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A Geometric Interpretation of Gravity Theory

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Abstract

A system of postulates which give a geometric meaning to the fundamental notions of gravity theory is introduced. The fundamental equation of geometric gravity theory is derived. The consistency of the system of postulates is shown. A series of examples adequately describing the gravitational field are considered. Mathematically rigorous definitions of a black hole and dark energy are given.

Keywords pseudo-Riemannian space, scalar curvature, metric gravity equation, spherically symmetric spatial black hole, dark energy

1 Introduction

Hilbert's sixth problem of axiomatizing those branches of physics in which mathematics is prevalent, posed by Hilbert in 1900 among other problems, has the status of being too vague. However, in the context of some particular area of physics, such as gravity theory, this problem can be stated rigorously.

The foundation of physical gravity theory is two fundamental physical concepts, of the gravitational field and of the mass of a body. In general relativity theory (GRT), which is essentially relativistic gravity theory, a substantial progress in the mathematical interpretation of one of the basic concepts of the theory, namely, that of the gravitational field, has been made. From the geometrical point of view, the gravitational field is interpreted as a metric pseudo-Riemannian 4-space. However, GRT does not provide such a precise mathematical definition of mass, or, to be more precise, the distribution density of mass. Thus, the GRT fundamental equation contains both a purely mathematical left-hand side, which is generated by a metric space, and a purely physical right-hand side, which is the energy-momentum tensor of the physical system under consideration. Note that GRT imposes no constraints on the choice of the energy-momentum tensor; this leads to the possibility of constructing unrealistic models and, thereby, provides evidence for the insufficiency of the principles on which GRT is founded.

Our purpose in this paper is to present a complete translation of gravity theory into the language of differential geometry, in which the physical notion of the mass distribution density has a purely geometric interpretation.

2 The First Postulates of Mathematical Gravity Theory

The notion of a metric space is the basis on which mathematical gravity theory is constructed. To construct the theory, it suffices to choose a pseudo-Riemannian 4-space of signature $(---+)$, on which a twice covariant symmetric nondegenerate tensor field $g_{ij}(x_1, K, x_4)$ is defined; we have

$$det\,|g_{ij}| \neq 0, g_{ij} = g_{ji}, i, j = 1, \ldots, 4$$

We refer to the tensor as the metric of the pseudo-Riemannian space. General requirements to a metric space are as follows: (1) smoothness, i.e., the continuity of all components of the metric on the entire space and the continuous differentiability of the components up to the second order with respect to all variables almost everywhere except, possibly, on singular sets; (2) the preservation of the metric signature at each point of the space.

Note that the metric smoothness condition is stated in a somewhat relaxed form in order to make it possible to consider pseudo-Riemannian spaces with discontinuous scalar curvature.

Note that the space structure and dimension chosen above are not regarded to be final. They are only sufficient for constructing geometric gravity theory; thus, we introduce them as sufficient conditions for constructing an adequate theory rather than as fundamental postulates.

We proceed to state two postulates of geometric gravity theory. As mentioned above, the main physical objects of gravity theory are the gravitational field and the distribution density of the matter mass. The objective of gravity theory is determining laws governing the interaction of the gravitational field with the distribution density of gravitational mass.

The first postulate of geometric gravity theory can be stated as follows.

Postulate 1. The gravitational field is a metric of a pseudo-Riemannian space.

This assertion is the basis of GRT and does not need any comments. Before stating the second postulate, we introduce the following notation[1-2]:

$\Gamma^k_{ij}$is the pseudo-Riemannian connection, which is defined by

$$\Gamma^k_{ij} = -\frac{1}{2}g^{kl}(\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^l}); \tag{1}$$

R_{ij} is the Ricci tensor, that is,

$$R_{ij} = \frac{\Gamma^k_{ij}}{\partial x^k} - \frac{\Gamma^k_{ik}}{\partial x^j} + \Gamma^k_{pk}\Gamma^p_{ij} - \Gamma^k_{pj}\Gamma^p_{ik}; \tag{2}$$

R is the scalar curvature, that is,

$$R = g^{ij}R_{ij}; \tag{3}$$

where the summation over repeated indices is implied; and is the contravariant metric tensor.

