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The Model of the Production Side of the Russian Economy

Nikolay Pilnik[1], Stanislav Radionov[1*]

[1] *Financial Research Institute of the Ministry of Finance of the Russian Federation, Moscow, Russia*

Abstract: We consider the model of the production side of the Russian economy. This model is derived as the solution of the nonlinear dynamic optimization problem of the macroeconomic agent we call Producer. This agent maximizes his discounted profit flow under technologic, demographic and financial constraints. We use the method of relaxation of complementary slackness conditions to transform the model to the more regular one and evaluate its parameters on the Russian macroeconomic data. We show that this model can successfully replicate the large set of Russian macroeconomic indicators such as gross domestic product, loans of producers, volume of fixed assets etc.

Keywords: dynamic optimization, complementary slackness conditions, production of GDP

1. INTRODUCTION AND LITERATURE REVIEW

We present the model of production side of the economy, which consists of all economic agents who produce the added value. In the context of general equilibrium models these agents can be considered as elements of macroeconomic agent Producer, similarly to agents Consumer, Bank, Government and others. This agent generates a supply of good and demand for labor and capital, which is typical for macroeconomic models, but also has connections with banking system in terms of loans both in national and foreign currency. To our knowledge, this is the first model which describes the process of production and interactions of producers with the banking system on such a detailed level. We consider the maximization problem of discounted profit flow of this macroeconomic agent under technologic, demographic and financial constraints. It leads to the nonlinear dynamic optimization problem. We solve it using Lagrange method and transform the solution by relaxation of the complementary slackness conditions (see [22]). This transformation leads to the nonlinear dynamic system. We estimate its parameters on the Russian macroeconomic data and find a set of parameters that allows to reproduce it with rather high accuracy.

In the modern literature, the works which are closest to ours are devoted to the estimation of production functions. This topic is considered as one of the most important in the empiric economic studies, because it deals with the fundamental economic process of formation of goods. The description of this process in terms of mathematical models is thus essential for the understanding of the functioning of the economy. Apart from theoretical interest, it also has crucial policy implications.

A large body of econometric literature is devoted to the estimation of production functions on the microdata (Cobb-Douglas function is usually used). The problem of obtaining reliable estimations turns out to be rather challenging because of simultaneity – if firm knows its productivity level when choosing inputs, it leads to the correlation between errors and

*Corresponding author: saradionov@edu.hse.ru

regressors. Another problem is that firms with low productivity levels are harder to observe because they cease to exist sooner. Several methods were developed to overcome these difficulties. Probably the most popular ones were proposed in [20] and [17], but they were questioned in [2]. Other methods were proposed, among others, in [7] and [15].

Modelling of gross domestic product in terms of production functions is also widely used in the literature, especially in the works dealing with output gaps and potential growth rates, see for example [11], [12], [14], [16]. These works adopt a rather straightforward methodology which could potentially also suffer from simultaneity problem. In the context of GDP estimation this problem was addressed, for example, in [23]. Description of GDP production in terms of production function is also ubiquitous in the modern macroeconomic dynamic stochastic general equilibrium (DSGE) models, however they usually use a very simple Cobb-Douglas production functions.

It should be noted that over the last years, a CES production function becomes more and more popular tool of production side of the economy, because empirical studies reject the hypothesis of elasticities of substitution between capital and labor being equal to one (the case of Cobb-Douglas function). The constant elasticity of substitution production function was derived in [4], and the numerous articles with estimation of CES production function on US data soon followed. In the first estimates the elasticity of substitution was indeed close to one (e.g., [10]), but the methodology used for these estimations was criticized. Use of more rigorous econometric methods led to estimates well below unity (e.g., [18], [19], [9], [13]). New arguments in favor of Cobb-Douglas function were presented in [6], but they were again challenged in [3]. It should also be noted that more flexible production function than Cobb-Douglas is required in many growth models to generate the plausible dynamics (e.g. [1]), and in policy analysis (e.g. [8], [5]). A thorough discussion of CES production function is presented in the special issue of Journal of Macroeconomics in June 2008.

2. THE MODEL: STATEMENT

We consider the model of the whole sector of the economy, which generates the added value, as a single agent. We will reduce it to the usual dynamic model that determines the supply of produced product, the demand of producer for loans, investments and labor, taking prices, wages, interest rates and other external factors on the market into account.

There are several non-standard features in the proposed model. First, we divide investments into two parts: investments in the maintenance of fixed assets and investments in building up fixed assets. This approach allows to explain the fluctuations of output with higher accuracy, perceiving these investments as transaction costs and capital costs respectively. Secondly, the production function depends on the volume of used fixed assets, adjusted on the level of investment in the maintenance of fixed assets. Thirdly, we use a similar scheme to describe the contribution of labor in output – similarly to the fixed assets, we include human capital in the production function, which can be increased due to special wage investments.

