Adv Syst Sci Appl 2024; **03**; 104-113
Published online at https://ijassa.ipu.ru.

The Approximation Matrix Method and its Comparison with the Analytical Hierarchy Process by T. Saaty

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Abstract: The article presents an optimization method for the formation of quantitative weights of objects (importance of criteria, priorities of alternatives) according to the initial expert judgment matrix in multi-criteria selection problems. Since the matrix of pairwise comparisons can be considered as some perturbation of the multiplicative matrix, the proposed method is based on the approximation of the original matrix of pairwise comparisons by the multiplicative matrix according to the matrix criterion of minimum distances between matrices. There is a one-to-one mapping between the elements of the weight vectors and the elements of the multiplicative matrix. For the first time, using a specific example using the matrix criterion, a relative estimate of the approximate solution of the Analytical Hierarchy Process by T. Saaty concerning the optimal solution obtained by the approximation matrix method is given. On account of the approximation matrix method being mathematically justified and due to the simplicity of finding optimal solutions, it can be recommended instead of the Analytical Hierarchy Process by T. Saaty.

Keywords: multi-criteria choice, normalized object weights, expert judgment matrix, multiplicative matrix, matrix criterion.

1. INTRODUCTION

When solving applied problems of multicriteria choice on a set of objects (alternatives, management decisions, options) presented in the form of preference relations, the problem of their expert measurement in the quantitative scale of relations arises. To date, many approaches and methods have been proposed to solve this problem, based on the resulting preference relations to narrow the set of non-dominant alternatives, as well as on paired comparisons of objects (solutions, criteria) [1, 2], which do not always allow us to identify a single alternative or management solution without attracting additional information.

However, when solving applied problems related to the measurement of objects in expert scales, as well as the formation of local weights of criteria presented in the form of a hierarchical tree of the importance of criteria, expert methods of evaluating and ranking objects are usually used [3]. Direct methods of expert evaluation of criteria weights have found application in the planning methodology through relative indicators of technical evaluation (PATTERN). Experts are asked to evaluate the normalized local weights of criteria at each level of the hierarchy on a quantitative scale, and then the global weights are found by multiplying local weights along the branches of a multi-level criteria tree [4].

Another expert approach based on the matrix of paired comparisons to assign "weights" to a finite set of compared objects is the Analytical Hierarchy Process (AHP) by T. Saaty which is now firmly established in the theory and practice of multicriteria selection problems [5–7].

Following the Analytical Hierarchy Process, experts form a so-called matrix of paired comparisons (judgments) of objects $V = [v_{ij}]$, $i, j = \overline{1, n}$, in the scale of relations, and then find the right eigenvector $\vec{w} = (w_1, \ldots, w_n)^T$ of this matrix, corresponding to the maximum eigenvalue. The desired weight vector is a vector whose elements are normalized by the sum of the elements of the right eigenvector.

Since the calculation of the vectors of weights of objects (criteria and alternatives) is performed by a numerical (approximate) method, T. Saaty aware of this, introduced a special numerical indicator: consistency index - the compatibility index of the judgment matrix V and the multiplicative matrix W obtained based on the values of the eigenvector, in the form of the Hadamard product [9]:

$$S.I. = \frac{1}{n^2} e^T V \circ W^T e, \; e^T = (1,1, \ldots, 1),$$

where V is the original judgment matrix, and $W = [w_{ij}] = \left[\frac{\widetilde{w}_i}{\widetilde{w}_j}\right]$ is a multiplicative square matrix whose elements are determined from the normalized elements of the right eigenvector $\vec{w}$ of the judgment matrix V. In this case, the ratio takes place:

$$w_{ij} = \frac{\widetilde{w}_i}{\widetilde{w}_j} = \frac{w_i / \sum_{l=1}^{n} w_l}{w_j / \sum_{l=1}^{n} w_l} = \frac{w_i}{w_j}, i, j = \overline{1, n}.$$

The *S.I.* Index It characterizes the degree of confidence in the results obtained with the help of AHP and is interpreted as a kind of measure of the deviation of the initial perturbed judgment matrix V from the multiplicative one W. In the work of T. Saaty [8, p. 76], it is shown that if we perform the Hadamard matrix product, then the compatibility index takes the form $S.I. = \frac{\lambda_{\max}}{n} > 1$, where $\lambda_{\max}$ is the maximum eigenvalue of the matrix V. With a sufficiently close approximation to the unit value of the index, the matrix of paired comparisons V is "close" to the multiplicative matrix W. If the consistency index exceeds a certain "threshold" value, then it is impossible to conclude the proximity of these matrices, and therefore it is not recommended to use AI in such cases. The hierarchy analysis method and its applications are described in many reviews, monographs, scientific articles, as well as works popularizing this method [10–15].

