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A Controllability Problem for Causal Functional Inclusions with an Infinite Delay and Impulse Conditions

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Abstract: In this paper we study the controllability problem in a Banach space for various classes of functional inclusions with causal operators with an infinite delay, and impulse effects. Basing on the topological degree theory for condensing multimaps, we prove a global theorem on the existence of trajectories for systems governed by functional inclusions. As an application, we obtain generalizations of existence theorems for the controllability problem for a semilinear first order functional differential inclusions of this type and a semilinear functional differential inclusions of a fractional order $0 < q < 1$.

Keywords: causal operator, functional inclusion, controllability problem, functional differential inclusion, fractional derivative, measure of non-compactness, fixed point, topological degree, condensing multioperator

1. INTRODUCTION

It is well known that the contemporary approach in the theory of control systems and in mathematical physics leads to models which are convenient to be described by using differential equations and inclusions. Recently, the attention of many researchers (see [1], [2], [3] and references therein) has been attracted to generalizations of differential equations and inclusions, namely to the class of functional equations and inclusions with causal operators. The term of a causal or Volterra operator in the sense of A.N. Tikhonov (see [4]), was used in mathematical physics to solve problems of differential equations, integro-differential equations, functional-differential equations with a finite or infinite delay, integral equations of Volterra type, functional equations of a neutral type, etc. (see, for example, [5]). The papers [6], [7], [8], [9] among others are devoted to the study of equations and inclusions with causal operators of various types, theorems on the existence of solutions, description of qualitative properties of solutions and various applications.

At the same time, in recent decades the interest to the theory of fractional-order differential equations has significantly increased, thanks to applications in various branches of applied mathematics, physics, engineering, biology, economics, etc. (see, for example, monographs [10], [11] papers [12], [13], [14], [15], [16], etc.). The boundary value problems of various types for fractional differential equations and inclusions were considered in the works [17], [18], [19], [20], [21], [22], [23].

In this paper we develop and generalize the results of papers [2] and [3], and we study the controllability problem in Banach spaces for various classes of functional inclusions with causal operators with an infinite delay and impulse effects. Basing on the topological degree

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theory for condensing multimaps we prove a global theorem on the existence of trajectories for systems governed by functional inclusions. As an application, we obtain generalizations of existence theorems for solutions of the controllability problem for a first order semilinear functional differential inclusions and a semilinear functional differential inclusions of a fractional order $0 < q < 1$.

2. PRELIMINARIES

2.1. Multivalued maps and measures of noncompactness

Let X be a metric space and Y a normed space. Introduce the following notation:
$P(Y)$ denotes the collection of all non-empty subsets of Y;
$Pb(Y)$ denotes the collection of all non-empty and bounded subsets of Y;
$C(Y)$ denotes the collection of all non-empty and closed subsets of Y;
$Cv(Y)$ denotes the collection of all non-empty, closed and convex subsets of Y;
$K(Y)$ denotes the collection of all non-empty and compact subsets of Y;
$Kv(Y)$ denotes the collection of all non-empty, compact and convex subsets of Y.
Let us recall some notations (see, for example, [24], [25]).

Definition 2.1:
A multivalued map (multimap) $\mathcal{F} : X \to P(Y)$ is said to be upper semicontinuous (u.s.c.) at a point $x \in X$, if for every open set $V \subset Y$ such that $\mathcal{F}(x) \subset V$, there exists a neighborhood $U(x)$ of x such that $\mathcal{F}(U(x)) \subset V$.

Definition 2.2:
A multivalued map (multimap) $\mathcal{F} : X \to P(Y)$ is called closed if its graph $G_{\mathcal{F}} = \{(x, y) : x \in X, y \in \mathcal{F}(x)\}$ is a closed subset of $X \times Y$.

Definition 2.3:
A multivalued map (multimap) $\mathcal{F} : X \to P(Y)$ is called quasicompact if its restriction to each compact subset $A \subset X$ is compact.

Lemma 2.1:
([24], Theorem 1.1.12). If $\mathcal{F} : X \to K(Y)$ a closed quasicompact multimap, Then $\mathcal{F}$ is u.s.c.

Definition 2.4:
For a given $p \geq 1$, a multifunction $G : [0, T] \to K(Y)$ is called:

- *L^p–integrable* if it admits an L^p–Bochner integrable selection, i.e., there exists a function $g \in L^p([0, T]; Y)$ such that $g(t) \in G(t)$ for a.e. $t \in [0, T]$;
- *L^p–integrably bounded* if there exists a function $\xi \in L^p([0, T])$ such that

$$\|G(t)\| \leq \xi(t)$$

for a.e. $t \in [0, T]$.

Let $\mathcal{E}$ be a Banach space

Lemma 2.2:
(see [24], Theorem 4.2.1.) Let a sequence of functions $\{\xi_n\} \subset L^1([0, T]; E)$ be L^1–integrably bounded. Suppose that

$$\chi(\{\xi_n\}(t)) \leq \alpha(t) \quad \text{a.e. } t \in [0, T]$$

for all $n = 1, 2, ...$, where $\alpha \in L^1_+([0, T])$. Then for every $\delta > 0$ there exist a compact set $K_\delta \subset E$, a set $m_\delta \subset [0, T]$ of a Lebesgue measure $m_\delta < \delta$, and a set of functions $G_\delta \subset L^1([0, T]; E)$ with values in K_δ, such that for every $n \geq 1$ there exists a function $b_n \in G_\delta$ for *which*

$$\|\xi_n(t) - b_n(t)\|_E \leq 2\alpha(t) + \delta, \qquad t \in [0,T] \setminus m_\delta.$$

Moreover, the sequence $\{b_n\}$ may be chosen so that $b_n \equiv 0$ on m_δ and this sequence is weakly compact.

Definition 2.5:
Let $(\mathcal{A}, \geq)$ be a partially ordered set. A function $\beta : Pb(\mathcal{E}) \to \mathcal{A}$ is called the measure of noncompactness (MNC) in $\mathcal{E}$ if for each $\Omega \in Pb(\mathcal{E})$ we have:

$$\beta(\overline{\text{co}}\,\Omega) = \beta(\Omega),$$

where $\overline{\text{co}}\,\Omega$ denotes the closure of the convex hull of Ω.