The formation of production capital (fixed assets) by the Producer is described by the following relations:

$$\frac{\mathrm{d}}{\mathrm{d}t} M(t) = Jm(t) - \delta_{am}(t) M(t), \tag{2.1}$$

$$0 \le Jm(t). \tag{2.2}$$

In this equation $M(t)$ are the fixed assets, $Jm(t)$ are the investments in building up fixed assets, $\delta_{am}(t)$ is the depreciation coefficient. This equation allows to calculate the series $Jm(t)$ based on statistics on the level of fixed assets and depreciation. The balance of investments

$$J(t) = Jm(t) + Ju(t), \tag{2.3}$$

where $J(t)$ is the overall level of investment (gross fixed capital formation) in base year prices, allows to calculate the investments in the maintenance of fixed assets $Ju(t)$. Further we assume that the prices of these investments may be different, therefore the balance of investments in nominal volumes can be written as

$$pJ(t) = p_m(t)\,Jm(t) + p_u(t)\,Ju(t)\,, \tag{2.4}$$

where $pJ(t)$ is the overall level of investments (gross fixed capital formation) in the current prices, $p_m(t)$ is the deflator of investments in building up fixed assets, $p_u(t)$ is the deflator of investments in the maintenance of fixed assets.

The formation of human capital is described in a manner similar to the description of fixed assets:

$$\frac{\mathrm{d}}{\mathrm{d}t}H(t) = \frac{Jh(t)}{R_{\max}(t)} - \delta_{ah}(t)\,H(t)\,, \tag{2.5}$$

$$0 \leq Jh(t)\,, \tag{2.6}$$

where $H(t)$ is the human capital of the economically active (but not necessarily employed) population per capita, $Jh(t)$ is the investments in building up human capital, $\delta_{ah}(t)$ is the depreciation coefficient. Everywhere further in the model we will distinguish the total economically active population $R_{\max}(t)$ and the number of employed $R(t)$. We assume that human capital is distributed among all participants of the economically active population, but only the part corresponding to the employed workers is used.

The statistical data provides information on the total amount of payments made by the producer to its employees – wage and mixed income $W(t)$. Further, we assume that this flow (as well as investment costs) consists of two parts: payments to employees $w_r(t)\,R(t)$ and investments in human capital $w_h(t)\,Jh(t)$. Hence, investment in human capital actually leads to the productivity increase of the economically active population in the economy, but does not lead to its growth.

The most important element of the Producer's problem is the production function, which describes the relationship between the use of factors of production and the resulting output. When modeling the Russian economy, linear production function can be used. The procedure its estimation is quite simple and can be is carried out by standard econometric tools. Moreover, when instead of the indicator of fixed capital, the indicator of investments in fixed capital is used, it demonstrates a fairly high accuracy on the Russian data. However, it cannot be used in the optimization problem because investments will not enter the first order conditions of the producer and it is impossible to obtain an expression for it.

Another approach is using the Cobb-Douglas function, which is popular, for example, in DSGE models. But another problem arises here: on the Russian data, the use of capital in such a production function leads to a poor model accuracy, and using of investments instead of capital is debatable from a theoretical point of view. That is why the presented Producer model separates the long-term process of capital replacement and the short-term process of changing the load of this capital. Such a technique, as will be shown below, allows to obtain a sufficiently high accuracy of the data reproduction without restrictions of the linear production function and with explicit use of capital in the model. Thereby we use the following production function:

$$Y(t) = A\,(u_M(t)\,M(t))^{\alpha}\,(u_H(t)\,H(t)\,R_{\max}(t))^{\beta}\,, \tag{2.7}$$

where $u_M(t)$ and $u_H(t)$ are the utilization rates, respectively, of fixed assets and human capital. We define these indicators in the following non-linear way:

$$u_M(t) = \left(\frac{Ju(t)}{M(t)}\right)^{b},\, u_H(t) = \left(\frac{R(t)}{R_{\max}(t)}\right)^{a}. \tag{2.8}$$

We substitute (2.8) into (2.7) and also substitute A with several normalization coefficients in order to avoid the dimensionality problem. As a result, the production function takes the form

$$Y(t) = Y0 \left(\frac{Ju(t)}{J0}\right)^{b\alpha} \left(\frac{M(t)}{M0}\right)^{\alpha(1-b)} \left(\frac{H(t)}{H0}\right)^{\beta} \left(\frac{R(t)}{R0}\right)^{a\beta} \left(\frac{R_{\max}(t)}{Rmax0}\right)^{(1-a)\beta} . \quad (2.9)$$

Depending on the market conditions, however, the volume of production $0 \leq Y(t)$ and the volume of products sold $0 \leq Yp(t)$ may vary. The Producer can thus form the stock of the product in size $0 \leq Z(t)$, which can be calculated as

$$\frac{\mathrm{d}}{\mathrm{d}t} Z(t) = Y(t) - Yp(t) . \quad (2.10)$$

Denote the loans attracted by the Producer by $L(t)$. The average terms for which loans are attracted will be denoted by $(\beta_l(t))^{-1}$. The variable $\beta_l(t)$ will be referred below as the inverse duration and interpreted as the average frequency of the loans return. Then the dynamics of loans is described by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} L(t) = K(t) - \beta_l(t) L(t) , \quad (2.11)$$

where $0 \leq K(t)$ is the flow of newly attracted loans. The Producer pays interest on borrowed funds $r_L(t) L(t)$, where $r_L(t)$ is the effective interest rate on loans.