However, it can be clearly stated that the method of hierarchy analysis by T. Saaty is approximate since the method of determining the weight vector is based on numerical (approximate) methods for calculating the roots of a polynomial. The problem of finding the roots of polynomials of degree $n \geq 5$ is unsolvable in radicals. Therefore, in T. Saaty's method, for the number of objects at least five, the procedure for finding the eigenvalues of the matrix of degrees of the superiority of the importance of criteria or preferences of alternatives is carried out using numerical approximate methods for finding the roots of a polynomial implemented in the package Expert Choice [16].

V.D. Noghin states that "the value of the compatibility index can only indirectly judge the magnitude of the final "model" error: it can never be precisely determined by anyone. This is the specificity of this heuristic approach" [10, p. 1194].

The article suggests a more efficient method of the approximation matrix (MAM) for forming optimal object weights based on the matrix criterion of distance to the original matrix of judgments than the method of analyzing hierarchies of T. Saaty.

2. STATEMENT OF THE PROBLEM OF APPROXIMATION OF MATRICES OF JUDGMENTS

In multi-criteria applied problems the aggregation mechanism is usually represented in the form of an additive candle of object ratings (alternatives, variants) $a_l \in A = \{a_l | l = \overline{1, n_A}\}$, according to criteria with weights of importance in the form [1]:

$$F(a_l, w_1, \ldots, w_n) = \sum_{j=1}^{n} w_j f_j(a_l), \sum_{j=1}^{n} w_j = 1,$$

where $f_j(a_l)$ is the object score a_l in the resulting scale according to the criterion $f_j, j = \overline{1,n}$;

$w_j = w(f_j)$ is the quantitative (normalized) weight f_j of the criterion.

Let us consider the formulation of the formation of object weights by the criterion of proximity to the original matrix of paired comparisons of a multiplicative matrix. Let us have as initial data: $V = [v_{ij}]$ – the initial expert matrix of judgments about the relative importance of objects (criteria) $f_j \in F, i,j = \overline{1,n}$; W_n – the set of multiplicative square matrices *n* of the nth order over the field of real numbers. As a measure of proximity $d(V,W)$ between the original $V = [v_{ij}]$ and the multiplicative $W = [w_{ij}]$ matrix, where $W \in W_n$, we take the square of the Euclidean l_2-norm equal to the difference of these matrices [9]:

$$d(V,W) = \|V - W\|_E^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}(v_{ij} - w_{ij})^2 . \tag{2.1}$$

Then the mathematical formulation of the problem of choosing a multiplicative matrix W that approximates the original matrix $V = [v_{ij}]$, i.e., the closest approach to the original expert judgment matrix, is reduced to minimizing the indicator (2.1) in the form

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(v_{ij} - w_{ij})^2 \rightarrow \min_{W \in W_n}, \tag{2.2}$$

provided that the matrix elements are multiplicative $W = [w_{ij}]$:

$$w_{ij} = w_{ik}w_{kj}, \forall\, i,j,k = \overline{1,n}. \tag{2.3}$$

Due to condition (2.3), it is not analytically possible to obtain a solution to the original problem using one of the classical optimization methods, for example, the Lagrange multiplier method. Let us use the properties of the multiplicative matrix and solve this problem by reducing the original problem to an equivalent problem.

3. SPECIAL LINEAR PROPERTIES OF A MULTIPLICATIVE MATRIX

To solve the original problem (2.2)–(2.3), it is necessary to find elements of the multiplicative $W = [w_{ij}], \forall i,j = \overline{1,n}$, matrix, that provides a minimum for the quadratic criterion $d(V,W)$ (2.1) under condition (2.3). It turns out that for a multiplicative matrix, the relationship between the elements of the columns of the matrix and the elements of the right eigenvector is valid. To do this, consider two statements. The work of B.G. Mirkin [17, p. 183–184] provided that the matrix $B = [b_{ij}], \forall\, i,j = \overline{1,n}$ is over traditional if there exists a positive vector $x = (x_1,\dots,x_N)$ such that that $b_{ik} = \frac{x_i}{x_k}$ and the vector x is a point of equilibrium process: $\bar{q}^t = \frac{1}{\lambda^t}B\bar{q}^t$, $t = 1, 2,\dots$, which in the limit leads to a private vector $\bar{q} = \lim_{t\to\infty} \bar{q}^t$. We show that the multiplicative matrix has several other properties that will be useful in reducing the original, analytically unsolvable problem (2.2)–(2.3) in the framework of classical optimization methods to an equivalent one.