A measure of noncompactness β is called:

1) *monotone* if for each $\Omega_0, \Omega_1 \in Pb(\mathcal{E})$, from $\Omega_0 \subseteq \Omega_1$ follows $\beta(\Omega_0) \leq \beta(\Omega_1)$.
2) *nonsingular*, if for each $a \in E$ and each $\Omega \in Pb(\mathcal{E})$ we have $\beta(\{a\} \cup \Omega) = \beta(\Omega)$.

If $\mathcal{A}$ is a cone in a Banach space, the MNC β is called:

3) *regular*, if $\beta(\Omega) = 0$ is equivalent to the relative compactness of $\Omega \in Pb(\mathcal{E})$;
4) *real*, if $\mathcal{A}$ is the set of all real numbers $\mathbb{R}$ with the natural ordering.

As the example of a real MNC obeying all above properties, we can consider the Hausdorff MNC $\chi(\Omega)$:

$$\chi(\Omega) = \inf\{\varepsilon > 0, \text{for which } \Omega \text{ has a finite } \varepsilon\text{-net in } \mathcal{E}\ \}.$$

As other examples, consider the measures of noncompactness defined in the space of continuous functions $C([a,b];E)$ with values in the Banach space E:

(1) *the modulus of fiber noncompactness:*

$$\varphi(\Omega) = \sup_{t\in[a,b]} \chi_E(\Omega(t)),$$

where χ_E is the Hausdorff MNC in E and $\Omega(t) = \{y(t) : y \in \Omega\}$;

(2) *the fading modulus of fiber noncompactness:*

$$\gamma(\Omega) = \sup_{t\in[a,b]} e^{-Lt}\chi_E(\Omega(t)),$$

where $L > 0$ is a given number;

(3) *the modulus of equicontinuity:*

$$\text{mod}_C(\Omega) = \lim_{\delta\to 0} \sup_{y\in\Omega} \max_{|t_1-t_2|\leq\delta} \|y(t_1) - y(t_2)\|.$$

These measures of noncompactness satisfy all the above properties, except for the regularity.

Definition 2.6:
A multimap $\mathcal{F} : X \subseteq \mathcal{E} \to K(\mathcal{E})$ is called condensing with respect to a MNC β (or β–condensing) if for each bounded set $\Omega \subseteq X$ which is not relatively compact, we have:

$$\beta(F(\Omega)) \ngeq \beta(\Omega).$$

Let $\mathcal{D} \subset \mathcal{E}$ a non-empty closed convex subset, V a non-empty bounded open subset of $\mathcal{D}$, β a monotone nonsingular MNC in $\mathcal{E}$ and $\mathcal{F} : \overline{V} \to Kv(\mathcal{D})$ be a u.s.c. β-condensing map such that $x \notin \mathcal{F}(x)$ for all $x \in \partial V$, where $\overline{V}$ and ∂V denote the closure and the boundary of the set V in the relative topology of $\mathcal{D}$.

In such a setting, the (relative) topological degree

$$\deg_{\mathcal{D}}\left(i - \mathcal{F}, \overline{V}\right)$$

of the corresponding vector field $i - \mathcal{F}$, satisfying the standard properties is defined (see, for example, [24], [25]). In particular, the condition

$$\deg_{\mathcal{D}}\left(i - \mathcal{F}, \overline{V}\right) \neq 0$$

implies that the fixed points set $Fix\mathcal{F} = \{x : x \in \mathcal{F}(x)\}$ is a nonempty subset of V.

Application of topological degree theory leads to the following fixed point principles, which will be used in the what follows.

Theorem 2.1:
([24], Corollary 3.3.1). Let $\mathcal{M}$ be a convex closed bounded subset of $\mathcal{E}$ and $\mathcal{F} : \mathcal{M} \to Kv(\mathcal{M})$ a β–condensing multimap, where β is a monotone nonsingular MNC in $\mathcal{E}$. Then the fixed point set $\operatorname{Fix}\mathcal{F}$ is non-empty.

Theorem 2.2:
([24], Theorem 3.3.4). Let $V \subset \mathcal{D}$ be a bounded open neighborhood of a point $a \in V$ and $\mathcal{F} : \overline{V} \to Kv(\mathcal{D})$ a u.s.c. β-condensing multimap, where β is a monotone nonsingular MNC in $\mathcal{E}$, satisfying the boundary condition

$$x - a \notin \lambda(\mathcal{F}(x) - a)$$

for all $x \in \partial V$ and $0 < \lambda \leq 1$. Then $Fix\mathcal{F} \neq \emptyset$ is a non-empty compact set.

2.2. Phase space

We will use the axiomatic definition of the phase space $\mathcal{B}$, introduced by J.K. Hale and J. Kato (see. [26], [27]).The space $\mathcal{B}$ we will be considered as a linear topological space of functions defined on $(-\infty, 0]$ with values in a Banach space E endowed with the seminorm $\|\cdot\|_{\mathcal{B}}$.

For all function $x\colon (-\infty, T] \to E$, where $T > 0$, and every $t \in (-\infty, T]$, x_t is a function from $(-\infty, 0]$ to E, defined as

$$x_t(\theta) = x(t + \theta),\ \theta \in (-\infty, 0].$$

We will be assume that $\mathcal{B}$ satisfies the following axioms:

($\mathcal{B}1$) if the function $x : (-\infty; T] \to E$ is continuous on $[0; T]$ and $x_0 \in \mathcal{B}$, then for each $t \in [0; T]$:
- (i) $x_t \in \mathcal{B}$;
- (ii) the function $t \mapsto x_t$ is continuous;
- (iii) $\|x_t\|_{\mathcal{B}} \leq K(t) \sup\limits_{0 \leq \tau \leq t} \|x(\tau)\| + H(t)\|x_0\|_{\mathcal{B}}$, where the functions $K, H : [0; \infty) \to [0; \infty)$ are independent of x, K is strictly positive and continuous and H is locally bounded.

($\mathcal{B}0$) there exists $l > 0$ such that $\|\psi(0)\|_E \leq l\|\psi\|_{\mathcal{B}}$, for all $\psi \in \mathcal{B}$.

Notice that under these conditions the space C_{00} of all continuous functions from $(-\infty, 0]$ to E with compact support into phase space $\mathcal{B}$([27], Proposition 1.2.1).

In addition, we will assume that the following condition is satisfied:

($\mathcal{BC}1$) if a uniformly bounded sequence $\{\psi_n\}_{n=1}^{+\infty} \subset C_{00}$ converges to a function ψ compactly (i.e. uniformly on each compact subset $(-\infty, 0]$), then $\psi \in \mathcal{B}$ and $\lim\limits_{n \to +\infty} \|\psi_n - \psi\|_{\mathcal{B}} = 0$.

