Central Limit Theorems for the Single Point Catalytic Super-Brownian Motion with Immigration

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Abstract
We establish the central limit theorems for the single point catalytic super-Brownian motion with deterministic immigration and single point catalytic super-Brownian motion immigration on the Schwartz space and a weighted Sobolev space. For the catalytic immigration case, the weak convergence depends on the branching rates of both immigration part and non-immigration part.
Keywords Super-Brownian motion, catalytic, conditional log-Laplace functional, immigration, central limit theorem

1 Introduction and Main Results

Catalytic super-Brownian motion is the superprocess with Brownian motion as underlying spatial motion,whose branching occurs only in the presence of some catalysts. Dawson and Fleischmann [1] considered the case of single point catalytic super-Brownian motion, namely, the underlying particles move as independent Brownian motions in $\mathbb{R}$. The life time of each particle is exponentially distributed. When it dies, it splits according to critical branching only if they pass 0, which is called the (single) catalyst point. Fleischmann and Xiong [2], Yang and Zhang [3] and Li and Wang [4] proved the large deviation, moderate deviation and central limit theorem for the single point catalytic super-Brownian motion, respectively. If we suppose the situation where there are additional particles added, we need to consider the process with immigration. Superprocesses with immigration are studied by many authors; see e.g. [5-10].

In the present paper,first we will prove the central limit theorem for the single point catalytic super-Brownian motion with deterministic immigration on Schwartz space, and then extend the result to a weighted Sobolev space. We shall also investigate the processes with catalytic immigration. In this case, the weak convergence depends not only on the branching rate ϱ of non-immigration part, but also on the branching rate ϱ_0 of immigration part.

1.1 Notations and Preliminaries

First we introduce some notations. Let $p \geq 2$, $h_p(x) = (1 + x^2)^{-\frac{p}{2}}$ and $C_p(\mathbb{R})$ denote the set of all real-valued continuous functions φ on $\mathbb{R}$ such that $\varphi(x)/h_p(x)$ has a finite limit as $|x| \to \infty$. Equipped with the norm $|||\varphi|||_p := \sup\{|\varphi(x)|/h_p(x) : x \in \mathbb{R}\}, C_p(\mathbb{R})$ is a Banach space. Let $C_p^+(\mathbb{R})$ denote all the positive functions of $C_p(\mathbb{R})$ and $\mathbf{M}_p(\mathbb{R})$ be the set of all measures μ on $\mathbb{R}$ such that $\langle \mu, h_p \rangle < \infty$. Suppose that $|||\mu|||_p = \langle \mu, h_p \rangle$. Let $C^\infty(\mathbb{R})$ be the set of bounded infinitely differentiable functions on $\mathbb{R}$ with bounded derivatives. Let $\mathscr{S}(\mathbb{R}) \subset C^\infty(\mathbb{R})$ denote the Schwartz space of rapidly decreasing functions on $\mathbb{R}$ and $\mathscr{S}_+(\mathbb{R})$ be the collection of non-negative elements of $\mathscr{S}(\mathbb{R})$. That is, each $f \in \mathscr{S}(\mathbb{R})$ is infinitely differentiable and for each non-negative integer k and each non-negative integer α we have

$$\lim_{|x| \to \infty} x^k \frac{d^\alpha}{dx^\alpha} f(x) = 0.$$

Now we introduce some basic results in the following subsection, which can be found in [11]. We define the Hilbertian norms $\{q_0, q_1, q_2, \cdots\}$ on $\mathscr{S}(\mathbb{R})$ by

$$q_n(f)^2 = \sum_{k=0}^{n} \int_{\mathbb{R}} (1 + x^2)^n (f^{(k)}(x))^2 dx.$$

The Hermite polynomials on $\mathbb{R}$ are given by

$$g_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}, \quad k = 0, 1, 2, \cdots.$$

Based on those we define the Hermite functions

$$h_k(x) = \frac{1}{\sqrt[4]{\pi} \sqrt{2^k k!}} e^{-x^2/2} g_k(x), \quad k = 0, 1, 2 \cdots.$$

Then $h_k \in \mathscr{S}(\mathbb{R})$ and $\{h_k : k \geq 0\}$ is a complete orthonormal system in $L^2(\mathbb{R})$. Let $\langle \cdot, \cdot \rangle$ denote the inner product of $L^2(\mathbb{R})$. For $f \in \mathscr{S}(\mathbb{R})$ we write $f = \sum\limits_{k=0}^{\infty} \langle f, h_k \rangle h_k$ and define

$$\|f\|_n^2 = \sum_{k=0}^{\infty} (2k+1)^{2n} \langle f, h_k \rangle^2 \tag{1}$$

for $n = 0, \pm 1, \pm 2, \cdots$. Let $H_n(\mathbb{R})$ be the completion of $\mathscr{S}(\mathbb{R})$ with respect to $\| \cdot \|_n$. By approximation we can extend $\langle \cdot, \cdot \rangle$ to a bilinear form between $H_{-n}(\mathbb{R})$ and $H_n(\mathbb{R})$. Let $\langle \cdot, \cdot \rangle_n$ denote the inner product of $H_n(\mathbb{R})$. For $g, f \in H_n(\mathbb{R})$ we have

$$\langle g, f \rangle_n = \sum_{k=0}^{\infty} (2k+1)^{2n} \langle g, h_k \rangle \langle f, h_k \rangle = \langle \pi_n g, f \rangle,$$

where

$$\pi_n g = \sum_{k=0}^{\infty} (2k+1)^{2n} \langle g, h_k \rangle h_k \in H_n(\mathbb{R}).$$

Then $H_{-n}(\mathbb{R})$ and $H_n(\mathbb{R})$ are dual spaces with the duality $\langle \cdot, \cdot \rangle$.