In addition to loans in national currency (rubles), the producer also attracts loans $vL(t)$ in dollars. The dynamics of foreign currency loans and deposits is described by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} vL(t) = vK(t) - \beta_{vl}(t) vL(t) . \quad (2.12)$$

Here $\beta_{vl}(t)$ is the reverse durations of currency loans. Newly attracted foreign currency loans are denoted by $0 \leq vK(t)$. The effective interest rate on foreign currency loans is denoted by $r_{vl}(t)$.

During its activities, the producer pays four types of taxes: value added tax, labor tax (social contributions), property tax and income tax. The rates of these taxes are denoted by $\tau_y(t)$, $\tau_r(t)$, $\tau_{tm}(t)$, $\tau_{pr}(t)$. For servicing operations related to loans and investments, the producer uses current account $N(t)$. It is assumed that its volumes in the balance sheet are proportional to ruble and foreign currency loans, fixed assets and human capital. These proportions are defined by the coefficients ν_l, ν_{vl}, ν_m, ν_h:

$$\begin{aligned} N(t) \geq \nu_l L(t) + \nu_{vl} w_{vl}(t) vL(t) + \nu_m p_m(t) M(t) + \\ + \nu_h w_h(t) (1 + \tau_h(t)) R_{\max}(t) H(t) , \end{aligned} \quad (2.13)$$

where $w_{vl}(t)$ denotes the exchange rate.

Other expenses of the Producer are denoted by $OC_o(t)$ and are considered as the exogenous variable. Thus, the financial balance of the producer can be written as

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} N(t) = K(t) - \beta_l(t) L(t) - r_l(t) L(t) - (1 + \tau_{pr}(t)) \Pr(t) + \\ &w_{vl}(t) (vK(t) - \beta_{vl}(t) vL(t) - r_{vl}(t) vL(t)) - OC_o(t) \\ &+ (1 - \tau_y(t)) p_y(t) Yp(t) - p_j(t) Ju(t) - p_m(t) Jm(t) - \\ &\tau_{tm}(t) p_{tm}(t) M(t) - (1 + \tau_r(t)) w_r(t) R(t) - (1 + \tau_h(t)) w_h(t) Jh(t) , \end{aligned} \quad (2.14)$$

where $\Pr(t)$ is the profit after taxation.

The above relationships represent the limitations imposed on the Producer's ability to choose the values of its planned variables (controls):

$$\begin{aligned}&H(t), Jh(t), Jm(t), Ju(t), K(t), L(t), M(t), N(t), \Pr(t),\\&R(t), Yp(t), Z(t), vK(t), vL(t), Y(t).\end{aligned} \tag{2.15}$$

According to the principle of rational expectations, when planning its control variables, the producer can rely on an accurate forecast of information variables:

$$\begin{aligned}&OC_o(t), R_{\max}(t), \beta_l(t), \beta_{vl}(t), \delta_{ah}(t), \delta_{am}(t), p_j(t), p_m(t), p_{tm}(t),\\&p_y(t), r_l(t), r_{vl}(t), \tau_h(t), \tau_{pr}(t), \tau_r(t), \tau_{tm}(t), \tau_y(t), w_h(t), w_r(t), w_{vl}(t).\end{aligned}$$

The planned variables of the Producer are thus a function of current and future values of information variables. The goal of the producer in the model is to maximize the total discounted utility from the undistributed profit after taxation with discount rate being equal to the deflator:

$$\int_{t0}^{T} \frac{\mathrm{e}^{-\Delta t}}{1-\beta}\left(\frac{\Pr(t)}{p_y(t)}\right)^{1-\beta} \mathrm{d}t. \tag{2.16}$$

The problem is supplemented with terminal condition, which can be interpreted as a growth condition for some linear combination of the phase variables:

$$\Omega(t0)\,\gamma \le \Omega(T), \tag{2.17}$$

where $\Omega(t) = aH(t)\,H(t) + aL(t)\,L(t) + aM(t)\,M(t) + aN(t)\,N(t) + aZ(t)\,Z(t) + avL(t)\,vL(t)$. This is an analogue of no Ponzi condition written in terms of the Producer's own capital which is a difference between its assets and liabilities.

To derive the solution of the maximization problem of the function (2.16) under constraints (2.1) – (2.14) and the terminal condition (2.17) with respect to the variables (2.15), we write down a Lagrange functional and find its saddle point. The dual variables are denoted by $\Phi 2(t)$,$\Phi 4(t)$,$\Phi 7(t)$,$\Phi 9(t)$. The obtained system, which is a set of sufficient conditions for optimality, is presented below divided into several groups.