The condition ($\mathcal{BC}1$) implies that the Banach space of bounded continuous functions $BC = BC((-\infty, 0]; E)$ is continuously embedded into $\mathcal{B}$. More precisely, the following assertion is true.

Theorem 2.3:
([27], Proposition 7.1.1).

(i) $BC \subset \overline{C_{00}}$, *where* $\overline{C_{00}}$ *denote the closure of* C_{00} *in* $\mathcal{B}$;
(ii) *if a uniformly bounded sequence* $\{\psi_n\}$ *in* BC *converges to a function* ψ *compactly on* $(-\infty, 0]$, *then* $\psi \in \mathcal{B}$ *and* $\lim\limits_{n\to+\infty} \|\psi_n - \psi\|_{\mathcal{B}} = 0$;
(iii) *there exists* $L > 0$ *such that* $\|\psi\|_{\mathcal{B}} \leq L\|\psi\|_{BC}$ *for all* $\psi \in BC$.

Finally, we will assume that the following condition is satisfied:

($\mathcal{BC}2$) if $\psi \in BC$ and $\|\psi\|_{BC} \neq 0$, then $\|\psi\|_{\mathcal{B}} \neq 0$.

This assumption implies that the space BC, endowed with $\|\cdot\|_{\mathcal{B}}$ is a normed space. We will denote it as $\mathcal{BC}$.

We may consider the following examples of phase spaces satisfying all the above properties:

(1) for $\gamma > 0$ let $\mathcal{B} = C_\gamma$ be the space of continuous functions $\varphi : (-\infty; 0] \to E$, having a limit $\lim\limits_{\theta\to-\infty} e^{\gamma\theta}\varphi(\theta)$ with

$$\|\varphi\|_{\mathcal{B}} = \sup_{-\infty<\theta\leq 0} e^{\gamma\theta}\|\varphi(\theta)\|.$$

(2) (Spaces of "fading memory") Let $\mathcal{B} = C_\rho$ be the space of functions $\varphi : (-\infty; 0] \to E$ such that
   (a) φ is continuous on $[-r; 0], r > 0$;
   (b) φ is Lebesgue measurable on $(-\infty; r)$ and there exists a nonnegative Lebesgue integrable function $\rho : (-\infty; -r) \to \mathbb{R}^+$ such that $\rho\varphi$ Lebesgue integrable on $(-\infty; r)$; moreover, there exists a locally bounded function $P : (-\infty; 0] \to \mathbb{R}^+$ such that, for all $\xi \leq 0$, $\rho(\xi + \theta) \leq P(\xi)\rho(\theta)$ a.e. $\theta \in (-\infty; -r)$.

   Then,

$$\|\varphi\|_{\mathcal{B}} = \sup_{-r\leq\theta\leq 0} \|\varphi(\theta)\| + \int_{-\infty}^{-r} \rho(\theta)\|\varphi(\theta)\|d\theta.$$

A simple example of such a space is given by $\rho(\theta) = e^{\mu\theta}$, $\mu \in \mathbb{R}$.

2.3. Causal multioperators with infinite delay

Let E be a separable Banach space. By $L^p([0, T]; E)$, $1 \leq p \leq \infty$, we denote the Banach space of all Bochner summable functions $f : [0, T] \to E$ with the usual norm.

For each subset $\mathcal{N} \subset L^p([0, T]; E)$ and $\tau \in (0, T)$ we define restriction $\mathcal{N}$ on $[0, \tau]$ as

$$\mathcal{N}|_{[0,\tau]} = \{f|_{[0,\tau]} : f \in \mathcal{N}\}.$$

We split the segment $[0, T]$ by points $0 < t_1 < ... < t_m < T, m \geq 1$. For a function $c : [0, T] \to E$ we will denote

$$c(t_k^+) = \lim_{\xi\to 0+} c(t_k + \xi),$$

$$c(t_k^-) = \lim_{\xi\to 0-} c(t_k + \xi),$$

for $1 \leq k \leq m$.

For a function $g : (-\infty, T] \to E$, we will assume that its restriction to $[0, T]$ belongs to the space $\mathcal{PC}([0, T]; E)$ of functions $z : [0, T] \to E$, continuous on $[0, T] \setminus \{t_1, \cdots, t_m\}$ and such that the left and right limits $z(t_k^+)$ and $z(t_k^-)$, $1 \leq k \leq m$, exist and $z(t_k^-) = z(t_k)$.

It is easy to see that the space $\mathcal{PC}([0, T]; E)$, endowed with the norm

$$\|z\|_{\mathcal{PC}} = \sup_{t \in [0,T]} \|g(t)\|_E,$$

is a Banach space. The classical space of continuous function $C([0, T], E)$ is its closed subspace.

We denote by $\mathcal{C}((-\infty; T]; E)$ the normed space of piecewise continuous functions $x : (-\infty; T] \to E$, endowed with the norm

$$\|x\|_{\mathcal{C}} = \|x_0\|_{\mathcal{BC}} + \|x \mid_{[0;T]} \|_{\mathcal{PC}}.$$

Definition 2.7:
A multivalued map $\mathcal{Q} : \mathcal{C}((-\infty, T]; E) \multimap L^p([0, T]; E)$ is said to be a causal multioperator, if for each $\tau \in (0, T)$ and for every $u(\cdot), v(\cdot) \in \mathcal{C}((-\infty, T]; E)$ the condition $u \mid_{(-\infty,\tau]} = v \mid_{(-\infty,\tau]}$ implies that $\mathcal{Q}(u) \mid_{[0,\tau]} = \mathcal{Q}(v) \mid_{[0,\tau]}$.

Let us give examples of causal multioperators.

Example 2.1:
We assume that the multimap $F : [0, T] \times \mathcal{BC} \times E \to Kv(E)$ satisfies the following conditions:

- $(F1)$ *for each $(\psi, \phi) \in \mathcal{BC} \times E$ the multifunction $F(\cdot, \psi, \phi) : [0, T] \to Kv(E)$ admits a measurable selection;*
- $(F2)$ *for a.e. $t \in [0, T]$ the multifunction $F(t, \cdot, \cdot) : \mathcal{BC} \times E \to Kv(E)$ is u.s.c.;*
- $(F3)$ *there exists a function $\alpha \in L^p_+[0, T]$, $1 \leq p \leq \infty$, such that*

$$\|F(t, \psi, \phi)\|_E := \sup\{\|z\|_E : z \in F(t, \psi, \phi)\} \leq \alpha(t)(1 + \|\psi\|_{\mathcal{BC}} + \|\phi\|_E)$$

for a.e. $t \in [0, T]$ and $(\psi, \phi) \in \mathcal{BC} \times E$.