Lemma 1.1 *For every $n \geq 0$ there is a constant $c(n) > 0$ such that*

$$q_n(f) \leq c(n)\|f\|_n \quad \text{and} \quad \|f\|_n \leq c(n)q_{2n}(f), \quad f \in \mathscr{S}(\mathbb{R}).$$

The sequence of norms defined by (1) induces a topology on the set $H_\infty := \bigcap_{k=0}^{\infty} H_n(\mathbb{R})$, which is compatible with the metric ρ defined by

$$\rho(f, g) = \sum_{k=0}^{\infty} \frac{\|f - g\|_k}{2^k(1 + \|f - g\|_k)}.$$

Then $(\mathscr{S}(\mathbb{R}), \rho)$ (written as $\mathscr{S}(\mathbb{R})$ for simplicity) is a nuclear space. This implies

$$\mathscr{S}'(\mathbb{R}) = \bigcup_{n=0}^{\infty} H_{-n}(\mathbb{R}) \supset \cdots \supset H_{-2}(\mathbb{R}) \supset H_{-1}(\mathbb{R}) \supset H_0(\mathbb{R})$$
$$\supset H_1(\mathbb{R}) \supset H_2(\mathbb{R}) \supset \cdots \supset \bigcap_{n=0}^{\infty} H_n(\mathbb{R}) = \mathscr{S}(\mathbb{R}).$$

A subset B of the nuclear space $\mathscr{S}(\mathbb{R})$ is said to be bounded if it is bounded in each norm $\|\cdot\|_n$, that is, $\sup_{x \in B} \|x\|_n < \infty$ for each $n \geq 0$. For each bounded set $B \subset \mathscr{S}(\mathbb{R})$ we define the semi-norm p_B on $\mathscr{S}'(\mathbb{R})$ by

$$p_B(f) = \sup\{|f(x)| : x \in B\}, \quad f \in \mathscr{S}'(\mathbb{R}).$$

We endow $\mathscr{S}'(\mathbb{R})$ with the topology generated by the collection of semi-norms $\{p_B : B \subset \mathscr{S}(\mathbb{R}) \text{ is bounded}\}$, which is called the strong topology. Then $\mathscr{S}'(\mathbb{R})$ is a nuclear space.

1.2 Models

For a process X taking its value in $\mathbf{M}_p(\mathbb{R})$, let $\mathbf{P}_{r,\nu}$ denote its conditional law given $X_r = \nu$. Suppose that p is the heat kernel in $\mathbb{R}$ with constant $\varsigma > 0$:

$$\frac{1}{\sqrt{2\pi\varsigma t}} \exp\left\{ -\frac{a^2}{2\varsigma t} \right\}, \quad t > 0,\ a \in \mathbb{R} \cdots. \tag{2}$$

For $\varphi \in C_p^+(\mathbb{R})$ fixed, $u(t, z; \varrho)$ denotes the unique non-negative solution to the log-Laplace equation

$$u(t, z; \varrho) = P_t\varphi(z) - \varrho \int_0^t p_{t-r}(z)u^2(r, 0; \varrho)dr, \quad t \geq 0,\ z \in \mathbb{R} \tag{3}$$

and $\omega(t,z;\varrho,\varrho_0)$ denotes the unique non-negative solution to the log-Laplace equation

$$\begin{aligned} &\omega(t,z;\varrho,\varrho_0) \\ =&\int_0^t P_{t-r}\left[u(r,\cdot;\varrho)\right](x)dr - \varrho_0\int_0^t p_{t-r}(z)\omega^2(r,0;\varrho,\varrho_0)dr, \quad t\geq 0,\ z\in\mathbb{R}\cdots. \end{aligned} \tag{4}$$

where ϱ and ϱ_0 are positive constants and $\{P_t : t\geq 0\}$ denotes the Brownian semigroup corresponding to (2). Let A be the generator of $\{P_t : t\geq 0\}$. In the following we suppose $\mu,\eta\in\mathbf{M}_p(\mathbb{R})$. We say $\xi^{\mu,\varrho}=\{\xi_t^{\mu,\varrho} : t\geq 0\}$ is a single point catalytic super-Brownian motion, if $\xi_0^{\mu,\varrho}=\mu$, and for $r\geq 0$ and $\nu\in\mathbf{M}_p(\mathbb{R})$,

$$-\log\mathbf{P}_{r,\nu}\exp\left\{-\langle\xi_t,\varphi\rangle\right\} = \langle\nu, u(t-r,\cdot;\varrho)\rangle, \quad 0\leq r\leq t,\ \varphi\in C_p^+(\mathbb{R})\cdots. \tag{5}$$

Suppose $Z=\{Z_t : t\geq 0\}$ is the single point catalytic super-Brownian motion with $Z_0=\mu$, deterministic immigration controlled by η and log-Laplace functional given by

$$\begin{aligned} &-\log\mathbf{P}_{r,\nu}e^{-\langle Z_t,\varphi\rangle} \\ =&\langle\nu, u(t-r,\cdot;\varrho)\rangle + \int_r^t\langle\eta, u(t-s,\cdot;\varrho)\rangle ds, \quad 0\leq r\leq t,\ \varphi\in C_p^+(\mathbb{R})\cdots. \end{aligned} \tag{6}$$