The first group are equations for primal variables, including equations from the original problem statement:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}H(t) = \frac{Jh(t)}{R_{\max}(t)} - \delta_{ah}(t)\,H(t),\\
&\frac{\mathrm{d}}{\mathrm{d}t}M(t) = Jm(t) - \delta_{am}(t)\,M(t),\\
&\frac{\mathrm{d}}{\mathrm{d}t}L(t) = K(t) - \beta_l(t)\,L(t),\\
&\frac{\mathrm{d}}{\mathrm{d}t}vL(t) = vK(t) - \beta_{vl}(t)\,vL(t),\\
&\frac{\mathrm{d}}{\mathrm{d}t}\Pr(t) = \left(\frac{\rho(t)-\Delta}{\eta} - \frac{\frac{\mathrm{d}}{\mathrm{d}t}\tau_{pr}(t)}{(1+\tau_{pr}(t))\,\eta} + \frac{\frac{\mathrm{d}}{\mathrm{d}t}p_y(t)}{p_y(t)}\left(1-\eta^{-1}\right)\right)\Pr(t),\\
&N(t) = \nu_l L(t) + \nu_{vl} w_{vl}(t)\,vL(t) + \nu_m p_m(t)\,M(t) +\\
&\nu_h w_h(t)\,(1+\tau_h(t))\,R_{\max}(t)\,H(t),
\end{aligned}$$

The second group are differential equations for dual variables

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi 4\left(t\right)=\left(\beta_{vl}\left(t\right)+\rho\left(t\right)-\frac{\frac{\mathrm{d}}{\mathrm{d}t}w_{vl}\left(t\right)}{w_{vl}\left(t\right)}\right)\Phi 4\left(t\right)+\left(1-\nu_{vl}\right)\rho\left(t\right)-r_{vl}\left(t\right)-\frac{\frac{\mathrm{d}}{\mathrm{d}t}w_{vl}\left(t\right)}{w_{vl}\left(t\right)},$$
$$\left(1+\tau_r\left(t\right)\right)w_r\left(t\right)R\left(t\right)=a\beta\,\left(1-\tau_y\left(t\right)\right)p_y\left(t\right)Y\left(t\right),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi 7\left(t\right)=\left(\delta_{am}\left(t\right)+\rho\left(t\right)-\frac{\frac{\mathrm{d}}{\mathrm{d}t}p_m\left(t\right)}{p_m\left(t\right)}\right)\Phi 7\left(t\right)-\left(1+\nu_m\right)\rho\left(t\right)\delta_{am}\left(t\right)-\tau_{tm}\left(t\right)+$$
$$\frac{\frac{\mathrm{d}}{\mathrm{d}t}p_m\left(t\right)}{p_m\left(t\right)}+\frac{\alpha\,\left(1-b\right)\left(1-\tau_y\left(t\right)\right)p_y\left(t\right)Y\left(t\right)}{p_m\left(t\right)M\left(t\right)},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi 2\left(t\right)=\left(\beta_l\left(t\right)+\rho\left(t\right)\right)\Phi 2\left(t\right)+\left(1-\nu_l\right)\rho\left(t\right)-r_l\left(t\right),$$
$$p_j\left(t\right)Ju\left(t\right)=b\alpha\,\left(1-\tau_y\left(t\right)\right)p_y\left(t\right)Y\left(t\right),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi 9\left(t\right)=\left(\delta_{ah}\left(t\right)+\rho\left(t\right)-\frac{\frac{\mathrm{d}}{\mathrm{d}t}\tau_h\left(t\right)}{1+\tau_h\left(t\right)}-\frac{\frac{\mathrm{d}}{\mathrm{d}t}w_h\left(t\right)}{w_h\left(t\right)}-\frac{\frac{\mathrm{d}}{\mathrm{d}t}R_{\max}\left(t\right)}{R_{\max}\left(t\right)}\right)\Phi 9\left(t\right)$$
$$-\delta_{ah}\left(t\right)-\left(\nu_h+\frac{\nu_h\frac{\mathrm{d}}{\mathrm{d}t}\tau_h\left(t\right)+1}{1+\tau_h\left(t\right)}\right)\rho\left(t\right)+\frac{\frac{\mathrm{d}}{\mathrm{d}t}w_h\left(t\right)}{w_h\left(t\right)}+\frac{\frac{\mathrm{d}}{\mathrm{d}t}R_{\max}\left(t\right)}{R_{\max}\left(t\right)}+$$
$$\frac{\beta\,\left(1-\tau_y\left(t\right)\right)p_y\left(t\right)Y\left(t\right)}{\left(1+\tau_h\left(t\right)\right)w_h\left(t\right)R_{\max}\left(t\right)H\left(t\right)},$$

Third group are complementary slackness conditions:

$$[\Phi 9\left(t\right)][Jh\left(t\right)],[\Phi 7\left(t\right)][Jm\left(t\right)],[\Phi 4\left(t\right)][vK\left(t\right)],[\Phi 2\left(t\right)][K\left(t\right)],$$
$$[\left(-\tau_y\left(t\right)\rho\left(t\right)+\rho\left(t\right)+\frac{\mathrm{d}}{\mathrm{d}t}\tau_y\left(t\right)\right)p_y\left(t\right)+\left(\frac{\mathrm{d}}{\mathrm{d}t}p_y\left(t\right)\right)\tau_y\left(t\right)-\frac{\mathrm{d}}{\mathrm{d}t}p_y\left(t\right)][Z\left(t\right)],$$

where $[a][b]$ means $0\le a, 0\le b, ab=0$.