3.1. The Relationship Between Normalized Column Elements of a Multiplicative Matrix and Elements of the Right Eigenvector

Theorem 3.1:

1. Between the elements w_{ij} of a multiplicative matrix $W = [w_{ij}], \forall i,j = \overline{1,n}$, and any pair (w_i, w_j) component of the right eigenvector $\vec{w} = (w_1,\dots,w_n)^T$ true bijective mapping $\frac{w_i}{w_j} \mapsto w_{ij}$, whose every attitude $\frac{w_i}{w_j}$ to one mapping element w_{ij} of the matrix W and back

$$w_{ij} \mapsto \frac{w_i}{w_j},$$

provided this is true equality:

$$w_{ij} = \frac{w_i}{w_j}, \qquad \forall\, i,j = \overline{1,n}. \tag{3.4}$$

2. The normalized elements $\widetilde{w}_{ij}$ *of the columns* $\vec{w}_j = (w_{1j}, \dots, w_{nj})^T$, $j = \overline{1,n}$, *coincide with each other and are equal to the normalized right eigenvector, i.e.*

$$\widetilde{\vec{w}}_j = \begin{pmatrix} \widetilde{w}_{1j} \\ \dots \\ \widetilde{w}_{nj} \end{pmatrix} = \begin{pmatrix} \widetilde{w}_1 \\ \dots \\ \widetilde{w}_n \end{pmatrix} = \widetilde{\vec{w}}, \qquad \forall\, j = \overline{1,n}, \tag{3.5}$$

where $\widetilde{w}_{ij} = \frac{w_{ij}}{\sum_{k=1}^{n} w_{ij}}$ *is the normalized element* j *of the j-th column of* $\vec{w}_j$*;* $\widetilde{w}_i = \frac{w_i}{\sum_{k=1}^{n} w_k}$ *is the normalized element of the i-th row of the right eigenvector* $\vec{w}$.

3. The multiplicative matrix has one basis row.

Proof. 1. Since by hypothesis the matrix $W = [w_{ij}]$ multiplicative then will provide that, if for any pair of numbers from $\{w_1, \dots, w_n\}$ the validity of the equation (3.4), then in this case the condition of multiplicative between the elements of the matrix $W = \left[\frac{w_i}{w_j}\right]$, i.e. thus there exists a one-to-one mapping between the elements:

$$w_{ij} \leftrightarrow \frac{w_i}{w_j}.$$

Indeed, if $w_{ik} = \frac{w_i}{w_k}$ and $w_{kj} = \frac{w_k}{w_j}$, then we have $w_{ik}w_{kj} = \frac{w_i}{w_k} \times \frac{w_k}{w_j} = \frac{w_i}{w_j} = w_{ij}$, that is, for all $i, j, k = \overline{1,n}$, the multiplicativity condition is satisfied.

2. Let the elements of the vector $\vec{w} = (w_1, \dots, w_n)^T$ satisfy equality (3.4). We show that $\vec{w} = (w1, \dots, w_n wn)^T$ is the right eigenvector of the matrix $W = \left[\frac{w_i}{w_j}\right], \forall i, j = \overline{1,n}$. Let us verify that $W\vec{w} = \lambda\vec{w}$ is valid:

$$W\vec{w} = \begin{pmatrix} w_1/w_1 & w_1/w_2 \dots & w_1/w_n \\ w_2/w_1 & w_2/w_2 \dots & w_2/w_n \\ \dots & \dots\dots & \dots \\ w_n/w_1 & w_n/w_2 \dots & w_n/w_n \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{pmatrix} = \begin{pmatrix} nw_1 \\ nw_2 \\ \dots \\ nw_m \end{pmatrix} = n \begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_m \end{pmatrix} = n\vec{w},$$

where $\lambda_{\max} = n$ is the maximum eigenvalue.

For arbitrary columns $\vec{w}_k$ и $\vec{w}_q$ wk *and* wq $(1 \le k, q \le n)$ of the matrice W, whose elements satisfy the multiplicativity condition w_{ij} (2.3), we normalize the elements.