Now we suppose that $X^\xi=\{X_t^\xi : t\geq 0\}$ is the single point catalytic super-Brownian motion with single point catalytic immigration determined by ξ^{η,ϱ_0}. Let $\mathbf{P}_{0,\nu,\eta}$ denote its conditional law given $X_0^\xi=\nu$ and $\xi_0=\eta$. By Theorem 3.2 of [12] we have the log-Laplace functional of X^ξ:

$$\begin{aligned} &-\log\mathbf{P}_{0,\nu,\eta}\exp\left\{-\langle X_t^\xi,\varphi\rangle\right\} \\ =&-\log\mathbf{P}_{0,\nu,\eta}\left[\mathbf{P}_{0,\nu,\eta}\exp\left\{-\langle X_t^\xi,\varphi\rangle\right\}\middle|\{\sigma(\xi_s : 0\leq s\leq t)\}\right] \\ =&-\log\mathbf{P}_{0,\eta}\exp\left\{-\langle\nu, u(t,\cdot;\varrho)\rangle - \int_0^t\langle\xi_s, u(t-s,\cdot;\varrho)\rangle ds\right\} \\ =&\langle\nu, u(t,\cdot)\rangle + \langle\eta,\omega(t,\cdot;\varrho,\varrho_0)\rangle, \quad \mu,\eta\in\mathbf{M}_p(\mathbb{R}),\ t\geq 0,\ \varphi\in C_p^+(\mathbb{R})\cdots. \end{aligned} \tag{7}$$

In the following, we breviate $u(t,\cdot;\varrho)$ and $\omega(t,\cdot;\varrho,\varrho_0)$ by $u(t,\cdot)$ and $\omega(t,\cdot)$, respectively. We construct the superprocesses Z, ξ^{η,ϱ_0} and X^ξ on the same probability space $(\Omega,\mathcal{F},\mathbf{P})$.

1.3 Main results

Let $\{Z_t^{(k)} : t \geq 0\}$ be the single point catalyst super-Brownian motion with deterministic immigration characterized by(3) and (6), with ϱ replaced by $1/k$, Let

$$W_t^{(k)} = k^{\frac{1}{2}} \left(Z_t^{(k)} - \mu P_t - \int_0^t \eta P_s ds \right).$$

Theorem 1.2 *As $k \to \infty$, the sequence $\{W_t^{(k)} : t \geq 0\}$ converges weakly to the Gaussian process $\{W_t : t \geq 0\}$ in $C([0,\infty), \mathscr{S}'(\mathbb{R}))$ with $W_0 = 0$ and Laplace functional given by*

$$\begin{aligned} &\mathbf{P}\exp\left\{ -\langle W_t, f\rangle \right\} \\ =&\exp\left\{ \int_0^t \langle \mu, p_{t-r}(\cdot)\rangle P_r^2 f(0) dr + \int_0^t ds \int_0^s \langle \eta, p_{s-r}(\cdot)\rangle P_r^2 f(0) dr \right\}, \end{aligned}$$

where $f \in \mathscr{S}_+(\mathbb{R})$.

Remark 1.3 *If $\eta \equiv 0$ in Theorem* **??**, *the result can be found in [4].*

Theorem 1.4 *For any $n \geq 3$, $C([0,\infty), \mathscr{S}'(\mathbb{R}))$ can be replaced by $C([0,\infty), H_{-n}(\mathbb{R}))$ in Theorem* **??**.

Similarly, we can establish the central limit theorem for the single point catalytic super-Brownian motion with immigration controlled by another single point catalytic super-Brownian motion. The weak convergence depends on ρ and ρ_0, which are the branching rates of non-immigration and immigration parts. To specify the effects of ϱ and ϱ_0 on the convergence of X^{ξ}, we suppose that $\varrho = \gamma_1 k^{-1}$, $\varrho_0 = \gamma_2 k^{-\beta} (\beta > 0)$, where $k \in \mathbb{N}$, γ_1, γ_2 are positive constants. Let $\{\xi_t^{(k)} : t \geq 0\}$ be defined by (3) and (5) with ϱ_0 replaced by $\gamma_2 k^{-\beta} (\beta > 0)$. Let $\{X_t^{(k)} : t \geq 0\}$ be defined accordingly by (7), (3) and(4) with ϱ, ϱ_0 replaced by $\gamma_1 k^{-1}$, $\gamma_2 k^{-\beta} (\beta > 0)$, respectively. Define

$$Y_t^k = k^{\alpha} \left(X_t^{(k)} - \mu P_t - t\eta P_t \right) \tag{8}$$

Theorem 1.5 *As $k \to \infty$, the sequence $\{Y_t^{(k)} : t \geq 0\}$ converges weakly to the Gaussian process $\{Y_t : t \geq 0\}$ in $C([0,\infty), \mathscr{S}'(\mathbb{R}))$ with $Y_0 = 0$ and Laplace functional given by*

$$\mathbf{P}\exp\left\{ -\langle Y_t, f\rangle \right\} = \exp\left\{ \int_0^t F_s(f) ds \right\}, \quad f \in \mathscr{S}_+(\mathbb{R}) \tag{9}$$

where $F_s(f)$ is determined by the following table:

β	α	$F_s(f)$
$(0,1)$	$\frac{1}{2}\beta$	$\gamma_2\langle\eta, p_{t-s}(\cdot)\rangle\,(sP_s f(0))^2$
1	$\frac{1}{2}$	$\gamma_1\langle\mu, p_{t-s}(\cdot)\rangle\,(P_s f(0))^2 + \gamma_1\int_0^s\langle\eta, p_{s-r}(\cdot)\rangle\,(P_r f(0))^2\,dr$ $+\gamma_2\langle\eta, p_{t-s}(\cdot)\rangle\,(sP_s f(0))^2$
$(1,\infty)$	$\frac{1}{2}$	$\gamma_1\langle\mu, p_{t-s}(\cdot)\rangle\,(P_s f(0))^2 + \gamma_1\int_0^s\langle\eta, p_{s-r}(\cdot)\rangle\,(P_r f(0))^2\,dr$

Theorem 1.6 *For any $n\ge 3$, $C([0,\infty),\mathscr{S}'(\mathbb{R}))$ can be replaced by $C([0,\infty), H_{-n}(\mathbb{R}))$ in Theorem* **??**.

Since the proofs are similar, we only show the case $\beta=1$ of Theorems **??**–**??**.

2 Proofs of of Theorem ?? and Theorem ?? (case $\beta = 1$)

In the proof of Theorem **??**, we need the following proposition.

Proposition 2.1 *As $k\to\infty$, the finite dimensional distributions of $\{Y_t^{(k)}: t\ge 0\}$ converge weakly to a $\mathscr{S}'(\mathbb{R})$-valued Gaussian process $\{Y_t: t\ge 0\}$ with $Y_0=0$ and Laplace functional determined by (9).*

Proof. To simplify the notations, we consider the two dimensional distributions of $Y^{(k)}$. First we calculate the Laplace transform of $(\langle Y_{t_1}^{(k)}, f\rangle, \langle Y_{t_2}^{(k)}, f\rangle)$. By the Markov property, (7), (3), (8), [10] and [12], for $f\in C_p^+(\mathbb{R})$, $t_2>t_1>0$, we get the Laplace transform of $(\langle Y_{t_1}^{(k)}, f\rangle, \langle Y_{t_2}^{(k)}, f\rangle)$:

$$\begin{aligned}
&\log \mathbf{P}_{0,\mu,\eta}\exp\Big\{-\theta_1\langle Y_{t_1}^{(k)}, f\rangle - \theta_2\langle Y_{t_2}^{(k)}, f\rangle\Big\}\\
=&\log \mathbf{P}_{0,\mu,\eta}\Bigg[\mathbf{P}_{0,\mu,\eta}\Bigg(\exp\Big\{-\langle X_{t_1}^{(k)}, \theta_1 k^{\frac12} f + v^{(k)}(t_2-t_1,\cdot;\theta_2)\rangle\\
&-\int_{t_1}^{t_2}\langle \xi_s^{(k)}, v^{(k)}(t_2-s,\cdot;\theta_2)\rangle ds\Big\}\Bigg|\Big\{\sigma(\xi_s^{(k)}: s\le t_2)\Big\}\Bigg)\Bigg]\\
&+k^{\frac12}\Big(\langle\mu P_{t_1},\theta_1 f\rangle + \langle\mu P_{t_2},\theta_2 f\rangle + t_1\langle\eta P_{t_1},\theta_1 f\rangle + t_2\langle\eta P_{t_2},\theta_2 f\rangle\Big)\\
=&\log \mathbf{P}_{0,\eta}\exp\Big\{-\langle\mu, u^{(k)}(t_1,\cdot;\theta_1,\theta_2)\rangle - \int_0^{t_1}\langle\xi_s^{(k)}, u^{(k)}(t_1-s,\cdot;\theta_1,\theta_2)\rangle ds\\
&-\int_{t_1}^{t_2}\langle\xi_s^{(k)}, v^{(k)}(t_2-s,\cdot;\theta_2)\rangle ds\Big\} + \langle\mu P_{t_1}, k^{\frac12}\theta_1 f\rangle\\
&+\langle\mu P_{t_2}, k^{\frac12}\theta_2 f\rangle + t_1\langle\eta P_{t_1}, k^{\frac12}\theta_1 f\rangle + t_2\langle\eta P_{t_2}, k^{\frac12}\theta_2 f\rangle.
\end{aligned}$$