The last condition is the transversality condition that defines terminal condition for conjugate differential equations:

$$\Omega\left(t0\right)\gamma=\Omega\left(T\right),$$

where

$$\Omega\left(t\right)=aH\left(t\right)H\left(t\right)+aL\left(t\right)L\left(t\right)+aM\left(t\right)M\left(t\right)+aN\left(t\right)N\left(t\right)+$$
$$+\,aZ\left(t\right)Z\left(t\right)+avL\left(t\right)vL\left(t\right).$$

We transform this system according to the method described in detail in [22]. First, we transform it to the discrete time by replacing derivatives with increments. Note that backward increments are used for direct variables and forward increments are used for dual variables. Second, some direct variables are replaced by their values in the previous period. Third, differential equations for the dual variables are replaced by expressions of corresponding separatrices. Forth, the complementary slackness conditions are replaced by their more regular approximations (the relaxation of complementary slackness conditions).

After all these transformations, we proceed to the dynamic system which can also be presented by several groups of expressions.

First group consists of several expressions defining growth rates of different exogenous variables (in different forms) and one more auxiliary variable:

$$g_{pm}(t)=\frac{p_m(t)-p_m(t-1)}{p_m(t-1)},\ g_{wh}(t)=\frac{w_h(t)-w_h(t-1)}{w_h(t-1)},$$
$$g_{py}(t)=\frac{p_y(t)-p_y(t-1)}{p_y(t-1)},\ g_{taupr}(t)=\frac{\tau_{pr}(t)-\tau_{pr}(t-1)}{\tau_{pr}(t-1)+1},$$
$$g_{tauh}(t)=\frac{\tau_h(t)-\tau_h(t-1)}{1+\tau_h(t-1)},\ g_{rmax}(t)=\frac{R_{\max}(t)-R_{\max}(t-1)}{R_{\max}(t-1)},$$
$$g_{ttauy}(t)=-\frac{\tau_y(t)-\tau_y(t-1)}{-\tau_y(t-1)+1},$$
$$B(t)=-OC_o(t)+Y(t)(1-\tau_y(t))\,p_y(t)-\Pr(t)(1+\tau_{pr}(t)).$$

Second group consists of expressions defining GDP and its components included in the model – different kinds of investments, change of stocks and profit:

$$Y(t)^{(-a\beta-b\alpha+1)}=Y0\left(\frac{b\alpha\,(1-\tau_y(t))\,p_y(t)}{p_j(t)\,J0}\right)^{b\alpha}\left(\frac{M(t-1)}{M0}\right)^{\alpha(1-b)}\times$$
$$\times\left(\frac{H(t-1)}{H0}\right)^{\beta}\left(\frac{a\beta\,(1-\tau_y(t))\,p_y(t)}{w_r(t)(1+\tau_r(t))\,R0}\right)^{a\beta}\left(\frac{R_{\max}(t)}{Rmax0}\right)^{(1-a)\beta},$$
$$(1+\tau_r(t))\,w_r(t)\,R(t)=a\beta\,(1-\tau_y(t))\,p_y(t)\,Y(t),$$
$$p_j(t)\,Ju(t)=b\alpha\,(1-\tau_y(t))\,p_y(t)\,Y(t),$$
$$Jh(t)=b_1(g_{rmax}(t)+g_{tauh}(t)+g_{wh}(t)-\delta_{ah}(t)-\rho(t))\,R_{max}(t)H(t-1)-$$
$$-a_1((-\nu_h-1)\rho(t)+g_{rmax}(t)+g_{tauh}(t)+g_{wh}(t)-\delta_{ah}(t))\,R_{\max}(t)\,H(t-1)-$$
$$a_1\frac{\beta\,(\tau_y(t)-1)(g_{tauh}(t)+1)\,p_y(t)\,Y(t)}{(1+\tau_h(t))\,w_h(t)}+\frac{cc_1B(t)}{w_h(t)(1+\tau_h(t))},$$
$$Jm(t)=b_2(\delta_{am}(t)-g_{pm}(t)+\rho(t))\,M(t-1)-$$
$$a_2((\nu_m+1)\rho(t)+\delta_{am}(t)+\tau_{tm}(t)-g_{pm}(t))\,M(t-1)-$$
$$\frac{\alpha\,(b-1)(\tau_y(t)-1)\,p_y(t)\,Y(t)}{p_m(t)}+\frac{cc_2B(t)}{p_m(t)},$$
$$Z(t)=Z(t-1)+(b_5-a_5(-g_{py}(t)-g_{ttauy}(t)+\rho(t)))\,Z(t-1)+\frac{cc_5B(t)}{p_y(t)(1-\tau_y(t))},$$
$$\Pr(t)=\left(\frac{-\Delta-g_{taupr}(t)+\rho(t)}{\eta}+\left(-\eta^{-1}+1\right)g_{py}(t)+1\right)\Pr(t-1).$$

Hereinafter the expressions without time dependence such as $Y0$, a, α etc. are the parameters which will be estimated in the next section.