From above conditions $(F1) - (F3)$ and $(\mathcal{B}1)$ it follows that the multimap $\mathcal{P}_F : \mathcal{C}((-\infty; T]; E) \to P(L^p([0, T]; E))$, given in the following way

$$\mathcal{P}_F(x) = \{f \in L^p([0, T]; E) : f(t) \in F(t, x_t, x(t)) \text{ a.e. } t \in [0, T]\}$$

is well defined (see, for example, [24], [25]). It is clear that the multioperator $\mathcal{P}_F$ is causal.

Example 2.2:
Let $F : [0, T] \times \mathcal{BC} \to Kv(E)$ be a multimap satisfying conditions $(F1) - (F3)$ from Example 2.1. Suppose that $\{K(t, s) : 0 \leq s \leq t \leq T\}$ is a continuous (with respect to the corresponding norm) family of bounded linear operators in E and $m \in L^1([0, T]; E)$ is a given function. Consider the Volterra integral multioperator $\mathcal{V} : \mathcal{C}((-\infty, T]; E) \multimap L^1([0, T]; E)$ defined as

$$\mathcal{V}(u)(t) = m(t) + \int_0^t K(t, s)F(s, u_s)ds,$$

i.e.,

$$\mathcal{V}(u) = \{y \in L^1([0, T]; E) : y(t) = m(t) + \int_0^t K(t, s)f(s)ds : f \in \mathcal{P}_F(u)\}.$$

It is also clear that the multioperator $\mathcal{V}$ is causal.

3. THE CONTROLLABILITY PROBLEM FOR FUNCTIONAL INCLUSIONS WITH THE CAUSAL OPERATORS

We will assume that the causal operator $\mathcal{Q} : \mathcal{C}\left((-\infty, T]; E\right) \to C\left(L^p\left([0,T];E\right)\right)$ satisfies the following conditions:

$(\mathcal{Q}1)$ $\mathcal{Q}$ is weakly closed in the following sense: conditions $\{u_n\}_{n=1}^{\infty} \subset \mathcal{C}\left((-\infty, T]; E\right)$, $\{f_n\}_{n=1}^{\infty} \subset L^p\left([0,T];E\right)$, $1 \le p \le \infty$, $f_n \in \mathcal{Q}(u_n)$, $n \ge 1$, $u_n \to u_0$, $f_n \overset{L^1}{\rightharpoonup} f_0$ implies $f_0 \in \mathcal{Q}(u_0)$;

$(\mathcal{Q}2)$ there exists a function $\alpha \in L_+^{\infty}([0,T])$ such that

$$\|\mathcal{Q}\left(u\right)\left(t\right)\|_E \le \alpha\left(t\right)\left(1 + \|u\|_{\mathcal{C}}\right), \quad \text{for a.e. } t \in [0,T],$$

for all $u \in \mathcal{C}((-\infty, T]; E)$;

$(\mathcal{Q}3)$ there exists a function $\omega : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$(\omega 1)$ for all $x \in \mathbb{R}_+ : \omega(\cdot, x) \in L_+^p([0,T])$, $1 \le p \le \infty$, ;

$(\omega 2)$ for a.e. $t \in [0,T]$ a function $\omega(t, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, nondecreasing and quasihomogeneous in the sense that $\omega(t, \lambda x) \le \lambda \omega(t,x)$ for all $x \in \mathbb{R}_+$ and $\lambda \ge 0$;

$(\omega 3)$ for each bounded set $\Delta \subset \mathcal{C}\left((-\infty, T]; E\right)$ we have

$$\chi\left(\mathcal{Q}\left(\Delta\right)\left(t\right)\right) \le \omega\left(t, \sup_{s \in [0,t]} \varphi\left(\Delta_s\right)\right) \text{ for a.e. } t \in [0,T],$$

where the set $\Delta_s = \{y_s : y \in \Delta\} \subset \mathcal{BC}$ and φ is the modulus of fiber noncompactness in $\mathcal{BC}$.

Note that the condition $(\omega 2)$ means that $\omega(t, 0) = 0$ for a.e.$t \in [0,T]$ and as an example of such a function we can consider $\omega(t,x) = k(t) \cdot x$, where $k(\cdot) \in L_+^p([0,T])$.

Consider a linear operator $\mathcal{S} : L^p([0,T];E) \to C([0,T];E)$, which is causal in the sense that for every $\tau \in (0,T]$ and $f, g \in L^p([0,T];E)$ condition $f(t) = g(t)$ for a.e. $t \in [0,\tau]$ implies $\left(\mathcal{S}f\right)(t) = \left(\mathcal{S}g\right)(t)$ for all $t \in [0,\tau]$. Following [24], we impose the next conditions on operator $\mathcal{S}$:

$(\mathcal{S}1)$ for $1 \le p < \infty$ there exist $D \ge 0$ such that

$$\|\mathcal{S}f(t) - \mathcal{S}g(t)\|_E^p \le D \int_0^t \|f(s) - g(s)\|_E^p ds$$

for all $f, g \in L^p([0,T];E)$ and $0 \le t \le T$;
if $p = \infty$ there exist $D_1 \ge 0$ such that

$$\|\mathcal{S}f(t) - \mathcal{S}g(t)\|_E \le D_1 \int_0^t \|f(s) - g(s)\|_E ds$$

for all $f, g \in L^{\infty}([0,T];E)$ and $0 \le t \le T$.

$(\mathcal{S}2)$ for an arbitrary compact set $K \subset E$ and a sequence $\{f_n\}_{n=1}^{\infty} \subset L^p\left([0,T];E\right)$, $1 \le p \le \infty$, such that $\{f_n(t)\}_{n=1}^{\infty} \subset K$ for all $t \in [0,T]$ the weak convergence $f_n \overset{L^1}{\rightharpoonup} f_0$ implies $\mathcal{S}f_n \to \mathcal{S}f_0$ in $C([0,T];E)$.

Also we suppose that $\mathcal{S}$ satisfies the relation:

$(\mathcal{S}3)$ $\quad \left(\mathcal{S}f\right)(0) = 0$ for each function $f \in L^p([0,T];E)$.