Since $w_{iq} = w_{ik}w_{kq}$, then $w_{ik} = w_{iq}/w_{kq}$, whence for any column numbers k, q we have:

$$\widetilde{w}_{ik} = \frac{w_{ik}}{\sum_{i=1}^{n} w_{ik}} = \frac{w_{iq}/w_{kq}}{\sum_{i=1}^{n} w_{ik}} = \frac{w_{iq}}{\sum_{i=1}^{n} w_{ik}w_{kq}} = \frac{w_{iq}}{\sum_{i=1}^{n} w_{iq}} = \widetilde{w}_{iq}, \forall\ i = \overline{1,n},$$

i.e., the normalized components of the columns of the matrix coincide with each other. On the other hand:

$$\widetilde{w}_{iq} = \frac{w_{iq}}{\sum_{i=1}^{n} w_{iq}} = \frac{w_i/w_q}{\sum_{i=1}^{n} w_i/w_q} = \frac{w_i}{\sum_{i=1}^{n} w_i} = \widetilde{w}_i, \forall\, i = \overline{1,n}.$$

Thus, the normalized components of the matrix columns are also equal to the normalized elements of the right vector of the multiplicative matrix.

3. Since the matrix $W = \left[\frac{w_i}{w_j}\right]$ by linear transformations is reduced to the form:

$$\begin{pmatrix} w_1/w_1 & w_1/w_2 \ldots & w_1/w_n \\ w_2/w_1 & w_2/w_2 \ldots & w_2/w_n \\ \ldots & \ldots\ldots & \ldots \\ w_n/w_1 & w_n/w_2 \ldots & w_n/w_n \end{pmatrix} \sim \prod_{k=1}^{n} w_k \begin{pmatrix} 1/w_1 & 1/w_2 \ldots & 1/w_n \\ 1/w_1 & 1/w_2 \ldots & 1/w_n \\ \ldots & \ldots\ldots & \ldots \\ 1/w_1 & 1/w_2 \ldots & 1/w_n \end{pmatrix} \sim \prod_{k=1}^{n} w_k \begin{pmatrix} 1/w_1 & 1/w_2 \ldots & 1/w_n \\ 0 & 0 \ldots & 0 \\ \ldots & \ldots\ldots & \ldots \\ 0 & 0 \ldots & 0 \end{pmatrix},$$

the rank of the multiplicative matrix is equal to one: $\operatorname{rg} W = 1$, i.e., the multiplicative matrix has one basic row. The theorem is proved. ■

3.2. Reducing the Elements of the Matrix Columns to Integer Values and Equality to the Right Eigenvector

Theorem 3.2:

Let the elements $q_{ik}, i, k = \overline{1, n}$, of the multiplicative matrix $W = [q_{ik}]$, $\forall\, i, k = \overline{1, n}$, be represented in general by integers and rational numbers in the form of irregular fractions.

Then, if the elements of q_{ik} any column $\vec{q}_k$, $k = \overline{1, n}$, of the matrix is multiplied by the least common multiple n_k denominators of the rational elements of the column, obtained in the result of this procedure, the integer elements $z_{ik} = n_k q_{ik}$, $\forall\, k = \overline{1, n}$, the columns of the matrix coincide with each other on any line number, and will be equal to the integer elements w_i right eigenvector $\vec{w} = (w_1, \ldots, w_i, \ldots w_n)^T$ corresponding to its own maximum eigenvalue $\lambda_{max} = n$:

$$\begin{pmatrix} z_{1k} \\ \cdots \\ z_{nk} \end{pmatrix} = \begin{pmatrix} w_1 \\ \cdots \\ w_n \end{pmatrix}, 1 \le k \le n. \tag{3.6}$$

Proof. By theorem 1, the elements of any multiplicative inverse-symmetric matrix can be brought to mind (3.4), namely: $q_{ik} = \frac{w_i}{w_k}$, where w_k are positive integer numbers – the elements of the right eigenvector, the least common multiple of positive integers to rational denominators of the elements of such a multiplicative matrix is equal to $n_k = w_k, \forall\, k = \overline{1, n}$.