Next group are the variables defining stocks of physical and human capital:

$$M(t)=Jm(t)-\delta_{am}(t)\,M(t-1)+M(t-1),$$
$$H(t)=\frac{Jh(t)}{R_{\max}(t)}-\delta_{ah}(t)\,H(t-1)+H(t-1).$$

The last group are the variables defining financial variables, both stocks and flows.

$$
\begin{aligned}
&vK(t) = b_3(\beta_{vl}(t) - g_{wvl}(t) + \rho(t))\, vL(t-1) \\
&- a_3((\nu_{vl} - 1)\rho(t) + r_{vl}(t) + g_{wvl}(t))\, vL(t-1) - \frac{cc_3 B(t)}{w_{vl}(t)}, \\
&K(t) = (b_4(\beta_l(t) + \rho(t)) - a_4((\nu_l - 1)\rho(t) + r_l(t)))\, L(t-1) + \\
&+ (-1 + cc_1 + cc_2 + cc_3 + cc_5)\, B(t), \\
&N(t) = \nu_l L(t-1) + \nu_{vl} w_{vl}(t)\, vL(t-1) + \nu_m p_m(t)\, M(t-1) + \\
&+ \nu_h w_h(t)(1 + \tau_h(t))\, R_{\max}(t)\, H(t-1), \\
&L(t) = K(t) - \beta_l(t)\, L(t-1) + L(t-1), \\
&vL(t) = vK(t) - \beta_{vl}(t)\, vL(t-1) + vL(t-1).
\end{aligned}
$$

3. THE MODEL: IDENTIFICATION AND RESULTS

The model is identified on the Russian macroeconomic data from 2005q1 to 2021q1. The data used in the model can be divided into four groups. First, it is GDP series in the current and fixed prices and its elements by use (gross fixed capital formation and changes of stocks) and incomes (wages and profits). This data is published by Rosstat, but it is currently presented by several partially overlapping series. We compose unified time series of GDP and its components using the assumption of preserving growth rates. GDP and its components have significant seasonalality, so the seasonality elimination procedure described in [21] is used. This procedure is chosen because it performs rather well for the short time series.

Second, several indicators from the balance of the Russian banking system published by the Bank of Russia are used. These indicators include volumes of loans to producers in rubles and in foreign currency, its interest rates and durations. Moreover, we use current accounts of non-financial organizations as the indicator of liquidity available to Producer. All these indicators do not have a significant seasonal component, so no additional transformation is required.

Third, the tax data published by the Federal Treasury is used. The taxes related to Producer include value added tax, property tax, profit tax and social payments included in the consolidated budget of Russia and budgets of non-budgetary funds. The treatment of this data is the most complicated because it has not only seasonal component but also in-year irregularity – the taxes not collected in the current quarter are paid in the next one. The algorithm similar to the one described in [21] is used for the effective tax rates calculated as the ratio of the collected tax to its base.

Finally, forth group consists of all the other indicators such as dollar to ruble exchange rate, volume of fixed assets and amortization rate. The last two indicators are published on a yearly basis, so they were decomposed into a quarterly data using the assumption of the fixed growth rate during the year.

The parameters of the model are estimated using the following approach. We consider growth rates of the following eight variables: Y, R, M, L, vL, N, Pr, W to the corresponding quarter of the previous year. For example, in case of GDP we have the following growth rates from the model and from data:

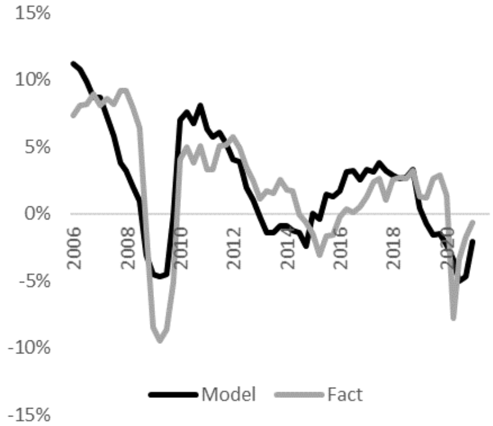
$$
grYmod(t) = \frac{Y(t) - Ystat(t-4)}{Ystat(t-4)}, grYstat(t) = \frac{Ystat(t) - Ystat(t-4)}{Ystat(t-4)}, \tag{3.18}
$$

where $Ystat$ is the statistical value of the real GDP, Y is the value calculated via the model. The function of errors is the sum of squares of $grYmod(t) - grYstat(t)$ for every t and similar expressions for other variables. It should be noted that flow variables such as K and vK are not considered because of their exceptionally high volatility. This leads to the problem of unconstrained optimization, because every parameter of the model can be an arbitrary real number, which is solved using lsqnonlin command from MATLAB Optimization package. We use the Monte Carlo method – initial points are generated in a rather wide region based on our a priori assumptions on the values of the model parameters and, after large number of iterations, we find the set of parameters which provides a good fit of the data and is also economically meaningful. Mean absolute percent errors of the model variables are presented in the table 3.1.