Notice, that condition $(\mathcal{S}1)$ implies that the operator $\mathcal{S}$ satisfies the Lipschitz condition:

$(\mathcal{S}1')$ $\|\mathcal{S}f - \mathcal{S}g\|_C \leq D\|f-g\|_{L^1}$.

Consider following important examples.

(i) Let a closed linear operator $A : D(A) \subset E \to E$ be the infinitesimal generata of a C_0-semigroup $\{e^{At}\}_{t\geq 0}$. The operator $\mathcal{L} : L^1([0,T];E) \to C([0,T];E)$ defined as

$$\mathcal{L}f(t) = \int_0^t e^{A(t-s)} f(s)ds$$

is a special case of the operator $\mathcal{S}$.
Note that taking $A = 0$ we obtain, as a special case, the usual integral operator$\mathcal{L}_I$: $L^1([0,T];E) \to C([0,T];E)$,

$$\mathcal{L}_I f(t) = \int_0^t f(s)ds;$$

(ii) Let $A : D(A) \to E$ be a closed linear operator E generating a C_0- semigroup $\{U(t)\}_{t\geq 0}$. The operator $G : L^p([0,T];E) \to C([0,T];E)$, $p > 1/q$, defined as

$$Gf(t) = \int_0^t (t-s)^{q-1}\mathcal{T}(t-s)f(s)ds, \quad 0 < q < 1,$$

where

$$\mathcal{T}(t) = q\int_0^\infty \theta\xi_q(\theta)U(t^q\theta)d\theta, \qquad \xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\Psi_q(\theta^{-1/q}),$$

$$\Psi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \theta \in \mathbb{R}^+,$$

is a special case of the operator $\mathcal{S}$.

Lemma 3.1:
([24], Lemma 4.2.1, [12], Lemma 3.4). The operators $\mathcal{L}$ and $\mathcal{G}$ satisfy conditions $(\mathcal{S}1)-(\mathcal{S}3)$.

Consider a control system governed by a functional inclusion with causal operators $\mathcal{Q}$ and $\mathcal{S}$, of the following form:

$$y(t) \in \mathcal{G}(t)\big(\psi(0) + \Sigma_{t_k<t} I_k(y(t_k))\big) + \mathcal{S}\circ\mathcal{Q}(y)(t) + \mathcal{S}\circ Bu(t),\ t \in [0,T], \tag{3.1}$$

$$y(t) = \psi(t),\ t \in (-\infty, 0], \tag{3.2}$$

$$I_k y(t_k) = y(t_k^+) - y(t_k),\ k = 1,...,m, \tag{3.3}$$

where $\psi \in \mathcal{BC}$ is a given function, $I_k : E \to E$ are the impulse functions, $B : U \to E$ is a linear bounded operator, U is a Banach space of the controls,

$$\mathcal{G}(t) = \int_0^\infty \xi_q(\theta)U(t^q\theta)d\theta,$$

and a control function $u \in L^{\infty}([0,T];U)$.

We will assume that the corresponding linear problem is solvable, that is, there exists an operator W^{-1} which is a right inverse operator for $W : L^{\infty}([0,T];U) \to E$ of the following form

$$Wu = \int_0^T (T-s)^{q-1}\mathcal{T}(T-s)Bu(s)ds$$

Let the operator W^{-1} satisfy the following regularity condition:

(W) there exists a function $\sigma \in L^{\infty}_{+}([0,T])$ such that for every bounded set $\Omega \subset E$ we have:

$$\chi_U\left(W^{-1}(\Omega)(t)\right) \le \sigma(t)\chi_E(\Omega) \text{ for a.e. } t \in [0,T],$$

where χ_U is the Hausdorff MNC in U.

We impose the following conditions on the impulse functions I_k:

$(I1)$ the functions $I_k : E \to E, \quad 1 \le k \le m$, are completely continuous.
$(I2)$ the functions $I_k : E \to E, \quad 1 \le k \le m$, are globally bounded, i.e. there exists $N > 0$ such that $\|I_k x\| \le N$ for all $x \in E$.

Lemma 3.2:
(see [12]) The operator functions $\mathcal{G}$ and $\mathcal{T}$ possess the following properties:

1) *For each $t \in [0,T]$, $\mathcal{G}(t)$ and $\mathcal{T}(t)$ are linear bounded operators, more precisely, for each $x \in E$ we have*

$$\|\mathcal{G}(t)x\|_E \le M\|x\|_E, \|\mathcal{T}(t)x\|_E \le \frac{qM}{\Gamma(1+q)}\|x\|_E,$$

where $M = \sup_{t\ge 0}\|U(t)\|$

2) *the operator functions $\mathcal{G}(\cdot)$ and $\mathcal{T}(\cdot)$ are strongly continuous, i.e., functions $t \in [0,T] \to \mathcal{G}(t)x$ and $t \in [0,T] \to \mathcal{T}(t)x$ are continuous for each $x \in E$.*

Suppose $\psi \in \mathcal{BC}$ is a given function. For a function $y \in \mathcal{PC}([0,T];E)$ such that $y(0) = \psi(0)$ we define the function $y[\psi] \in \mathcal{C}((-\infty,T];E)$ as

$$y[\psi](t) = \begin{cases} \psi(t), & -\infty \le t < 0, \\ y(t), & 0 \le t \le T. \end{cases}$$

We denote by $\mathcal{D}$ the closed convex subset of $\mathcal{PC}([0,T];E)$, consisting of all functions y satisfying the condition $y(0) = \psi(0)$.

Definition 3.1:
A function $y \in \mathcal{C}((-\infty,T];E)$ is called a mild solution of problem (3.1)-(3.3)*, if it satisfies conditions:*

(1) the function $y|_{[0,T]} \in \mathcal{D}$ and satisfies inclusion (3.1)*;*
(2) $y(t) = \psi(t)$, for $t \in (-\infty,0]$;
(3) $I_k y(t_k) = y(t_k^+) - y(t_k),\ k = 1,...,m$.

Now, the controllability problem which we solve in this paper may be formulated in the following way: for a given initial function $\psi \in \mathcal{BC}$ and a given $x_1 \in E$ we consider the existence of a mild solution y of problem (3.1)-(3.3) and a control u such that $y(t) = \psi(t),\ t \in (-\infty,0]$, and

$$y(T) = x_1. \tag{3.4}$$

The pair (y, u) consisting of an integral solution y of problem (3.1)-(3.3) and a control $u \in L^{\infty}([0,T];U)$ will be called the solution of controllability problem (3.1)-(3.4).

