\varepsilon>0$ there exists a continuous mapping $g:\Sigma\to\mathbb{R}^n$ such that

$$f(g(\sigma),\sigma)=0\quad\forall\,\sigma\in\Sigma,$$

$$|g(\sigma)|\le\frac{(1+\varepsilon)|f(0,\sigma)|}{\alpha(\bar r)}\quad\forall\,\sigma\in\Sigma.$$

Note that in [4] these assertions were proved under more general assumptions. In particular, it was assumed that the domain of f in the variable x as well as the target space are Banach spaces. However, the considered here weak form of implicit function theorem from [4] is enough for the subsequent constructions.

3. SOLVABILITY CONDITION FOR EQUATIONS

Let us turn back to equation (1.1). Assume that for every $x_2 \in \mathbb{R}^n$ the mapping $f(\cdot, x_2)$ is differentiable. For $t > 0$ put

$$a(t,r) := \inf\left\{ \operatorname{cov}\frac{\partial f}{\partial x}(x_1, x_2) : \ x_1 \in B_n(t), \ \ x_2 \in B_n(r) \right\},$$

$$b(r) := \sup_{x_2 \in B_n(r)} |f(0, x_2)|.$$

Theorem 3.1:
Assume that

(A) *the mapping $f(\cdot,\, \cdot)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^n$, for every $x_2 \in \mathbb{R}^n$ the mapping $f(\cdot, x_2) : \mathbb{R}^n \to \mathbb{R}^k$ is differentiable on $\mathbb{R}^n$, the mapping $\frac{\partial f}{\partial x}(\cdot,\, \cdot)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^n$.*

If there exists $\bar{r} > 0$ such that

$$b(\bar{r}) < \int_0^{\bar{r}} a(t, \bar{r})\, dt, \tag{3.2}$$

then there exists a point $\bar{x} \in B_n(\bar{r})$ such that such that

$$f(\bar{x}, \bar{x}) = 0.$$

Proof
Apply Theorem 2.1 to the mapping f with $\Sigma = B_n(\bar{r})$. We have

$$\alpha(t) = a(t, r) \quad \forall\, r > 0.$$

Therefore, assumption (3.2) implies that

$$\sup_{x_2 \in B_n(\bar{r})} |f(0, x_2)| = b(\bar{r}) < \int_0^{r} a(t, \bar{r})\, dt = \int_0^{+\infty} \alpha(t)\, dt.$$

Take an arbitrary $\varepsilon > 0$ such that

$$(1 + \varepsilon) b(\bar{r}) < \int_0^{\bar{r}} a(t, \bar{r})\, dt.$$

It follows from Theorem 2.1 that there exists a continuous mapping $g : B_n(\bar{r}) \to \mathbb{R}^n$ such that

$$f(g(x_2), x_2) = 0, \qquad \int_0^{|g(x_2)|} a(t, \bar{r})\, dt \le (1 + \varepsilon)|f(0, x_2)| \quad \forall\, x_2 \in B_n(\bar{r}). \tag{3.3}$$

Obviously the function $a(\cdot, \bar{r})$ is decreasing. Thus, the inequality in (3.3) and the assumption (3.2) imply that $|g(x_2)| \le r$ for every $x_2 \in B_n(\bar{r})$. So,

$$g(x_2) \in B_n(\bar{r}) \quad \forall\, x_2 \in B_n(\bar{r}).$$

Therefore, by virtue of continuity of g Brouwer's fixed-point theorem implies that there exists a point $\bar{x} \in B_n(r)$ such that

$$\bar{x} = g(\bar{x}).$$

We have

$$f(\bar{x}, \bar{x}) = f(g(\bar{x}), \bar{x}) = 0.$$

So, the point $\bar{x}$ is the desired one. □

Let us derive a stronger but simpler solvability condition for the equation (1.1).