Using Markov property, (3) and (8), we get

$$\begin{aligned}
&\log \mathbf{P}_{0,\nu,\eta} \exp\Big\{ -\theta_1\langle Y_{t_1}^{(k)}, f\rangle - \theta_2\langle Y_{t_2}^{(k)}, f\rangle\Big\}\\
=&\log \mathbf{P}_{0,\eta} \exp\Big\{ -\langle \mu, u^{(k)}(t_1,\cdot;\theta_1,\theta_2) - \int_0^{t_1}\langle \xi_s^{(k)}, u^{(k)}(t_1-s,\cdot;\theta_1,\theta_2)\rangle ds\\
&-\langle \xi_{t_1}^{(k)}, \omega^{(k)}(t_2-t_1,\cdot;\theta_2)\rangle\Big\}\\
&+\langle \mu P_{t_1}, k^{\frac{1}{2}}\theta_1 f\rangle + \langle \mu P_{t_2}, k^{\frac{1}{2}}\theta_2 f\rangle + t_1\langle \eta P_{t_1}, k^{\frac{1}{2}}\theta_1 f\rangle + t_2\langle \eta P_{t_2}, k^{\frac{1}{2}}\theta_2 f\rangle\\
=&k^{-1}\gamma_1\int_0^{t_2-t_1}\langle \mu, p_{t_2-r}(\cdot)\rangle\big[v^{(k)}(r,0;\theta_2)\big]^2 dr\\
&+k^{-1}\gamma_1\int_0^{t_1}\langle \mu, p_{t_1-r}(\cdot)\rangle\big[u^{(k)}(r,0;\theta_1,\theta_2)\big]^2 dr\\
&+k^{-1}\gamma_2\int_0^{t_2-t_1}\langle \eta, p_{t_2-l}(\cdot)\rangle[\omega^{(k)}(l,0;\theta_1,\theta_2)]^2 dl\\
&+k^{-1}\gamma_2\int_0^{t_1}\langle \eta, p_{t_1-r}(\cdot)\rangle[s^{(k)}(r,0;\theta_1,\theta_2)]^2 dr\\
&+k^{-1}\gamma_1\int_0^{t_2-t_1}dl\int_0^{l}\langle \eta, p_{t_2-r}(\cdot)\rangle[v^{(k)}(r,0;\theta_2)]^2 dr\\
&+k^{-1}\gamma_1 t_1\int_0^{t_2-t_1}\langle \eta, p_{t_2-r}(\cdot)\rangle[v^{(k)}(r,0;\theta_2)]^2 dr\\
&+k^{-1}\gamma_1\int_0^{t_1}dl\int_0^{l}\langle \eta, p_{t_1-l}(\cdot)\rangle[u^{(k)}(r,0;\theta_1,\theta_2)]^2 dr,
\end{aligned}$$

where $\theta_1,\theta_2\geq 0$, $v^{(k)}(\cdot,\cdot;\theta_2)$ is the non-negative solution to

$$v^{(k)}(r,x;\theta_2) = k^{\frac{1}{2}}\theta_2 P_r f(x) - \gamma_1 k^{-1}\int_0^r p_{r-l}(x)\big[v^{(k)}(l,0;\theta_2)\big]^2 dl,$$

$u^{(k)}(\cdot,\cdot;\theta_1,\theta_2)$ is the non-negative solution to

$$\begin{aligned}
u^{(k)}(r,x;\theta_1,\theta_2) &= P_r\big[k^{\frac{1}{2}}\theta_1 f + v^{(k)}(t_2-t_1,\cdot;\theta_2)\big](x)\\
&\quad - \gamma_1 k^{-1}\int_0^r p_{r-l}(x)\big[u^{(k)}(l,0;\theta_1,\theta_2)\big]^2 dl,
\end{aligned}$$

$\omega^{(k)}(\cdot,\cdot;\theta_2)$ is the non-negative solution to

$$\omega^{(k)}(r,x;\theta_2) = \int_0^r P_{r-l}\big[v^{(k)}(l,\cdot;\theta_2)\big](x)dl - \gamma_2 k^{-1}\int_0^r p_{r-l}(x)\big[\omega^{(k)}(l,0;\theta_2)\big]^2 dl,$$

and $s^{(k)}(\cdot,\cdot;\theta_1,\theta_2)$ is the non-negative solution to

$$\begin{aligned} s^{(k)}(r,x;\theta_1,\theta_2) &= P_r\big[\omega^{(k)}(t_2-t_1,\cdot;\theta_2)\big](x)+\int_0^r P_{r-l}\big[u^{(k)}(l,\cdot;\theta_1,\theta_2)\big](x)dl \\ &\quad -\gamma_2 k^{-1}\int_0^r p_{r-l}(x)\big[s^{(k)}(l,0;\theta_1,\theta_2)\big]^2 dl. \end{aligned}$$

It is easy to show that $k^{-1/2}v^{(k)}, k^{-1/2}u^{(k)}, k^{-1/2}\omega^{(k)}$ and $k^{-1/2}s^{(k)}$ are all convergent as $k\to\infty$. Then

$$\begin{aligned} &\lim_{k\to\infty}\log\mathbf{P}_{0,\mu,\eta}\exp\Big\{-\theta_1\langle Y_{t_1}^{(k)},f\rangle-\theta_2\langle Y_{t_2}^{(k)},f\rangle\Big\} \\ =&\gamma_1\theta_1^2\int_0^{t_1}\langle\mu,p_{t_1-r}(\cdot)\rangle\big[P_rf(0)\big]^2dr+\gamma_1\theta_2^2\int_0^{t_2}\langle\mu,p_{t_2-r}(\cdot)\rangle\big[P_rf(0)\big]^2dr \\ &+2\gamma_1\theta_1\theta_2\int_0^{t_1}\langle\mu,p_{t_1-r}(\cdot)\rangle P_rf(0)P_{t_2-t_1+r}f(0)dr \\ &+\gamma_1\theta_1^2\int_0^{t_1}dl\int_0^l\langle\eta,p_{t_1-r}(\cdot)\rangle\big[P_rf(0)\big]^2dr \\ &+\gamma_1\theta_2^2\int_0^{t_2}dl\int_0^l\langle\eta,p_{t_2-r}(\cdot)\rangle\big[P_rf(0)\big]^2dr \\ &+2\gamma_1\theta_1\theta_2\int_0^{t_1}dl\int_0^l\langle\eta,p_{t_1-r}(\cdot)\rangle P_rf(0)P_{t_2-t_1+r}f(0)dr \\ &+\gamma_2\theta_1^2\int_0^{t_1}\langle\eta,p_{t_1-r}(\cdot)\rangle\big[rP_rf(0)\big]^2dr+\gamma_2\theta_2^2\int_0^{t_2}\langle\eta,p_{t_2-r}(\cdot)\rangle\big[rP_rf(0)\big]^2dr \\ &+2\gamma_2\theta_1\theta_2\int_0^{t_1}dl\int_0^l\langle\eta,p_{t_1-r}(\cdot)\rangle r(t_2-t_1+r)P_rf(0)P_{t_2-t_1+r}f(0)dr. \end{aligned}$$