Consider the multioperator $\Gamma : \mathcal{D} \multimap \mathcal{D}$ defined as

$$\Gamma(y) = \{x \in \mathcal{D} : x(t) = \mathcal{G}(t)\big(\psi(0) + \Sigma_{t_k<t} I_k(y(t_k))\big) +$$

$$\mathcal{S} \circ \mathcal{Q}(y[\psi]) + \mathcal{S} \circ BW^{-1}\big(x_1 - \mathcal{G}(T)\psi(0) - \zeta \circ \mathcal{S} \circ \mathcal{Q}(y[\psi])\big) -$$

$$\mathcal{S} \circ BW^{-1}(\Sigma_{k=1}^{m} \mathcal{G}(T) I_k(y(t_k)))\},$$

where $f \in \mathcal{Q}(y[\psi])$ and the linear operator $\zeta : C([0;T];E) \to E$ is defined as $\zeta x = x(T)$.

It is clear that if the function y is a fixed point of the multioperator Γ, then the pair $(y[\psi], u)$ is a solution to controllability problem (3.1) - (3.4), therefore, our goal is to prove the existence of fixed point of the multioperator Γ.

Definition 3.2:
A sequence of functions $\{\xi_n\} \subset L^p([0,T];E)$ is called L^p–semicompact if it is L^p–integrably bounded, i.e.,

$$\|\xi_n(t)\|_E \leq v(t) \textit{ for a.e. } t \in [0,T] \textit{ and for all } n = 1, 2, ...,$$

where $v \in L^p([0,T])$, and the set $\{\xi_n(t)\}$ is relatively compact in E for a.e. $t \in [0,T]$.

Lemma 3.3:
(see. [24], Proposition 4.2.1.) Every L^p–semicompact sequence is weakly compact in $L^1([0,T];E)$.

We need the following properties of the multioperator $\mathcal{S} \circ \mathcal{Q}$. Since for every $1 < p \leq \infty : L^p([0,T];E) \subset L^1([0,T];E)$, we can formulate a modification of Theorem 5.1.1 from [24] in the following form.

Lemma 3.4:
Let $\mathcal{S} : L^p([0,T];E) \to C([0,T];E)$ be an operator satisfying conditions $(\mathcal{S}1)$ and $(\mathcal{S}2)$. Then for every L^p-semicompact sequence $\{f_n\}_{n=1}^{\infty} \subset L^p([0,T];E)$, the sequence $\{\mathcal{S}f_n\}_{n=1}^{\infty}$ is relatively compact in $C([0,T];E)$ and, moreover, the weak convergence $f_n \stackrel{L^1}{\rightharpoonup} f_0$ implies that $\mathcal{S}f_n \to \mathcal{S}f_0$ in $C([0,T];E)$.

Theorem 3.1:
Let the multioperator $\mathcal{Q}$ satisfy conditions $(\mathcal{Q}1)$–$(\mathcal{Q}3)$ and the operator $\mathcal{S}$ satisfy $(\mathcal{S}1)$, $(\mathcal{S}2)$. Then the composition $\mathcal{S} \circ \mathcal{Q} : \mathcal{C}((-\infty,T];E) \multimap C([0,T];E)$ is a u.s.c. multimap with compact values.

Proof
Let us show that the multioperator $\mathcal{S} \circ \mathcal{Q}$ is closed. Let $\{x_n\}_{n=1}^{\infty} \subset \mathcal{C}((-\infty,T];E)$, $\{y_n\}_{n=1}^{\infty} \subset C([0,T];E)$, $x_n \to x_0$, $y_n \in \mathcal{S} \circ \mathcal{Q}(x_n)$, $n \geq 1$, and $y_n \to y_0$. Take an arbitrary sequence $\{f_n\}_{n=1}^{\infty} \subset L^p([0,T];E)$ such that $f_n \in \mathcal{Q}(x_n)$, $y_n = \mathcal{S}(f_n)$, $n \geq 1$. From condition $(\mathcal{Q}2)$ it follows that the sequence $\{f_n\}_{n=1}^{\infty}$ is L^p-integrally bounded. The condition $(\mathcal{Q}3)$ means that

$$\chi\left(\{f_n(t)\}_{n=1}^{\infty}\right) \leq \omega\left(t, \sup_{s\in[0,t]} \varphi\left(\{(x_n)_s\}_{n=1}^{\infty}\right)\right) = \omega(t,0) = 0$$

for a.e. $t \in [0,T]$, and therefore the sequence $\{f_n\}_{n=1}^{\infty}$ is L^p-semicompact.

From Lemma 3.3 it follows that the sequence $\{f_n\}_{n=1}^{\infty}$ is weakly compact, so we can assume without loss of generality that $f_n \stackrel{L^1}{\rightharpoonup} f_0$. By Lemma 3.4 we have $y_n = \mathcal{S}f_n \to Sf_0 =$

y_0. On the other hand, applying the condition $(\mathcal{Q}1)$ we obtain $f_0 \in \mathcal{Q}(x_0)$ and moreover $y_0 \in \mathcal{S} \circ \mathcal{Q}(x_0)$, i.e., the multioperator $\mathcal{S} \circ \mathcal{Q}$ is closed.

For each $x \in \mathcal{C}((-\infty, T]; E)$, conditions $(\mathcal{Q}2)$ and $(\mathcal{Q}3)$ mean that the sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{Q}(x)$ is L^p-semicompact and, by Lemma 3.4, the sequence $\{\mathcal{S}f_n\}_{n=1}^{\infty} \subset C([0,T]; E)$ is relatively compact. The compactness of the set $\mathcal{S} \circ \mathcal{Q}(u)$ follows from its closedness.

Finally, if we consider the converging sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{C}((-\infty, T]; E)$ and the arbitrary sequence $\{f_n\}_{n=1}^{\infty} \subset L^p([0,T]; E,)$ such that $f_n \in \mathcal{Q}(x_n)$, then the sequence $\{\mathcal{S}f_n\}_{n=1}^{\infty} \subset C([0,T]; E)$ is relatively compact, which means that the multimap $\mathcal{S} \circ \mathcal{Q}$ is quasicompact and applying Lemma 2.1 we obtain that it is u.s.c. $\square$

Let us proceed to finding conditions under which the multioperator $\mathcal{S} \circ \mathcal{Q}$ will be condensing with respect to a corresponding MNC. For this we need the following statements.