Corollary 3.1:
Let the assumption **(A)** *hold. If there exist* $\bar{\alpha} > 0$ *and* $\bar{\beta} \geq 0$ *such that*

$$\bar{\beta} < \bar{\alpha} \leq \operatorname{cov} \frac{\partial f}{\partial x}(x_1, x_2) \quad \forall\, x_1 \in \mathbb{R}^n, \quad \forall\, x_2 \in \mathbb{R}^n,$$

$$|f(0, x_2)| \leq |f(0,0)| + \bar{\beta}|x_2| \quad \forall\, x_2 \in \mathbb{R}^n,$$

then there exists a point $\bar{x} \in \mathbb{R}^n$ *such that*

$$f(\bar{x}, \bar{x}) = 0, \quad |\bar{x}| \leq \frac{|f(0,0)|}{\bar{\alpha} - \bar{\beta}}. \tag{3.4}$$

Proof
We have

$$a(t, r) \geq \bar{\alpha}, \quad b(r) \leq |f(0,0)| + \bar{\beta} r \quad \forall\, t > 0, \quad \forall\, r \geq 0.$$

Take

$$r_j := \frac{|f(0,0)|}{\bar{\alpha} - \bar{\beta}} + \frac{1}{j}, \quad j = 1, 2, \dots .$$

By construction we have

$$b(r_j) \leq |f(0,0)| + \bar{\beta}\frac{|f(0,0)|}{\bar{\alpha} - \bar{\beta}} + \frac{\bar{\beta}}{j} < \bar{\alpha}\left(\frac{|f(0,0)|}{\bar{\alpha} - \bar{\beta}} + \frac{1}{j}\right) = \int_0^{r_j} a(t, r_j)\, dt.$$

Therefore, Theorem 3.1 implies that there exists a point $\bar{x}_j \in B_n(r_j)$ such that $f(\bar{x}_j, \bar{x}_j) = 0$ for every $j = 1, 2, \dots$. By virtue of the compactness of $B_n(r_1)$ there exists a subsequence $\{\bar{x}_{j_i}\}$ of the sequence $\{\bar{x}_j\}$ which converges to a point $\bar{x}$. Obviously, the point $\bar{x}$ satisfies the inequality in (3.4). Passing to the limit in the equalities $f(\bar{x}_{j_i}, \bar{x}_{j_i}) = 0$ as i to ∞ we obtain that $f(\bar{x}, \bar{x}) = 0$. □

4. SOLVABILITY CONDITION FOR SYSTEMS OF EQUATIONS

Consider now the following system

$$\begin{cases} f_1(x_1, x_2) = 0, \\ f_2(x_1, x_2) = 0. \end{cases} \tag{4.5}$$

Here $f_1, f_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ are given mappings. Let us derive solvability conditions for the system (4.5) analogous to those in Section 3.

Assume that f_1 is differentiable in x_1 and f_2 is differentiable in x_2. Given numbers $\bar{r}_1 > 0$ and $\bar{r}_2 > 0$, denote

$$a_1 := \inf\left\{\mathrm{cov}\frac{\partial f_1}{\partial x_1}(x_1, x_2) : \, x_1 \in B_n(\bar{r}_1), \quad x_2 \in B_n(\bar{r}_2)\right\},$$

$$b_1 := \sup_{x_2 \in B_n(\bar{r}_2)} |f_1(0, x_2)|,$$

$$a_2 := \inf\left\{\mathrm{cov}\frac{\partial f_1}{\partial x_1}(x_1, x_2) : \, x_1 \in B_n(\bar{r}_1), \quad x_2 \in B_n(\bar{r}_2)\right\},$$

$$b_2 := \sup_{x_1 \in B_n(\bar{r}_1)} |f_2(x_1, 0)|.$$

Theorem 4.1:
Assume that mappings $f_1(\cdot, \cdot)$ and $f_2(\cdot, \cdot)$ are continuous on $\mathbb{R}^n \times \mathbb{R}^n$, for every $x_1, x_2 \in \mathbb{R}^n$ the mappings $f_1(\cdot, x_2), f_2(x_1, \cdot) : \mathbb{R}^n \to \mathbb{R}^k$ are differentiable on $\mathbb{R}^n$, the mappings $\frac{\partial f_1}{\partial x_1}(\cdot, \cdot)$ and $\frac{\partial f_2}{\partial x_2}(\cdot, \cdot)$ are continuous on $\mathbb{R}^n \times \mathbb{R}^n$.