Recalling (9), we can obtain the result by the method of [12] Page 110. □

In the following, for fixed interval $I:=[0,T], T>0$, we introduce the Banach space $C_p^I(\mathbb{R})$ of all continuous maps u of I into $C_p(\mathbb{R})$ equipped with the norm $\|u\|_p^I:=\sup\{\|\|u(t)\|\|_p : t\in I\}$. The proofs of the following Lemmas **??**, **??** and **??** are essentially similar to those of [1, Lemma 2.5.2], [1, Lemma 2.6.2] and [1, Lemma 3.2.1], so we only present the results and omit the proofs here:

Lemma 2.2 *There are two positive constants $\varepsilon_1,\varepsilon_2$, such that for $|\theta|<\varepsilon_1$, there is a unique solution $u=u_\theta\in C_p^I(\mathbb{R})$ to the following equation*

$$u(t,x)=\theta P_tf(x)-\int_0^t p_{t-r}(x)\left[u(r,0)\right]^2dr,\quad 0\le t\le T,\ x\in\mathbb{R}$$

satisfying $\|u\|_p^I<\varepsilon_2$.

Set

$$v(t,x) := \theta P_t f(x) - u(t,x), \quad 0 \le t \le T,\ x \in \mathbb{R},\ |\theta| < \varepsilon_1.$$

Let $v^{(n)}$ denote the nth derivative of v with respect to θ, taken at $\theta = 0$. Put

$$\|Sf\|_T := \sup\left\{|P_t f(0)| : t \in (0,T]\right\}, \quad f \in C_p(\mathbb{R}),\ T > 0.$$

Lemma 2.3 *For each $f \in C_p(\mathbb{R}),\ 0 \le t \le T,\ n \ge 2$,*

$$v^{(2)}(t,x) = 2\int_0^t p_{t-r}(x)\,[P_r f(0)]^2\,dr, \quad x \in \mathbb{R}$$

and there is a constant c_n such that

$$|||v^{(n)}(t)|||_p \le n! c_n \|Sf\|_T^n \alpha(t)^{n-1},$$

where $\alpha(t) = t^{\frac{1}{2}} + t^{\frac{p+1}{2}}$.

Lemma 2.4 *For each $k \ge 1,\ n \ge 2,\ f \in C_p(\mathbb{R})$,*

$$\begin{aligned}\mathbf{P}_{0,\mu,\eta}\left[\left|\langle Y_t^{(k)}, f\rangle\right|^2\right] &= 2\int_0^t \langle \mu, p_{t-s}(\cdot)\rangle (s^2+1)\,[P_s f(0)]^2\,ds \\ &\quad +2\int_0^t ds \int_0^s \langle \eta, p_{t-r}(\cdot)\rangle\,[P_r f(0)]^2\,dr,\end{aligned}$$

and there exists a constant C_n such that

$$\left|\mathbf{P}_{0,\mu,\eta}\left[\langle Y_t^{(k)}, f\rangle\right]^n\right| \le C_n t^{\frac{n}{4}} \|Sf\|_1^n \sum_{i=1}^{n-1} \big(\,|||\,\mu\,|||_p + |||\,\eta\,|||_p\,\big)^i.$$

By the proofs of Proposition **??** and [1, Lemma 3.2.1], for all $t_2 > t_1 > 0$ and $\varphi, \psi \in C_p(\mathbb{R})$ we have

$$\mathbf{P}_{0,\mu,\eta}\left[\langle Y_{t_1}^{(k)}, \varphi\rangle + \langle Y_{t_2}^{(k)}, \psi\rangle\right] = 0 \tag{10}$$

and

$$\begin{aligned}&(-1)^n \mathbf{P}_{0,\mu,\eta}\left[\langle Y_{t_1}^{(k)}, \varphi\rangle + \langle Y_{t_2}^{(k)}, \psi\rangle\right]^n \\ =&\langle k\mu, \bar{u}_k^{(n)}(t_1,\cdot,\varphi,\psi)\rangle + \langle k\eta, \bar{s}_k^{(n)}(t_1,\cdot,\varphi,\psi)\rangle + \sum_{2\le j\le n-2}\binom{n-1}{j}\Big[\langle k\mu, \bar{u}_k^{(n-j)} \\ &(t_1,\cdot,\varphi,\psi)\rangle + \langle k\eta, \bar{s}_k^{(n-j)}(t_1,\cdot,\varphi,\psi)\rangle\Big](-1)^j \mathbf{P}_{0,\mu,\eta}\left[\langle Y_{t_1}^{(k)}, \varphi\rangle + \langle Y_{t_2}^{(k)}, \psi\rangle\right]^j,\end{aligned} \tag{11}$$

where $n \geq 2$, $\bar{v}_k(\cdot,\cdot,\psi)$ is the non-negative solution to

$$\bar{v}_k(r,x,\psi) = k^{-\frac{1}{2}} P_r\psi(x) - \int_0^r p_{r-l}(x)\big[\bar{v}_k(l,0,\psi)\big]^2 dl,$$