Postulate $2'$ (weak statement). If the mass density of the matter is nonzero at some point of the space, then so is the scalar curvature of the space at this point, and vice versa.

This postulate says nothing about the form of the dependence between these scalar functions. It only indicates the existence of a relation between them. To determine this relation, we use two principles; namely, the principle of minimal action and the correspondence principle.

These principles make it possible to derive the fundamental equation of geometric gravity theory and refine the statement of Postulate 2.

3 The Third Postulate, the Fundamental Equation of Gravity Theory, and the Strengthened Statement of Postulate $2'$

According to Postulate $2'$, there is a dependence between the scalar function of the matter mass density and the function of the space scalar curvature. To find this dependence, we introduce a composite scalar function $\chi(\rho)$, which depends on the matter mass density ρ in an unknown way, and a scalar curvature function $R(g_{ij})$, which is a composite function depending on the metric g_{ij}.

The general form of the gravitational field equations can be obtained by applying the principle of minimal action. The field equations are obtained as the EulerCLagrange equations under the variation of the field action. For the field action functional we take the quantity

$$S_g = \int_\Omega (R+\chi)\sqrt{|g|}d^4x, \qquad (4)$$

where $d\Omega = \sqrt{|g|}d^4x$ is the standard volume 4-form on the pseudo-Riemannian space,$g = det(g_{ij})$, and the integral is the whole 3-space (x^1, x^2, x^3) and over the interval $x_1^4 \le x^4 \le x_2^4$ of the time coordinate x_4.

Postulate 3. In geometric gravity theory, the following relation holds:

$$\frac{\delta S_g}{\delta g^{ij}} = 0.$$

Postulate 3 means that the sought dependence between the composite functions $R(g)$ and $\chi(\rho)$ minimizes the action functional (4) with respect to the contravariance metric of the pseudo-Riemannian space g^{ij} .

Theorem 1. If Postulate 3 holds for functional (4), then

$$R_{ij} - \frac{1}{2}Rg_{ij} = \frac{1}{2}\chi g_{ij}. \qquad (5)$$

Proof. Relation (4) implies

$$\frac{\delta S_g}{\delta g^{ij}} = \frac{\delta \int_\Omega R(g)\sqrt{|g|}d^4x}{\delta g^{ij}} + \frac{\delta \int_\Omega \chi(\rho)\sqrt{|g|}d^4x}{\delta g^{ij}}.$$

The first term is the Hilbert variational derivative[3]

$$\frac{\delta \int_{\Omega} R(g)\sqrt{|g|}d^4x}{\delta g^{ij}} = R_{ij} - \frac{R}{2}g_{ij}.$$

The second term can be represented in the form

$$\frac{\delta \int_{\Omega} \chi(\rho)\sqrt{|g|}d^4x}{\delta g^{ij}} = \frac{\int_{\Omega}(\delta\chi(\rho)\sqrt{|g|} + \chi(\rho)\delta\sqrt{|g|})d^4x}{\delta g^{ij}}.$$

Taking into account the relation $\delta\sqrt{|g|} = -\frac{1}{2}\sqrt{|g|}g_{ij}\delta g^{ij}$ and the fact that the variation of the scalar.

function $\chi(\rho)$, which does not depend on the metric g_{ij}, vanishes, we obtain

$$\frac{\delta \int_{\Omega} \chi(\rho)\sqrt{|g|}d^4x}{\delta g^{ij}} = -\frac{\chi(\rho)}{2}g_{ij}.$$

It follows that $\frac{\delta S_g}{\delta g^{ij}} = R_{ij} - \frac{R}{2}g_{ij} - \frac{\chi(\rho)}{2}g_{ij} = 0$, which completes the proof of the theorem.