As a result, for k-th column $\vec{q}_k = \left(\frac{w_1}{w_k}, \ldots, \frac{w_n}{w_k}\right)^T$, we find

$$n_j \vec{q}_k = n_k \times \begin{pmatrix} w_1/n_k \\ \cdots \\ w_n/n_k \end{pmatrix} = w_k \times \begin{pmatrix} w_1/w_k \\ \cdots \\ w_n/w_k \end{pmatrix} = \begin{pmatrix} w_1 \\ \cdots \\ w_n \end{pmatrix} = \vec{w}, \forall\, k = \overline{1, n},$$

that is, we proved the correctness of (3.6). It is easy to verify that $\vec{w}$ is the right eigenvector of the matrix W, which was required to prove. The theorem is proved. ■

Example 1. Consider a multiplicative matrix

$$W = \begin{pmatrix} 1 & 2 & 3 & 5 \\ \frac{1}{2} & 1 & \frac{3}{2} & \frac{5}{2} \\ \frac{1}{3} & \frac{2}{3} & 1 & \frac{5}{3} \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & 1 \end{pmatrix} \tag{3.7}$$

For the columns of the matrix, the lowest common multiples are:

$n_1 = w_1 = 2 \times 3 \times 5 = 30, n_2 = w_2 = 3 \times 5 = 15,$
$n_3 = w_3 = 2 \times 5 = 10, n_4 = 2 \times 3 = 6.$

From here we come to the matrix: $\widehat{W} = \begin{pmatrix} 30 & 30 & 30 & 30 \\ 15 & 15 & 15 & 15 \\ 10 & 10 & 10 & 10 \\ 6 & 6 & 6 & 6 \end{pmatrix}$. It is easy to verify that the right eigenvector $\vec{w} = (30, 15, 10, 6)^T$ of the matrix W (3.7), corresponding to the eigenvalue $\lambda = 4$ and the normalized elements of the columns of the matrix coincide with each other and are equal to the elements of the normalized right vector of the matrix:

$$\tilde{\vec{w}} = \left(\frac{30}{61}, \frac{15}{61}, \frac{10}{61}, \frac{6}{61}\right)^T.$$

The original matrix can also be represented as $W = \left[\frac{w_i}{w_j}\right]$:

$$W = \begin{pmatrix} 30/30 & 30/15 & 30/10 & 30/6 \\ 15/30 & 15/15 & 15/10 & 15/6 \\ 10/30 & 10/15 & 10/10 & 10/6 \\ 6/30 & 6/15 & 6/10 & 6/6 \end{pmatrix}.$$

4. REDUCING THE ORIGINAL PROBLEM TO AN EQUIVALENT ONE

Since, following (3.5), the elements of the i-th rows of the normalized vector columns of the multiplier and vector matrix are equal to the i-th normalized component of the right proper vector, namely:

$$\tilde{w}_{i1} = \cdots = \tilde{w}_{ij} = \cdots = \tilde{w}_{in} = \tilde{w}_i, \tag{4.8}$$

where $\sum_{i=1}^{n} \tilde{w}_i = 1$, to we have: $w_{ij} = \frac{w_i}{w_j} = \frac{w_i / \sum_{k=1}^{n} w_k}{w_j / \sum_{k=1}^{n} w_k} = \frac{\tilde{w}_i}{\tilde{w}_j}, \forall i, j = \overline{1, n}.$

We show that the multiplicativity condition (2.3) for the normalized row elements (4.8) of the matrix is satisfied:

$$w_{ik} w_{kj} = \frac{\tilde{w}_i}{\tilde{w}_k} \times \frac{\tilde{w}_k}{\tilde{w}_j} = \frac{\tilde{w}_i}{\tilde{w}_j} = w_{ij}, \;\; i, j, k = \overline{1, n}.$$

Thus, finding the elements of the approximating matrix $W = [w_{ij}]$ of the problem (2.2)–(2.3) is equivalent to finding the elements of a normalized column vector:

$$\tilde{\vec{w}} = (\tilde{w}_1, \dots, \tilde{w}_i, \dots, \tilde{w}_n)^T.$$

As the target indicator of the approximation problem, we take the square of the difference between the normalized elements of the original and multiplicative matrices in the form:

$$\rho(\tilde{V}, \tilde{W}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\tilde{v}_{ij} - \tilde{w}_i\right)^2 \tag{4.9}$$

Then the mathematical formulation of the original problem is reduced to finding the normalized right column vector of the approximating matrix and providing the minimum criterion (4.9):

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\tilde{v}_{ij} - \tilde{w}_i\right)^2 \to \min_{(\tilde{w}_1, \dots, \tilde{w}_n)}, \tag{4.10}$$

where $\tilde{v}_{ij} = \frac{v_{ij}}{\sum_{i=1}^{n} v_{ij}}$ are the normalized elements of the matrix $V = [v_{ij}]$ of judgments, $j = \overline{1, n}$.