$\bar{u}_k(\cdot,\cdot,\varphi,\psi)$ is the non-negative solution to

$$\bar{u}_k(r,x,\varphi,\psi) = P_r\big[k^{-\frac{1}{2}}\varphi + \bar{v}_k(t_2-t_1,\cdot,\psi)\big](x) - \int_0^r p_{r-l}(x)\big[\bar{u}_k(l,0,\varphi,\psi)\big]^2 dl,$$

$\bar{\omega}_k(\cdot,\cdot,\psi)$ is the non-negative solution to

$$\bar{\omega}_k(r,x,\psi) = \int_0^r P_{r-l}\big[\bar{v}_k(l,\cdot,\psi)\big](x)dl - \int_0^r p_{r-l}(x)\big[\bar{\omega}_k(l,0,\psi)\big]^2 dl,$$

and $\bar{s}_k(\cdot,\cdot,\varphi,\psi)$ is the non-negative solution to

$$\begin{aligned}\bar{s}_k(r,x,\varphi,\psi) &= P_r\big[\bar{\omega}_k(t_2-t_1,\cdot,\psi)\big](x) + \int_0^r P_{r-l}\big[\bar{u}_k(l,\cdot,\varphi,\psi)\big](x)dl \\ &\quad - \int_0^r p_{r-l}(x)\big[\bar{s}_k(l,0,\varphi,\psi)\big]^2 dl.\end{aligned}$$

Then by (10), (11) and calculations, we get the following estimate.

Lemma 2.5 *There exists a positive constant c_0 such that*

$$\mathbf{P}_{0,\mu,\eta}\left[\langle Y_t^{(k)}, P_h f\rangle - \langle Y_{t+h}^{(k)}, f\rangle\right]^6 \leq c_0 h^{\frac{3}{2}} \|Sf\|_1^6, 0 \leq t < t+h \leq 1,\ k \geq 1,\ f \in \mathscr{S}(\mathbb{R}).$$

Similarly to [1, Lemma 3.2.2], we have the sixth moment of the increments of the process $\{Y_t^{(k)} : t \geq 0\}$:

Lemma 2.6 *There is a positive constant c_0 such that*

$$\mathbf{P}_{0,\mu,\eta}\left[\langle Y_t^{(k)} - Y_s^{(k)}, f\rangle\right]^6 \leq c_0 (t-s)^{\frac{3}{2}}, \quad 0 \leq s < t \leq 1,\ k \geq 1,\ f \in \mathscr{S}_+(\mathbb{R}).$$

Proof. Applying the elementary inequality

$$|x+y|^n \leq 2^{n-1}(|x|^n + |y|^n), \quad x,y \in \mathbb{R},\ n \geq 0,$$

there are positive constants D_1, D_2, D_3 such that

$$\begin{aligned}&\mathbf{P}_{0,\mu,\eta}\left[\langle Y_t^{(k)} - Y_s^{(k)}, f\rangle\right]^6 \\ \leq\ & D_1\left(\mathbf{P}_{0,\mu,\eta}\left[\langle Y_s^{(k)}, P_{t-s}f - f\rangle\right]^6 + \mathbf{P}_{0,\mu,\eta}\left[\langle Y_s^{(k)} P_{t-s} - Y_t^{(k)}, f\rangle\right]^6\right) \\ \leq\ & D_2\left(\|S(P_{t-s}f - f)\|_1^6 + (t-s)^{\frac{3}{2}}\|Sf\|_1^6\right) \\ \leq\ & D_3\left(\|P_{t-s}f - f\|_\infty^6 + (t-s)^{\frac{3}{2}}\|f\|_\infty^6\right)\end{aligned}$$

by Lemmas **??** and **??**. Since $\frac{P_h f-f}{h} \longrightarrow \frac{\varsigma}{2}f'' = Af$ as $h \to 0$, $h \mapsto \frac{P_h f-f}{h}$ is strongly continuous on $h \in [0,1]$. This implies that there exists a positive constant D_4 such that $\|P_h f - f\|_\infty \le hD_4$ for $h \in [0,1]$. Let $c_0 = D_3(D_4^6 + \|f\|_\infty^6)$. We finish the proof. □

Lemma **??** and Kolmogorov's criterion lead to

Lemma 2.7 *For any $f \in \mathscr{S}(\mathbb{R})$, the sequence $\{\langle Y_t^{(k)}, f\rangle : t \ge 0; k \ge 1\}$ is tight in $C([0,\infty),\mathbb{R})$.*

Proof of Theorem **??**. By Lemma **??** and Proposition **??**, the result follows from [13, Theorem 6.15]. □

Proposition 2.8 *For each $k \ge 1$ and $f \in C_p(\mathbb{R})$,*

$$M_t^{(k)}(f) := \langle Y_t^{(k)}, f\rangle - \int_0^t \langle Y_s^{(k)}, Af\rangle ds - \int_0^t \left(\langle \xi_r^{(k)}, k^{\frac{1}{2}} f\rangle - \langle \eta, P_r k^{\frac{1}{2}} f\rangle\right) dr, \quad t \ge 0$$

is a continuous martingale.