Equation (5) implies the presence of a linear dependence between the functions R and χ. Indeed, multiplying the right- and left-hand sides of equation (5) by the contravariant metric tensor g^{ij} and convolving both sides over the indices i and j, we obtain the relation

$$\chi = -\frac{R}{2}. \tag{6}$$

According to (6), the scalar curvature R turns out to depend on the matter mass density ρ. However, we do not know the particular form of this dependence so far. To determine it, we apply the correspondence principle, which can be stated as follows: If the metric g_{ij} weakly converges to the Minkowski metric η_{ij} and, moreover, $g_{44} = 1 + \frac{a\varphi}{c^2} + o(\frac{1}{c^3})$ and $g_{ij} = -\delta_{ij} + o(\frac{1}{c^3})$ for $i, j = 1, \ldots, 4, i \neq j$, where c^{-1} is a small parameter and φ is a twice differentiable function, then equation (5) must degenerate into the Poisson equation for Newtonian gravity theory, which is

$$\Delta\varphi = 4\pi G\rho, \tag{7}$$

where Δ is the Laplace operator, φ is the Newtonian gravitational potential, G is the gravitational constant, and c is the speed of light. Simple calculations show that equation (5) does degenerate into the Poisson equation (7) under the condition

$$\rho = \frac{c^2}{32\pi G}R. \tag{8}$$

Relation (8) can be regarded as a stronger statement of Postulate 2'.

Postulate 2 (strong statement). The matter mass distribution density in space is directly proportional to the scalar curvature of the pseudo-Riemannian space: $\rho = æR$, where $æ = \frac{c^2}{32\pi G}$.It follows from Postulate 2 and equation (5) , based on Postulate 3, that the fundamental equation of gravity theory can be represented in the form

$$R_{ij} - \frac{1}{2}Rg_{ij} = -\frac{8\pi G}{c^2}\rho g_{ij}, i,j = 1,\dots,4, \tag{9}$$

or in the form of the system of two relations

$$R_{ij} = \frac{R}{4}g_{ij}, i,j = 1,\dots,4, \tag{10}$$

$$R = \frac{32\pi G}{c^2}\rho$$

the first of which is a direct consequence of Postulate 3 and the second, of Postulate 2.

Thus, physical theory of gravitational fields can be translated into the language of differential geometry. According to Postulates 1 and 2, the fundamental physical concepts of gravitational theory, such as gravitational field and matter mass density, are interpreted in the geometric as the metric and the scalar curvature (up to proportionality), respectively, of a pseudo-Riemannian space.

4 Consistency and Physical Adequacy of Postulates 1-3

To verify the consistency of the system of postulates stated above and the physical adequacy of these postulates, consider a model of a spherically symmetric space with a ball of radius r_1 at the center of symmetry. The ball is uniformly filled with a matter of mass ρ and constant density; outside the ball, there is no matter. From the geometric point of view, we have a spherically symmetric pseudo-Riemannian space with constant scalar curvature $R = \frac{32\pi G}{c^2}\rho$ inside a ball of radius r_1 and vanishing scalar curvature outside the ball. The problem is to determine the metric of such a space.

Hereafter, we always assume that the system units used for measuring physical quantities is chosen so that $G, c = 1$.

The general form of a stationary spherically symmetric metric of a pseudo-Riemannian 4-space in spherical coordinates t, r, θ, φ is[4]

$$dS^2 = g_{44}(r)dt^2 + g_{11}(r)dr^2 + g_{22}(r)(d\theta^2 + \sin^2\theta d\varphi^2), \tag{11}$$

where g_{11} and g_{22} are negative unknown functions and g_{44} is a positive function, which depend on the variable r, and $r, t \in (0,\infty), r \in (0,\infty), \theta \in [0,\pi], \varphi \in [0, 2\pi)$.

The components of metric (11) must satisfy system (10) . Using relations (1) and (2), we can reduce system (10) for metric (11) to the form

$$(\frac{g'_{22}}{g_{22}})' + \frac{1}{2}(\frac{g'_{44}}{g_{44}})' - \frac{1}{2}(\frac{g'_{11}}{g_{11}})(\frac{g'_{22}}{g_{22}} + \frac{g'_{44}}{2g_{44}}) + \frac{1}{2}(\frac{g'_{22}}{g_{22}})^2 + \frac{1}{4}(\frac{g'_{44}}{g_{44}})^2 = \frac{R}{4}g_{11},$$
$$\frac{1}{2}(\frac{g'_{22}}{g_{11}})' + \frac{g'_{22}}{2g_{11}}(\frac{g'_{11}}{2g_{11}} + \frac{g'_{22}}{g_{22}} + \frac{g'_{44}}{2g_{44}}) - \frac{1}{2g_{11}g_{22}}(g'_{22})^2 - 1 = \frac{R}{4}g_{22}, \quad (12)$$
$$-\frac{1}{2}(\frac{g'_{44}}{g_{11}})' - \frac{g'_{44}}{2g_{11}}(\frac{g'_{11}}{2g_{11}} + \frac{g'_{22}}{g_{22}} + \frac{g'_{44}}{2g_{44}}) + \frac{1}{2g_{11}g_{44}}(g'_{44})^2 = \frac{R}{4}g_{44},$$