The reasonableness of the transition from the original perturbed matrix $V = [v_{ij}]$ to the normalized $\tilde{V} = [\tilde{v}_{ij}]$ is based on the fact that the rank and magnitude of the relation between the elements of the same vector column are preserved for the original and normalized matrix.

4.1. Theorem on the Optimal Solution:

Theorem 4.3:

The optimal solution to the problem (4.10) is a normalized vector

$$\vec{\widetilde{w}}_* = \begin{pmatrix} \widetilde{w}_1^* \\ \dots \\ \widetilde{w}_n^* \end{pmatrix},$$

the components of which are taken as the coefficients of the importance the criteria $\widetilde{w}_i^* = \widetilde{w}(f_i)$ *and calculated by the formula:*

$$\widetilde{w}_i^* = \frac{1}{n}\sum_{j=1}^{n} \tilde{v}_{ij},\ \forall\, i = \overline{1,n},$$

while delivering a minimum of the indicator (4.9).

The elements of the optimal approximation matrix $W_* = [w_{ij}^*]$ *are determined by the formula:*

$$w_{ij}^* = \frac{\widetilde{w}_i^*}{\widetilde{w}_j^*}.$$

As an estimate of the approximation matrix to the original judgment matrix, we take the Euclidean matrix norm [9]:

$$d(V, W_*) = \|V - W_*\|_E = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(v_{ij} - w_{ij}^*\right)^2} \tag{4.11}$$

The proof is trivial and is based on necessary and sufficient conditions for the existence of an optimal solution of a function of many variables.

4.2. Algorithm for Generating Optimal Weights for Criteria

Step 1. Normalization of elements of the original matrix of pairwise comparisons of the importance of criteria by the sum $\sum_{i=1}^{n} v_{ij}$ of elements columns of the original matrix of judgments: $\tilde{v}_{ij} = \frac{v_{ij}}{\sum_{i=1}^{n} v_{ij}}$ are the normalized elements of the judgment matrix $V = [v_{ij}]$.

Step 2. The calculation for each row of the normalized matrix $\tilde{V} = [\tilde{v}_{ij}]$ of judgments of its average value, we take it as the normalized weights of the criteria, namely:

$$\widetilde{w}_i^* = \frac{1}{n}\sum_{j=1}^{n} \tilde{v}_{ij},\ \ \widetilde{w}_i^* = \widetilde{w}(f_i)\ \forall\, i = \overline{1,n}.$$

Step 3. Recovery of the elements of the multiplicative matrix by the optimal normalized weights of the criteria: $W_{MAM} = \left(\frac{\widetilde{w}_i^*}{\widetilde{w}_j^*}\right)$.

Step 4. Estimation of the proximity between the elements of the original matrix of pairwise comparisons and the optimal multiplicative matrix by the formula $d(V, W_*)$ (4.11).

5. COMPARISON OF THE EFFECTIVENESS OF THE METHOD WITH THE ANALYTICAL HIERARCHY PROCESS BY T. SAATY

Let us compare the error of calculating the weights of the importance of objects (criteria, alternatives) by T. Saaty's method with the optimization method of the approximation matrix for the formation of object weights in multi-criteria problems. As multiplicative matrices consider the square matrix $W_{AHP} = \left(\frac{\tilde{w}_i}{\tilde{w}_j}\right)_{i,j=\overline{1,n}.}$ and $W_{MAM} = \left(\frac{\tilde{w}_i^*}{\tilde{w}_j^*}\right)_{i,j=\overline{1,n}.}$, in which the elements are determined by the vector of priorities, found by the method of analysis of hierarchies and using approximate matrices.

To do this, we will use the data from the example of buying a house by a family with average incomes, given in the work of T. Saaty (see [6, p. 41–44]).

The problem consists in choosing one house from three available alternatives {A, B, C} based on eight factors that serve as criteria for the multi-criteria selection problem.

Table 5.1 shows expert estimates of pairwise comparison of the importance of criteria and the priority vector calculated from the maximum eigenvalue following the method of hierarchy analysis by T. Saaty ($\lambda_{\max} = 8{,}811$, compatibility index $S.I. = \frac{8{,}811}{8} \approx 1{,}101$).

The summed elements of the judgment matrix and the vector of weights of the importance of criteria found from these data using the approximation matrix method are presented in Table 5.2.