If

$$b_1 < a_1\bar{r}_1, \quad b_2 < a_2\bar{r}_2, \tag{4.6}$$

then there exists a solution $(\bar{x}_1, \bar{x}_2) \in B_n(\bar{r}_1) \times B_n(\bar{r}_2)$ to the system (4.5), i.e.

$$\begin{cases} f_1(\bar{x}_1, \bar{x}_2) = 0, \\ f_2(\bar{x}_1, \bar{x}_2) = 0. \end{cases}$$

Proof
Take $\varepsilon > 0$ such that

$$\frac{(1+\varepsilon)b_1}{a_1} \le \bar{r}_1, \quad \frac{(1+\varepsilon)b_2}{a_2} \le \bar{r}_2.$$

The existence of such number ε follows from the assumption (4.6).

Since $b_1 < a_1\bar{r}_1$, applying Corollary 2.1 to $f = f_1$ and $\Sigma = B_n(\bar{r}_2)$ we obtain that there exists a continuous mapping $g_1 : B_n(\bar{r}_2) \to \mathbb{R}^n$ such that

$$f_1(g_1(x_2), x_2) = 0 \quad \forall\, x_2 \in B_n(\bar{r}_2),$$

$$|g_1(x_2)| \le \frac{(1+\varepsilon)|f_1(0, x_2)|}{a_1} \le \frac{(1+\varepsilon)b_1}{a_1} \le \bar{r}_1 \quad \forall\, x_2 \in B_n(\bar{r}_2).$$

Since $b_2 < a_2\bar{r}_2$, applying Corollary 2.1 to $f = f_2$ and $\Sigma = B_n(\bar{r}_1)$ we obtain that there exists a continuous mapping $g_2 : B_n(\bar{r}_1) \to \mathbb{R}^n$ such that

$$f_2(g_2(x_1), x_1) = 0 \quad \forall\, x_1 \in B_n(\bar{r}_1),$$

$$|g_2(x_1)| \le \frac{(1+\varepsilon)|f_2(x_1, 0)|}{a_1} \le \frac{(1+\varepsilon)b_2}{a_2} \le \bar{r}_2 \quad \forall\, x_1 \in B_n(\bar{r}_1).$$

Consider the mapping

$$g : B_n(\bar{r}_1) \to B_n(\bar{r}_1), \quad g(x_1) = g_1(g_2(x_1)), \quad x_1 \in B_n(\bar{r}_1).$$

This mapping is well-defined, since the above relations imply $g_2(x_1) \in B_n(\bar{r}_1)$ and $g_1(g_2(x_1)) \in B_n(\bar{r}_1)$ for all $x_1 \in B_n(\bar{r}_1)$. Moreover, g is continuous since it is a composition of continuous mappings g_1 and g_2. Therefore, it follows from Brouwer's fixed point theorem that there exists a point $\bar{x}_1 \in B_n(\bar{r}_1)$ such that $\bar{x}_1 = g(\bar{x}_1)$. Take $\bar{x}_2 := g_2(\bar{x}_1)$. Let us show that $(\bar{x}_1, \bar{x}_2)$ is a desired point.

Obviously $\bar{x}_2 \in B_n(\bar{r}_2)$. Moreover,

$$f_1(\bar{x}_1, \bar{x}_2) = f_1(g(\bar{x}_1), \bar{x}_2) = f_1(g_1(g_2(\bar{x}_1)), g_2(\bar{x}_1)) = 0,$$

$$f_2(\bar{x}_1, \bar{x}_2) = f_2(\bar{x}_1, g_2(\bar{x}_1)) = 0.$$

So, $(\bar{x}_1, \bar{x}_2)$ is a desired point. □

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REFERENCES

1. Arutyunov, A.V., Avakov, E.R. & Zhukovskiy, S.E. (2009) Covering mappings and their applications to differential equations unsolved for the derivative, *Differ. Equations*, **45**(5), 627–649.
2. Arutyunov, A., de Oliveira, V.A., Pereira, F.L., Zhukovskiy, E. & Zhukovskiy, S. (2015) On the solvability of implicit differential inclusions, *Appl. Anal.*, **94**(1), 129–143.
3. Granas, A. & Dugundji, J. (2003) *Fixed Point Theory*, N.Y., USA: Springer.
4. Arutyunov, A.V. & Zhukovskiy, S.E. (2021) On Global Solvability of Nonlinear Equations with Parameters, *Doklady Mathematics*, **103**(1), 57–60.