where $g'_{ij} = \frac{dg_{ij}}{dr}$, $R > 0$,at $r \leq r_1$, and $R = 0$ at $r > r_1$. For the sake of generality, we assume that the unknown scalar function R in system (12) depends on the r coordinate.

The following theorem is valid.

Theorem 2. If $g_{22}(0) = 0$ and $g_{44}(0) < \infty$, then the scalar curvature $R(r)$ does not depend on r in the domain where it is nonzero, and system (12) has a unique solution depending on one free parameter.

If, in addition,$g_{44}(0) = 1$, then the solution of system (12) is unique; moreover, at $r \leq r_1$, where $r_1 < \sqrt{\frac{12}{R}}$, it coincides with the de Sitter metric[5]

$$dS^2 = (1 - \frac{R}{12}r^2)dt^2 - \frac{dr^2}{1 - \frac{R}{12}r^2} - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (13)$$

and at $r > r_1$, it coincides with the Schwarzschild metric[6]

$$dS^2 = (1 - \frac{R}{12}\frac{r_1^3}{r})dt^2 - \frac{dr^2}{1 - \frac{R}{12}\frac{r_1^3}{r}} - r^2(d\theta^2 + \sin\theta^2 d\varphi^2). \quad (14)$$

Proof. System of equations (12) can be simplified by passing to the generalized spherical coordinates $x_1, \ldots, x_4$,where $x_1 = \frac{r^3}{3}$,$x_2 = -\cos\theta$,$x_3 = \varphi$,and $x_4 = t$. Let us introduce the following new notation for the components of the metric tensor:

$$g_{11} = -(3x_1)^{\frac{4}{3}} f_1(x_1), g_{22} = -f_2(x_1), g_{44} = f_4(x_1).$$

In this notation, metric (11) takes the following form in the generalized spherical coordinates:

$$dS^2 = f_4 dx_4{}^2 - f_1 dx_1{}^2 - f_2(\frac{dx_2{}^2}{1 - x_2{}^2} + (1 - x_2)dx_3{}^2). \quad (15)$$

Using the arbitrariness of the scaling multiplier of the coordinate x_1 , we can achieve

$$f_1 f_2{}^2 f_4 = 1. \quad (16)$$

System (10) for metric (15) can be represented as

$$-\frac{1}{2}(\frac{f_1'}{f_1})' + \frac{1}{2}(\frac{f_2'}{f_2})^2 + \frac{1}{4}(\frac{f_1'}{f_1})^2 + \frac{1}{4}(\frac{f_4'}{f_4})^2 = -\frac{R}{4}f_1,$$

$$\frac{1}{2}(\frac{f_2'}{f_1})' - \frac{1}{2f_1f_2}(f_2')^2 - 1 = -\frac{R}{4}f_2, \tag{17}$$

$$-\frac{1}{2}(\frac{f_4'}{f_1})' + \frac{1}{2f_1f_4}(f_4')^2 = \frac{R}{4}f_4.$$

Condition (16) gives the additional equation

$$\frac{f_1'}{f_1} + \frac{2f_2'}{f_2} + \frac{f_4'}{f_4} = 0, \tag{18}$$

where $f_i' = \frac{df_i}{dx_i}$.