Lemma 3.5:
Let a sequence of functions $\{f_n\}_{n=1}^{\infty} \subset L^p([0,T]; E)$ be L^p-integrally bounded and there exist a function $\upsilon \in L^p_+([0,T])$ such that

$$\chi\left(\{f_n(t)\}_{n=1}^{\infty}\right) \leq \upsilon(t) \text{ for a.e. } t \in [0,T].$$

If an operator $\mathcal{S}$ satisfies conditions $(\mathcal{S}1)$ and $(\mathcal{S}2)$, then for $1 \leq p < \infty$ we have

$$\chi\left(\{\mathcal{S}f_n(t)\}_{n=1}^{\infty}\right) \leq \left(4^p D \int_0^t \upsilon^p(s)ds\right)^{1/p},$$

and for $p = \infty$

$$\chi\left(\{\mathcal{S}f_n(t)\}_{n=1}^{\infty}\right) \leq 2D_1 \int_0^t \upsilon(s)ds,$$

where D, D_1 are the constants from condition $(\mathcal{S}1)$.

Proof
For $\epsilon > 0$ we take $\delta \in (0, \epsilon)$, such that for every $m \subset [0,T]$, with the measure $meas(m) < \delta$, we have:

$$\int_m |\upsilon(s)|^p < \epsilon, \text{ for } 1 \leq p < \infty,$$

and respectively for $p = \infty$:

$$\int_m |\upsilon(s)| < \epsilon.$$

Taking m_δ and b_n corresponding to $\{f_n\}$ from Lemma 2.2, we, by using property $(\mathcal{S}1)$, obtain that the sequence $\{\mathcal{S}(b_n)\}$ is relatively compact in $C([0,T]; E)$.

Let $1 \leq p < \infty$, then the following estimates hold:

$$\|\mathcal{S}(f_n)(t) - \mathcal{S}(b_n)(t)\|_E^p \leq D \int_0^t \|f_n(s) - b_n(s)\|_E^p \, ds \leq$$

$$D \int_{[0,t]\setminus m_\delta} \|f_n(s) - b_n(s)\|_E^p \, ds + D \int_{[0,t]\cap m_\delta} \|f_n(s)\|_E^p \, ds \leq$$

$$D \int_{[0,t]\setminus m_\delta} [2\upsilon(s) - \delta]^p \, ds + D \int_{m_\delta} |\upsilon(s)|^p \, ds \leq D \int_0^t |2\upsilon(s) + \epsilon|^p \, ds + \epsilon D \leq$$

$$D \int_0^t 2^p \, |2\upsilon(s)|^p \, ds + D \int_0^t 2^p \epsilon^p ds + \epsilon D \leq 4^p D \int_0^t \upsilon^p(s)ds + 2^p \epsilon^p T D + \epsilon D.$$

Therefore, the relatively compact set $\mathcal{S}G_\delta(t)$ is a $\left(4^p D\int_0^t \upsilon^p(s)ds + 2^p\epsilon^p TD + \epsilon D\right)^{1/p}$ - net for the set $\{\mathcal{S}(f_n)(t)\}$. Since $\epsilon > 0$ is arbitrary, we obtain the conclusion of the lemma for the case $1 \leq p < \infty$.

If $p = \infty$, then the following estimates hold:

$$\|\mathcal{S}(f_n)(t) - \mathcal{S}(b_n)(t)\|_E \leq D_1 \int_0^t \|f_n(s) - b_n(s)\|_E \, ds \leq$$

$$D_1 \int_{[0,t]\setminus m_\delta} \|f_n(s) - b_n(s)\|_E \, ds + D_1 \int_{[0,t]\cap m_\delta} \|f_n(s)\|_E \, ds \leq$$

$$D_1 \int_{[0,t]\setminus m_\delta} |2\upsilon(s) - \delta| ds + D_1 \int_{m_\delta} |\upsilon(s)| \, ds \leq 2D_1 \int_0^t \upsilon(s)ds + \epsilon TD_1 + \epsilon D_1.$$

Thus, the relatively compact set $\mathcal{S}G_\delta(t)$ is a $2D_1 \int_0^t \upsilon(s)ds + \epsilon D_1(T+1)$ - net for the set $\{\mathcal{S}(f_n)(t)\}$. Since $\epsilon > 0$ is arbitrary, we obtain the conclusion of the lemma also for the case $p = \infty$. □

Let M_1, M_2 be positive constants, such that

$$\|B\| \leq M_1, \quad \left\|W^{-1}\right\| \leq M_2. \tag{3.5}$$

Consider the measure of noncompactness ν in the space $\mathcal{PC}([0,T];E)$ with values in the cone $\mathbb{R}^2_+$. On a bounded subset of $\Omega \subset \mathcal{PC}([0,T];E)$ we define the values of ν as follows:

$$\nu(\Omega) = \left(\gamma\left(\Omega\right), mod_{\mathcal{C}}\left(\Omega\right)\right),$$

where mod_C is the modulus of equicontinuity, γ is the fading modulus of fiber noncompactness

$$\gamma(\Omega) = \sup_{t\in[0,T]} e^{-Lt}\chi(\Omega(t)).$$

The constant $L > 0$ is chosen so that

$$\max\{q_1, q_2\} < 1,$$

where

$$q_1 = \sup_{t\in[0,T]} \left(4D^{1/p}\left(1 + 4M_1\sigma D^{1/p}T^{1/p}\right)\int_0^t e^{-Lp(t-s)}\omega^p\left(s,1\right)ds\right)^{1/p},$$

$$q_2 = \sup_{t\in[0,T]} \left(2D_1\left(1 + 2M_1\sigma D_1 T\right)\int_0^t e^{-L(t-s)}\omega\left(s,1\right)ds\right),$$

where the constants D, D_1 are from condition $(\mathcal{S}1)$, ω is a function from condition $(\mathcal{Q}3)$.

It is easy to see that the MNC ν is monotone, nonsingular, and algebraically semi-additive. It follows from the Arzela–Ascoli theorem that it is also regular.

Theorem 3.2:
Let a causal multioperator $\mathcal{Q} : \mathcal{C}((-\infty,T];E) \multimap L^p([0,T];E)$ satisfy conditions $(\mathcal{Q}2)$ and $(\mathcal{Q}3)$ and for a causal operator $\mathcal{S} : L^p([0,T];E) \to C([0,T];E)$ the conditions $(\mathcal{S}1)$–$(\mathcal{S}3)$ be satisfied. Then, under conditions $(I1)$, $(I2)$, (W) the multioperator Γ is ν-condensing.