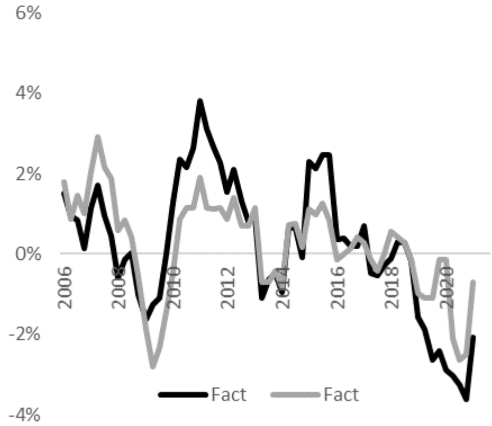
Table 3.1. Mean absolute percent errors of the model variables growth rates

Y	R	M	L	vL	N	Pr	W
2.46	0.82	1.62	2.96	6.43	6.86	3.03	2.85

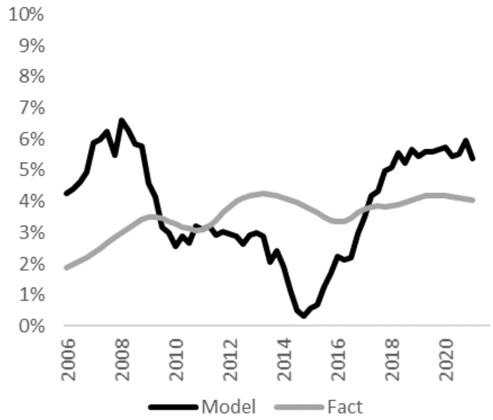
Moreover, the accuracy of data fit is shown on the plots below.



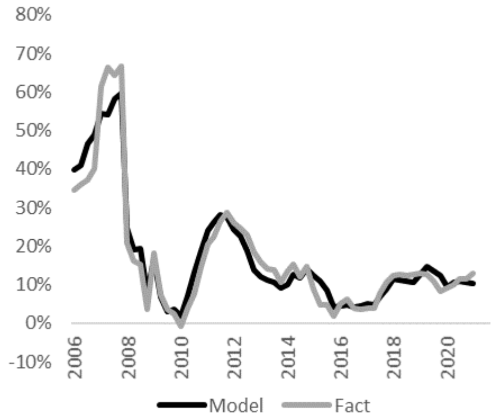
(a) GDP growth rate



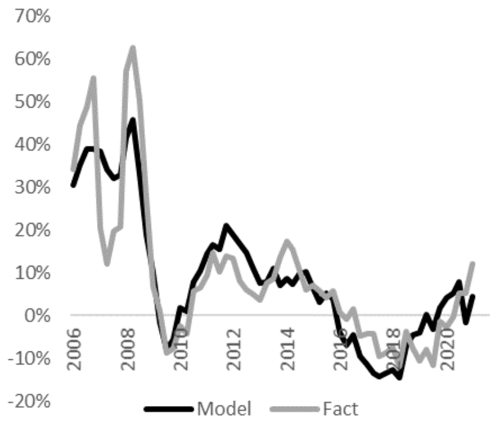
(b) Number of employed growth rate



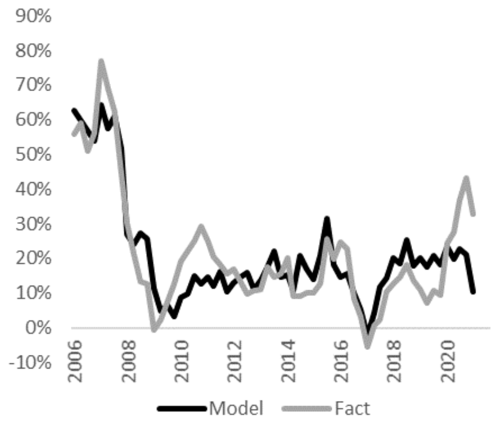
(c) Fixed assets growth rate



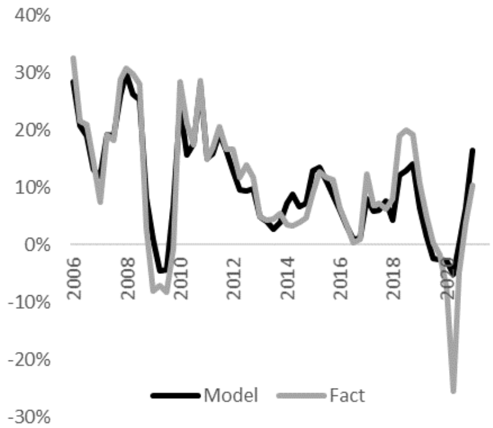
(d) Ruble loans growth rate



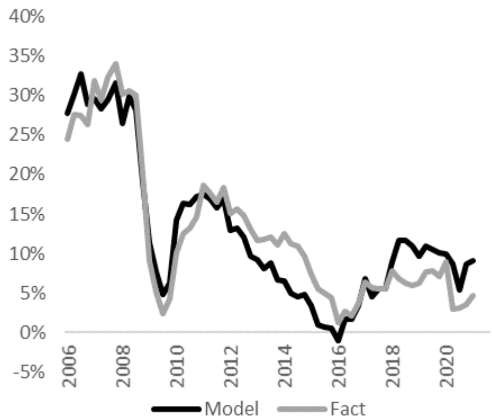
(e) Foreign currency loans growth rate



(f) Current accounts growth rate



(g) Profit growth rate



(h) Wages growth rate

As we can see in the plots above and the table 3.1, the model performs rather well for all model variables. The percent error for vL is relatively high because the values of this variable are very small, so small errors in absolute terms may lead to high relative errors. The percent error for N is relatively high because of the volatility of this variable.