System (17), (18) contains four unknown functions f_1 ,f_2 ,f_4 , and R ; only two of these functions, say f_2 and R , can be regarded to be independent. This follows from relations (3) and (16). Let us express the unknown functions f_1 and f_4 in terms of f_2 and R . For this purpose, note that the third equation in system (17) can be represented as

$$\frac{1}{2f_1f_4}f_1'f_4' - \frac{1}{2}(\frac{f_4'}{f_4})' = \frac{R}{4}f_1. \tag{19}$$

Adding the first equation in system (17) to equation (19) and using (18) ,we obtain the relation

$$(\frac{f_2'}{f_2})' + \frac{3}{2}(\frac{f_2'}{f_2})^2 = 0.$$

Twice integrating it, we obtain $f_2 = \lambda(3x_1 + \alpha)^{\frac{2}{3}}$, where λ and α are arbitrary constants of integration. By the assumption of the theorem, we have $f_2(0) = 0$, which implies $\alpha = 0$ and

$$f_2 = \lambda(3x_1)^{\frac{2}{3}}. \tag{20}$$

We seek $\frac{f_4'}{f_4}$ in (19) in the form $\frac{f_4'}{f_4} = c(x_1)f_1$. Substituting the last relation into equation (19) , we obtain

$$f_4' = \frac{C - \frac{1}{2}\int Rdx_1}{\lambda^2(3x_1)^{\frac{4}{3}}},$$

where C is a constant.

Integrating both sides of this relation, we arrive at the formula

$$f_4 = \beta - \frac{C}{\lambda^2(3x_1)^{\frac{1}{3}}} + \frac{1}{2\lambda^2}\frac{\int Rdx_1}{(3x_1)^{\frac{1}{3}}} - \frac{1}{2\lambda^2}\int \frac{R}{(3x_1)^{\frac{1}{3}}}dx_1, \tag{21}$$

where β is constant of integration. By the assumption of the theorem, the function $f_4(0)$ is bounded; therefore, considering a solution in some neighborhood of zero, we must set $C = 0$, and the formula for f_4 takes the form

$$f_4 = \beta + \frac{\int R(x_1)dx_1}{2\lambda^2(3x_1)^{\frac{1}{3}}} - \frac{1}{2\lambda^2}\int \frac{R(x_1)}{(3x_1)^{\frac{1}{3}}}dx_1. \tag{22}$$

Condition (16) implies

$$f_1 = \frac{1}{\lambda^2(3x_1)^{\frac{4}{3}}}\frac{1}{f_4}. \tag{23}$$

The functions f_1 ,f_2, and f_4 specified by (20), (22), and (23) satisfy system (17) of differential equations only if

$$\beta\lambda^3 = 1,$$

$$\int \frac{R(x_1)}{(3x_1)^{\frac{1}{3}}}dx_1 = \frac{R(x_1)}{2}(3x_1)^{\frac{2}{3}}. \tag{24}$$

It follows from relation (24) that the function $R(x_1)$ must be constant in the domain where it is nonzero. Thus, inside the ball of radius r_1 , the components of metric (25) have the form

$$f_2 = \lambda(3x_1)^{\frac{2}{3}}, f_4 = \frac{1}{\lambda^3} - \frac{R}{12}\frac{(3x_1)^{\frac{2}{3}}}{\lambda^2}, f_1 = \frac{1}{\lambda^2(3x_1)^{\frac{4}{3}}}\frac{1}{f_4}. \tag{25}$$

Outside the ball, the scalar curvature vanishes, and relation (21) implies that the components of metric (15) have the form

$$f_2 = \lambda(3x_1)^{\frac{2}{3}}, f_4 = \frac{1}{\lambda^3} - \frac{C}{\lambda^2(3x_1)^{\frac{1}{3}}}. \tag{26}$$

These components satisfy also system (17),(18). The constant C in (26) is found from the condition that the metric components must be continuous on the entire space, which implies $C = \frac{R}{12}{r_1}^3$.

System (17), (18) has the unique solution (25), (26), which contains one free parameter λ. Taking into account the additional condition $g_{44}(0) = 1$ in the theorem, which implies $\lambda = 1$, and passing to the usual spherical coordinates, we see that relations (13) and (14) hold. This completes the proof of the theorem.

Thus, in the framework of the proposed axiomatics, we have constructed a mathematically rigorous model of a spherically symmetric space adequately describing the spherically symmetric gravitational field generated by a spherical gravitational source with constant mass density.

Theorem 2 are easy to extend to a more general case.

5 Stationary Spherically Symmetric Spaces with Scalar Curvature having Finitely or Countably Many Discontinuities of the First Kind and a Mathematically Rigorous Definition of Spherically Symmetric Black Holes

Consider a stationary spherically symmetric space endowed with a metric of the general form (11). Suppose that the scalar curvature of this space is a piecewise smooth function $R(r)$ with at most countably many discontinuities of the first kind at points $r_1, r_2, \ldots, r_n, \ldots$ numbered in increasing order. For such a space, the following theorem is valid.