Table 5.1. Initial matrix V of judgments and priority vector according to AHP [6, p. 42]

Factor (criteria)	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	Priority vector, $\vec{w}_{AHP}$
Size, f_1	1	5	3	7	6	6	1/3	1/4	0,175
Transport, f_2	1/5	1	1/3	5	3	3	1/5	1/7	0,062
Environment, f_3	1/3	3	1	6	3	4	1/2	1/5	0,103
Age, f_4	1/7	1/5	1/6	1	1/3	1/4	1/7	1/8	0,019
Yard, f_5	1/6	1/3	1/3	3	1	1/2	1/5	1/6	0,034
Facilities, f_6	1/6	1/3	1/4	4	2	1	1/5	1/6	0,041
State, f_7	3	5	2	7	5	5	1	1/2	0,221
Finance, f_8	4	7	5	8	6	6	2	1	0,348

Table 5.2. Normalized matrix $\tilde{V} = \frac{v_{ij}}{\sum_{i=1}^{n} v_{ij}}$ of judgments and a vector of priorities by MAM

Factor (criteria)	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	Priority vector, $\vec{w}_{MAM}$
Size, f_1	0,11	0,23	0,25	0,17	0,23	0,23	0,07	0,10	0,174
Transport, f_2	0,02	0,05	0,03	0,12	0,11	0,12	0,04	0,06	0,068
Environment, f_3	0,04	0,14	0,08	0,15	0,11	0,16	0,11	0,08	0,108
Age, f_4	0,02	0,01	0,01	0,02	0,01	0,01	0,03	0,05	0,021
Yard, f_5	0,02	0,02	0,03	0,07	0,04	0,02	0,04	0,07	0,038
Facilities, f_6	0,02	0,02	0,02	0,10	0,08	0,04	0,04	0,07	0,047
State, f_7	0,33	0,23	0,17	0,17	0,19	0,19	0,22	0,20	0,212
Finance, f_8	0,44	0,32	0,41	0,20	0,23	0,23	0,44	0,39	0,333

To evaluate the accuracy, we restore multiplicative matrices of pairwise relations by priority vectors:

$$\vec{w}_{AHP} = (0{,}175; 0{,}062; 0{,}103; 0{,}019; 0{,}034; 0{,}041; 0{,}221; 0{,}348), \tag{5.12}$$

$$\vec{w}_{MAM} = (0{,}174; 0{,}068; 0{,}108; 0{,}021; 0{,}038; 0{,}047; 0{,}212; 0{,}333), \tag{5.13}$$

obtained by the method of calculating the eigenvector and the method of approximating the matrix of pairwise comparisons using the minimum distance criterion, and compare the results.

Tables 5.3 and 5.4 present multiplicative matrices $\vec{w}_{AHP}, \vec{w}_{MAM}$, with priority vectors $\vec{w}_{AHP}$ (5.12) and $\vec{w}_{MAM}$ (5.13) formed from normalized weights.

Table 5.3. Multiplicative matrix W_{AHP}, formed from normalized weights of the priority vector $\vec{w}_{AHP}$

Factor (criteria)	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
Size, f_1	1,00	2,82	1,70	9,21	5,15	4,27	0,79	0,50
Transport, f_2	0,35	1,00	0,60	3,26	1,2	1,51	0,28	0,18
Environment, f_3	0,59	1,66	1,00	5,42	3,03	2,51	0,47	0,30
Age, f_4	0,11	0,31	0,18	1,00	0,56	0,46	0,09	0,05
Yard, f_5	0,19	0,55	0,33	1,79	1,00	0,83	0,15	0,10

Facilities, f_6	0,23	0,66	0,40	2,16	1,21	1,00	0,19	0,12
State, f_7	1,26	3,56	2,15	11,63	6,50	5,39	1,00	0,64
Finance, f_8	1,99	5,61	3,38	18,32	10,24	8,49	1,57	1,00

Table 5.4. Multiplicative matrix W_{MAM} formed from normalized weights of the priority vector $\vec{w}_{MAM}$

Factor (criteria)	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
Size, f_1	1,00	2,54	1,62	8,37	4,63	3,70	0,82	0,52
Transport, f_2	0,39	1,00	0,64	3,30	1,83	1,46	0,32	0,21
Environment, f_3	0,62	1,57	1,00	5,17	2,86	2,29	0,51	0,32
Age, f_4	0,12	0,30	0,19	1,00	0,55	0,44	0,10	0,06
Yard, f_5	0,22	0,55	0,35	1,81	1,00	0,80	0,18	0,11
Facilities, f_6	0,27	0,69	0,44	2,26	1,25	1,00	0,22	0,14
State, f_7	1,22	3,10	1,98	10,21	5,65	4,52	1,00	0,64
Finance, f_8	1,91	4,86	3,10	16,02	8,86	7,08	1,57	1,00