4. CONCLUSION

The model of the macroeconomic agent Producer is presented. To our knowledge, this is the first model which describes not only the technologic and demographic, but also financial constraints of firms and other agents who produce the added value. Application of the relaxation of complementary slackness conditions method allows to transform the model to a nonlinear dynamic system. The model is estimated on the Russian macroeconomic data, and the set of parameters was found which allows to reproduce the dynamics of Russian macroeconomic indicators with rather high accuracy.

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REFERENCES

[1] D. Acemoglu, S. Johnson, J. Robinson, and Y. Thaicharoen. Institutional causes, macroeconomic symptoms: Volatility, crises and growth. *Journal of monetary economics*, 50(1):49–123, 2003.

[2] D. A. Ackerberg, K. Caves, and G. Frazer. Identification properties of recent production function estimators. *Econometrica*, 83(6):2411–2451, 2015.

[3] P. Antras. Is the us aggregate production function cobb-douglas? new estimates of the elasticity of substitution. *Contributions in Macroeconomics*, 4(1).

[4] K. J. Arrow, H. B. Chenery, B. S. Minhas, and R. M. Solow. Capital-labor substitution and economic efficiency. *The review of Economics and Statistics*, pages 225–250, 1961.

[5] D. Backus, E. Henriksen, and K. Storesletten. Taxes and the global allocation of capital. *Journal of Monetary Economics*, 55(1):48–61, 2008.

[6] E. R. Berndt. Reconciling alternative estimates of the elasticity of substitution. *The review of economics and statistics*, pages 59–68, 1976.

[7] R. Blundell and S. Bond. Gmm estimation with persistent panel data: an application to production functions. *Econometric reviews*, 19(3):321–340, 2000.

[8] R. S. Chirinko. Corporate taxation, capital formation, and the substitution elasticity between labor and capital. *National Tax Journal*, 55(2):339–355, 2002.

[9] R. M. Coen. Tax policy and investment behavior: Comment. *The American Economic Review*, 59(3):370–379, 1969.

[10] P. J. Dhrymes and P. Zarembka. Elasticities of substitution for two-digit manufacturing industries: A correction. *The Review of Economics and Statistics*, pages 115–117, 1970.

[11] B. Dunaev. Calculating gross domestic product as a function of labor and capital. *Cybernetics and Systems Analysis*, 40(1):86–96, 2004.

[12] D. Ecfin. The production function approach to calculating potential growth and output gaps estimates for eu member states and the us. 2006.

[13] R. Eisner and M. I. Nadiri. Investment behavior and neo-classical theory. *The review of economics and statistics*, pages 369–382, 1968.

[14] N. P. Epstein and C. Macchiarelli. *Estimating Poland's Potential Output: a Production Function Approach*. Number 10-15. International Monetary Fund, 2010.

[15] A. Gandhi, S. Navarro, and D. Rivers. On the identification of production functions: How heterogeneous is productivity? Technical report, CIBC Working Paper, 2011.

[16] T. Kawamoto, T. Ozaki, N. Kato, K. Maehashi, et al. Methodology for estimating output gap and potential growth rate: An update. 2017.

[17] J. Levinsohn and A. Petrin. Estimating production functions using inputs to control for unobservables. *The review of economic studies*, 70(2):317–341, 2003.

[18] R. E. Lucas Jr and L. A. Rapping. Real wages, employment, and inflation. *Journal of political economy*, 77(5):721–754, 1969.

[19] G. Maddala. Productivity and technological change in the bituminous coal industry, 1919-54. *Journal of Political Economy*, 73(4):352–365, 1965.

[20] G. S. Olley and A. Pakes. The dynamics of productivity in the telecommunications equipment industry. *Econometrica: Journal of the Econometric Society*, pages 1263–1297, 1996.

[21] N. Pilnik, I. Pospelov, and I. Stankevich. Ob ispolzovanii fiktivnikh peremennikh dlya resheniya problemy sezonnosti v modelyah obshego ravnovesiya (on the use of dummy variables for the solution of the seasonality problem in the general equilibrium models). *HSE Economic Journal*, 19(2):249–270, 2015.

[22] N. Pilnik, S. Radionov, and A. Yazikov. The model of the russian banking system with indicators nominated in rubles and in foreign currency. In Y. Evtushenko, M. Jaćimović, M. Khachay, Y. Kochetov, V. Malkova, and M. Posypkin, editors, *Optimization and Applications*, pages 427–438, Cham, 2019. Springer International Publishing.

[23] A. Swamy and B. Fikkert. Estimating the contributions of capital and labor to gdp: An instrumental variable approach. *Economic Development and Cultural Change*, 50:693–708, 02 2002.