Theorem 3. If the components of metric (11) satisfy the conditions $g_{22}(0) = 0$ and $g_{44}(0) = 1$, then the scalar curvature of the spherically symmetric space under consideration is a piecewise constant function taking the constant values $R(r) = R_k$ at $r_{k-1} < r \leq r_k$ for $k = 1, \ldots, n, \ldots$, where $r_0 = 0$.

The metric of such a space is everywhere continuous, provided that $\frac{2m(r)}{r} < 1$ for $r \in (0, \infty)$, and has the form

$$dS^2 = (1 - \frac{2m(r)}{r})dt^2 - \frac{dr^2}{1 - \frac{2m(r)}{r}} - r^2(d\theta^2 + \sin^2\theta d\varphi), \tag{27}$$

where $m(r) = \int\limits_0^r \sum\limits_{k>0} \frac{R_k}{4}[\theta(x - r_{k-1}) - \theta(x - r_k)]x^2 dx.$

The components of metric (27) satisfy system (12) of differential equations everywhere except at a finite or countable set of points $r_1, \ldots, r_n, \ldots$.

Proof. This theorem is proved by the same method as Theorem 2. As in Theorem 2, we show that, under the assumptions of Theorem 3, the components of metric (11) in the spherical coordinate system have the form

$$g_{44} = 1 - \frac{1}{4r}\int\limits_0^r R(x)x^2 dx, g_{22} = -r^2, g_{11} = -{g_{44}}^{-1}, \tag{28}$$

where R is a piecewise smooth function with at most countably many discontinuities of the first kind at points $r_1, r_2, \ldots$. The functions in (28) satisfy system (12) only under the condition

$$\frac{dR}{dr} = 0, \tag{29}$$

which is an analogue of condition (24) in the proof of Theorem 2. Relation (29) implies that $R(r)$ must take constant values in its domains of continuity. Suppose that $R(r)$ takes a value R_k at $r_{k-1} < r \leq r_k$; then

$$R(r) = \sum_{k=1} R_k(\theta(r - r_{k-1}) - \theta(r - r_k)),$$

which implies the assertion Theorem 3.

The components of metric (27) are everywhere continuous if $\frac{2m(r)}{r} < 1$ for $r \in (0, \infty)$. If $\frac{2m(r)}{r} = 1$ at some $r = r_g$. In this case, there are two possibilities: either the spherically symmetric space is bounded by a hypersphere with radial parameter r_g (if $\frac{2m(r)}{r} > 1$ at $r > r_g$), or this space can be extended (if $\frac{2m(r)}{r} < 1$ at $r > r_g$). In the latter case, the space contains a spherically symmetric body of radius r_g with generally nonuniform mass distribution density, and on the boundary of this body, the metric exhibits an irregular behavior.

We refer to such bodies as spherically symmetric stationary black holes. Below we give the definition of the simplest stationary black hole.

Definition 1. A globular body of radius r_g with constant mass density satisfying the condition $R = \frac{12}{{r_g}^2}$ is called a stationary spherical black hole. The sphere of radius r_g being the surface of a black hole is called its horizon level, or the Schwarzschild sphere.

It follows from the definition that $r_g = \sqrt{\frac{12}{R}}$. Note that the signature of the space inside a black hole and outside it remains invariable, which agrees with condition (2) on the metric of a pseudo-Riemannian space. All components of metric (27) are continuously differentiable on the entire space except on the surface of the black hole, on which the behavior of metric (27) is irregular, namely, $g_{11}(r_g) = -\infty$. The mathematical properties of a stationary spherically symmetric black hole in the formalism suggested here substantially differ from the properties of black holes investigated in the framework of GRT[7].

If the spherically symmetric space has the same scalar curvature R at all points, then the space is the de Sitter closed elliptic space[5] determined by metric (27) of the form

$$dS^2 = (1 - \frac{R}{12}r^2)dt^2 - \frac{dr^2}{1 - \frac{R}{12}r^2} - r^2(d\theta^2 + \sin^2\theta d\varphi)$$

with $r < \sqrt{\frac{12}{R}}$. This space is bounded by a sphere with radial parameter

$$r < \sqrt{\frac{12}{R}}.$$

Such a model corresponds to a homogeneous closed space filled with a matter with constant mass density.