We find the values of the norm of the difference between the original matrices and the multiplicative matrices of relations formed from the values of the priority vector (the importance of objects):

$$d_{AHP} = \|V - W_{AHP}\| = \sqrt{200{,}66} \approx 14{,}2; d_{MAM} = \|V - W_{MAM}\| = \sqrt{139{,}61} \approx 11{,}8.$$

Let us determine the accuracy of the solution d_{AHP} by the method of T. Saaty concerning the optimal d_{MAM} obtained in the framework of the optimization problem by the criterion (4.11). Using the ε-approximation formula, we find that:

$$\varepsilon = \frac{|d_{AHP} - d_{MAM}|}{d_{MAM}} \times 100\ \% = \frac{14{,}2 - 11{,}8}{11{,}8} \times 100\ \% \approx 20{,}3\ \%.$$

Comparison of methods for the accuracy of obtaining the weights of objects for the original perturbed matrix of the 8th order is shown in Fig. 5.1.

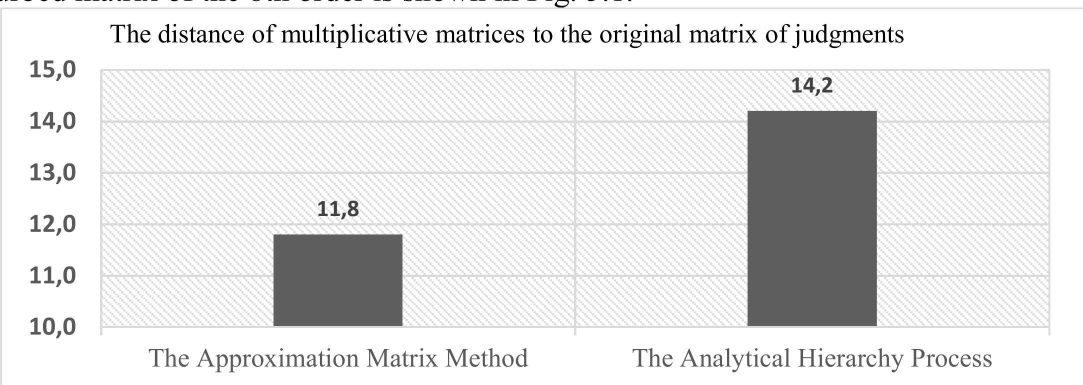

Fig 5.1. Comparison of methods based on the accuracy of obtaining object weights

It follows that the error estimate is 20,3 % and the AHP solution that differs from the optimal one by this value cannot be considered satisfactory. Thus, T. Saaty's method of finding priorities for the importance of criteria and objects based on the eigenvector of the matrix of pairwise comparisons should be attributed to approximate, and not to exact, as the author of AHP declares.

7. CONCLUSION

The paper explores the problem of forming quantitative weights of objects based on the matrix of pairwise comparisons in the relationship scale. Since in the hierarchy analysis process of T. Saaty, the method of determining the weight vector is carried out using polynomials, the problem of finding the roots of polynomials of degree $n \geq 5$ is unsolvable in radicals.

Therefore, in T. Saaty's method, for the number of objects at least five, the procedure for finding the eigenvalues of the matrix of degrees of the superiority of the importance of criteria or preferences of alternatives is carried out using numerical methods for finding the roots of a polynomial implemented in the package Expert Choice [17].

On the other hand, and because of the inverse symmetry of the elements of the matrix of judgments $V = [v_{ij}]$, $i, j = \overline{1, n}$, obtained by sequential comparison of all pairs of objects, the expert has to answer $\frac{n(n-1)}{2}$ questions about the values v_{ij}. In this regard, the T. Saaty's method is justified only for a small number of criteria and objects.

In this article, using a concrete example, it is shown that the analytical hierarchy method is approximate and at the same time, its error is estimated relative to the optimal solution obtained by the method of the approximation matrix for the formation of object weights in multi-criteria problems. Since the presented method is mathematically justified, and also because of the computational simplicity of forming object weights, it can be recommended instead of the method T. Saaty in solving applied problems.

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