Proof. Fix $k \ge 1$. By the Markov property of $\{X_t^{(k)} : t \ge 0\}$, for $t > s \ge 0$,

$$\begin{aligned}&\mathbf{P}_{0,\mu,\eta}\left(\langle X_t^{(k)}, f\rangle - \int_s^t \langle X_r^{(k)}, Af\rangle dr \Big| \mathscr{F}_s\right)\\ =&\langle X_s^{(k)}, P_{t-s}f\rangle + (t-s)\langle \xi_s^{(k)}, P_{t-s}f\rangle\\ &-\int_s^t \left(\langle X_s^{(k)}, P_{r-s}Af\rangle + (r-s)\langle \xi_s^{(k)}, P_{r-s}Af\rangle\right) dr.\end{aligned}$$

Then by [14, Proposition 1.5], we have

$$\begin{aligned}&\mathbf{P}_{0,\mu,\eta}\left(\langle X_t^{(k)}, f\rangle - \int_0^t \langle X_r^{(k)}, Af\rangle dr \Big| \mathscr{F}_s\right)\\ =&\langle X_s^{(k)}, f\rangle - \int_0^s \langle X_r^{(k)}, Af\rangle dr + \int_0^{t-s} \langle \xi_s^{(k)}, P_r f\rangle dr\end{aligned}$$

and

$$\langle \mu, P_t f\rangle + t\langle \eta, P_t f\rangle - \int_0^t \Big(\langle \mu, P_r Af\rangle + r\langle \eta, P_r Af\rangle\Big) dr = \langle \mu, f\rangle + \int_0^t \langle \eta, P_r f\rangle dr.$$

Recalling $Y_t^{(k)} = k^{\frac{1}{2}}(X_t^{(k)} - \mu P_t - t\eta P_t)$, we have

$$\mathbf{P}_{0,\mu,\eta}\left[M_t^{(k)}(f) \Big| \mathscr{F}_s\right] = M_s^{(k)}(f).$$

□

Lemma 2.9 *There is a locally bounded function $t \to C(t)$ on $[0,\infty)$ such that*

$$\sup_{k\geq 1} \mathbf{P}_{0,\mu,\eta}\left\{\sup_{0\leq t\leq T}\left|\langle Y_t^{(k)}, f\rangle\right|^2\right\} \leq C(T)\|f\|_2^2, \quad T\geq 0,\ f\in H_2(\mathbb{R}).$$

Proof. We only consider $f \in \mathscr{S}(\mathbb{R})$. For each $k \geq 1$,

$$\begin{aligned} &\mathbf{P}_{0,\mu,\eta}\left\{\sup_{0\leq t\leq T}\left|\langle Y_t^{(k)}, f\rangle\right|^2\right\} \\ \leq\ & 3\ \mathbf{P}_{0,\mu,\eta}\left\{\sup_{0\leq t\leq T} M_t^{(k)}(f)^2\right\} + 3\ \mathbf{P}_{0,\mu,\eta}\left\{\left[\int_0^T \left|\langle Y_s^{(k)}, Af\rangle\right| ds\right]^2\right\} \\ & +3\ \mathbf{P}_{0,\eta}\left\{\left[\int_0^T \left|\langle \xi_s^{(k)}, k^{\frac{1}{2}} f\rangle - \langle \eta, P_s k^{\frac{1}{2}} f\rangle\right| ds\right]^2\right\}. \end{aligned}$$

Using Hölder inequality, Fubini theorem and (5), we have

$$\begin{aligned} &\mathbf{P}_{0,\eta}\left\{\left[\int_0^T \left|\langle \xi_s^{(k)}, k^{\frac{1}{2}} f\rangle - \langle \eta, P_s k^{\frac{1}{2}} f\rangle\right| ds\right]^2\right\} \\ \leq& 2\ T\int_0^T ds\int_0^s \langle \eta, p_{s-r}(\cdot)\rangle\,(P_r f(0))^2\, dr. \end{aligned}$$

By Proposition **??**, $M_t^{(k)}(f)$ is a continuous martingale. Then by Doob's martingale inequality, Hölder inequality and Fubini theorem, we have

$$\begin{aligned} &\mathbf{P}_{0,\mu,\eta}\left\{\sup_{0\leq t\leq T} M_t^{(k)}(f)^2\right\} \\ \leq\ & 12\ \mathbf{P}_{0,\mu,\eta}\left\{\left|\langle Y_T^{(k)}, f\rangle\right|^2\right\} + 12\ T\left\{\int_0^T \mathbf{P}_{0,\mu,\eta}\left[\left|\langle Y_s^{(k)}, Af\rangle\right|^2\right] ds\right\} \\ & +24\ T\int_0^T ds\int_0^s \langle \eta, p_{s-r}(\cdot)\rangle\,(P_r f(0))^2\, dr. \end{aligned}$$

By Lemma **??**, there is a constant C such that

$$[P_r f(0)]^2 \leq \frac{1}{\sqrt{2\pi\varsigma r}} q_0(f)^2 \leq \frac{C}{\sqrt{2\pi\varsigma r}}\|f\|_0^2.$$

It is easy to show that $\langle \mu, p_{s-r}(\cdot)\rangle \leq \frac{1}{\sqrt{2\pi\varsigma(s-r)}}\left(\|\|\mu\|\|_p + \mu([-\sqrt{2}p\pi\varsigma s, \sqrt{2}p\pi\varsigma s])\right)$. Then by Lemmas **??** and **??**, there is a locally bounded function $t \to C_1(t)$ on

$[0, \infty)$ such that $\mathbf{P}_{0,\mu,\eta}\left[\left|\langle Y_s^{(k)}, f\rangle\right|^2\right] \leq C_2(s)\|f\|_0^2$ and $\mathbf{P}_{0,\mu,\eta}\left[\left|\langle Y_s^{(k)}, Af\rangle\right|^2\right] \leq C_2(s)\|f\|_2^2$ for any all $s \geq 0$. This finishes the proof. □

Proof of Theorem **??**. Using [13, Corollary 6.16], the result follows from Proposition **??**, Lemmas **??** and **??**. □

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