Consider yet another parameter of the spherically symmetric space generated by a ball of radius r_1 with constant mass density ρ_1 and a matter with constant mass density ρ_2 filling the whole space outside the ball. The scalar curvature of the space inside the ball is calculated by $R_1 = 32\pi\rho_1$ and outside the ball, by

$R_2 = 32\pi\rho_2$. The components of metric (27) for each space have the form

$$g_{44} = 1 - \frac{R_1}{12} r^2$$

at $r < r_1$,

$$g_{44} = 1 - \frac{R_1 - R_2}{12} \frac{{r_1}^3}{r} - \frac{R_2}{12} r^2$$

at $r \geq r_1$, and

$$g_{22} = -r^2, g_{11} = -{g_{44}}^{-1}(r)$$

if $\frac{R_1}{12}{r_1}^2 < 1$. The spherically symmetric space is bounded by a sphere of radius r_2, which is the least positive root of the cubic equation

$$r^3 - \frac{12}{R_2} r + \frac{R_1 - R_2}{R_2} {r_1}^3 = 0.$$

A criterion for the transformation of the ball into a black hole is $\frac{R_1}{12}{r_1}^2 = 1$.

6 A Model of a Nonstationary Spherically Symmetric Pseudo-Riemannian Space with Constant Scalar Curvature and the Definition of Dark Energy

In the preceding section, we described a stationary spherically symmetric space with discontinuous scalar curvature. Here, we consider a mathematical model of a spherically symmetric pseudo-Riemannian space with nonstationary Fridman-type metric[8]

$$dS^2 = {dx_4}^2 - r^2(x_4)({dx_1}^2 + \sin^2 x_1({x_2}^2 + \sin^2 x_2 {dx_3}^2)), \qquad (30)$$

where r is the curvature radius of the three-dimensional hypersphere, which depends on the parameter x_4; $x_1 \in (0, 2\pi)$; $x_2 \in (0, \pi)$; $x_3 \in (0, 2\pi)$; and $x_4 \in (0, \infty)$.

The following theorem is valid.

Theorem 4. System (10) for metric (30) reduces to the single second-order nonlinear differential equation

$$r\frac{d^2 r}{{dx_4}^2} - (\frac{dr}{dx_4})^2 - 1 = 0$$

for the unknown function $r(x_4)$. This equation has a real solution $r = r_0 \cosh \frac{x_4}{r_0}$,where r_0 is a some constant. The scalar curvature of the space is everywhere constant and has the form $R = \frac{12}{{r_0}^2}$.

Proof. According to (30), the nonzero components of the covariant metric tensor g_{ij} have the form

$$g_{11} = -r^2, g_{22} = -r^2\sin^2 x_1, g_{33} = -r^2\sin^2 x_1 \sin^2 x_2, g_{44} = 1. \qquad (31)$$

The components of the contravariant metric tensor g^{ij} have the form

$$g^{11} = -r^{-2}, g^{22} = -\frac{1}{r^2\sin^2 x_1}, g^{33} = -\frac{1}{r^2\sin^2 x_1 \sin^2 x_2}, g^{44} = 1. \quad (32)$$

Substituting the metric components (31) and (32) into relation (1), we find all nonzero elements of the pseudo-Riemannian connection; these are

$$\Gamma_{22}{}^1 = -\sin x_1 \cos x_1, \Gamma_{33}{}^1 = -\sin x_1 \cos x_1 \sin^2 x_2, \Gamma_{14}{}^1 = \frac{1}{r}\frac{dr}{dx_4},$$
$$\Gamma_{12}{}^2 = \cot x_1, \Gamma_{33}{}^2 = -\sin x_2 \cos x_2, \Gamma_{24}{}^2 = \frac{1}{r}\frac{dr}{dx_4}, \quad (33)$$
$$\Gamma_{13}{}^3 = \cot x_1, \Gamma_{23}{}^3 = \cot x_2, \Gamma_{34}{}^3 = \frac{1}{r}\frac{dr}{dx_4},$$
$$\Gamma_{11}{}^4 = r\frac{dr}{dx_4}, \Gamma_{22}{}^4 = r\frac{dr}{dx_4}\sin^2 x_1, \Gamma_{33}{}^4 = r\frac{dr}{dx_4}\sin^2 x_1 \sin^2 x_2.$$

Using the components (33) of the connection and applying formula (2), we obtain all nonzero components of the Ricci tensor R_{ij}:

$$R_{11} = -2(\frac{dr}{dx_4})^2 - r\frac{d^2r}{dx_4{}^2} - 2, R_{22} = \sin^2 x_1 R_{11}, \quad (34)$$
$$R_{33} = \sin^2 x_1 \sin^2 x_2 R_{11}, R_{44} = \frac{3}{r}\frac{d^2r}{dx_4{}^2}.$$

It follows from (10), (32), and (34) that the scalar curvature is

$$R = \frac{6}{r}\frac{d^2r}{dx_4{}^2} + \frac{6}{r^2}(\frac{dr}{dx_4})^2 + \frac{6}{r^2}. \quad (35)$$

By virtue of (31), (34), and (35), system (10) degenerates into a single equation of the form

$$r\frac{d^2r}{dx_4{}^2} - (\frac{dr}{dx_4})^2 - 1 = 0. \quad (36)$$

We seek a solution of equation (36) in the form

$$r = a_1 e^{\beta_1 x_4} + a_2 e^{-\beta_2 x_4},$$

where a_1, a_2, β_1, and are some constants. Then equation (36) transforms into the relation $a_1 a_2 (\beta_1 + \beta_2)^2 = e^{(\beta_2 - \beta_1)x_4}$, which implies $\beta_1 = \beta_2$ and $a_1 a_2 = \frac{1}{4\beta_1{}^2}$. Since $r > 0$, it follows that $a_1 > 0$ if $\beta_1 > 0$, and we can set $a_1 = \frac{e^a}{2\beta_1}$ and $a_2 = \frac{e^{-a}}{2\beta_1}$. If $\frac{dr}{dx_4}|_{x_4=0} = 0$, then $a = 0$. Setting $a_1 = r_0$, we finally obtain $r(x_4) = r_0 \cosh \frac{x_4}{r_0}$.

There exists yet another, complex, solution of equation (36), namely, $r = ix_4$, but we are interested only in real positive solutions. Substituting the solution $r(x_4) = r_0 \cosh \frac{x_4}{r_0}$ into (35), we obtain the following expression $R = \frac{12}{r_0{}^2}$ for the scalar curvature, which proves the theorem.

At first glance, the physical interpretation of the assertion of Theorem 4 may seem rather contradictory. On the one hand, the hyperspherical space expands by an almost exponential law, i.e., the curvature radius increases as $r = r_0 \cosh \frac{x_4}{r_0}$, while the scalar curvature R itself does not depend on the parameter x_4 and remains everywhere constant. According to Postulate 2, this means that the mass density of the matter uniformly filling the expanding space remains constant as well.

In reality, these results involve no contradiction. At present, the existence of a new type of matter in our universe, which is known as dark energy, has been reliably established in cosmology; this matter fills uniformly whole space and is characterized by constant mass density not depending on the time parameter x_4. The model suggested above can be regarded as an example confirming the existence of this type matter with such unusual physical properties.

7 Conclusion

The axiomatization of gravity theory proposed in this paper and the fundamental gravity equation obtained on the basis of these axioms make it possible to demonstrate the effectiveness of the suggested approach for a number of physical examples considered in the paper. It suffices to mention that the problem of constructing an everywhere continuous spherically symmetric stationary metric of a pseudo-Riemannian space with discontinuous scalar curvature, which is solved in general form in Section 4, still remains unsolved in the framework of GRT, in which solving this problem involves fundamental difficulties. In the framework of the formalism suggested here, the concepts of a stationary black hole and dark energy are defined more rigorously from the mathematical point of view. Importantly, dark energy arises in a natural way as one of the solutions of the system (10) of gravity equations, which, unlike in GRT[9], does not require introduce any additional empirical constants into the main equation, such as the cosmological constant and its comparatively recent interpretation as the mass density of dark energy[10].

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