Proof
By Lemma 3.2 and conditions $(I1),(I2)$, it is suffices to prove the assertion of the theorem for the multioperator

$$\mathcal{S}\circ\mathcal{Q}+\mathcal{S}\circ BW^{-1}\big(x_1-\mathcal{G}(T)\psi(0)-\zeta\circ S\circ\mathcal{Q}\big).$$

Let $\Omega\subset\mathcal{D}$ be a bounded set such that

$$\nu\left(\mathcal{S}\circ\mathcal{Q}\left(\Omega[\psi]\right)+\mathcal{S}\circ BW^{-1}\big(x_1-\mathcal{G}(T)\psi(0)-\zeta\circ S\circ\mathcal{Q}(\Omega[\psi])\big)\right)\geq\nu\left(\Omega\right). \qquad (3.6)$$

Let us show that the set Ω is relatively compact.
Inequality (3.6) means that

$$\gamma(\{\mathcal{S}\circ\mathcal{Q}\left(\Omega[\psi]\right)+\mathcal{S}\circ BW^{-1}\big(x_1-\mathcal{G}(T)\psi(0)-\zeta\circ S\circ\mathcal{Q}(\Omega[\psi])\big)\})\geq\gamma(\Omega). \qquad (3.7)$$

Applying the condition $(\mathcal{Q}3)$ and by using the properties of the function ω, we obtain for a.e. $t\in[0,T]$

$$\chi\left(\{f(t):f\in\mathcal{Q}\left(\Omega[\psi]\right)\}\right)\leq\omega\left(t,\sup_{s\in[0,t]}\varphi\left(\{y[\psi]_s:y\in\Omega\}\right)\right)=\omega\left(t,\varphi\left(\{y|_{[0,t]}:y\in\Omega\}\right)\right)=$$

$$\omega\left(t,e^{Lt}e^{-Lt}\varphi\left(\{y|_{[0,t]}:y\in\Omega\}\right)\right)\leq\omega\left(t,e^{Lt}\gamma\left(\{y|_{[0,t]}:y\in\Omega\}\right)\right)\leq$$
$$\omega\left(t,e^{Lt}\gamma\left(\Omega\right)\right)\leq\omega\left(t,e^{Lt}\right)\cdot\gamma\left(\Omega\right).$$

At first, we consider the case $1\leq p<\infty$. By Lemma 3.5 we have for each $t\in[0,T]$:

$$\chi\left(\{\mathcal{S}f(t):f\in\mathcal{Q}\left(\Omega[\psi]\right)\}\right)\leq\left(4^pD\int_0^t\omega^p\left(s,e^{Ls}\right)ds\cdot\gamma^p\left(\Omega\right)\right)^{1/p}\leq$$

$$4D^{1/p}\left(\int_0^t e^{pLs}\omega^p\left(s,1\right)ds\right)^{1/p}\cdot\gamma\left(\Omega\right).$$

Further,

$$\chi\left(\{BW^{-1}\big(x_1-\mathcal{G}(T)\psi(0)-\zeta\circ\mathcal{S}f(t):f\in\mathcal{Q}\left(\Omega[\psi]\right)\}\right)\leq$$
$$M_1\sigma\chi\left(\{\zeta\circ\mathcal{S}f(t):f\in\mathcal{Q}\left(\Omega[\psi]\right)\}\right)=$$
$$M_1\sigma\chi\left(\{\mathcal{S}f(T):f\in\mathcal{Q}\left(\Omega[\psi]\right)\}\right)\leq M_1\sigma\left(4^pD\int_0^T e^{pLs}\omega^p\left(s,1\right)ds\cdot\gamma^p\left(\Omega\right)\right)^{1/p}=$$
$$4M_1\sigma D^{1/p}\left(\int_0^T e^{pLs}\omega^p\left(s,1\right)ds\right)^{1/p}\cdot\gamma\left(\Omega\right),$$

where $\sigma=\sup_{t\in[0,T]}\sigma(t)$.
Using Lemma 3.5 again, we have for each $t\in[0,T]$:

$$\chi\left(\mathcal{S}\circ BW^{-1}\big(x_1-\mathcal{G}(T)\psi(0)-\zeta\circ\mathcal{S}f(t):f\in\mathcal{Q}\left(\Omega[\psi]\right)\right)\leq$$
$$\left(4^pD\int_0^t 4^pM_1^p\sigma^pD\Big(\int_0^T e^{pLs}\omega^p\left(s,1\right)ds\Big)d\tau\cdot\gamma^p\left(\Omega\right)\right)^{1/p}\leq$$

$$16D^{2/p}M_1\sigma T^{1/p}\Big(\int_0^T e^{pLs}\omega^p\left(s,1\right)ds\Big)^{1/p}\cdot\gamma\left(\Omega\right).$$

Inequality (3.7) and the last inequality imply the following

$$\gamma(\Omega)\leq\sup_{t\in[0,T]}\left(4D^{1/p}\left(1+4M_1\sigma D^{1/p}T^{1/p}\right)\int_0^t e^{-Lp(t-s)}\omega^p\left(s,1\right)ds\right)^{1/p}\gamma\left(\Omega\right)=q_1\cdot\gamma\left(\Omega\right),$$

therefore

$$\gamma\left(\Omega\right)=0,$$

thus

$$\varphi\left(\Omega[\psi]_t\right)=0$$

for all $t\in[0,T]$.

Let us turn to the case $p=\infty$. By Lemma 3.5 we have for each $t\in[0,T]$:

$$\chi\left(\{\mathcal{S}f(t):f\in\mathcal{Q}\left(\Omega[\psi]\right)\}\right)\leq 2D_1\int_0^t\omega\left(s,e^{Ls}\right)ds\cdot\gamma\left(\Omega\right)\leq 2D_1\int_0^t e^{Ls}\omega\left(s,1\right)ds\cdot\gamma\left(\Omega\right);$$

$$\chi\left(\{BW^{-1}\big(x_1-\mathcal{G}(T)\psi(0)-\zeta\circ\mathcal{S}f(t):f\in\mathcal{Q}\left(\Omega[\psi]\right)\}\right)\leq$$
$$M_1\sigma\chi\left(\{\zeta\circ\mathcal{S}f(t):f\in\mathcal{Q}\left(\Omega[\psi]\right)\}\right)=$$
$$M_1\sigma\chi\left(\{\mathcal{S}f(T):f\in\mathcal{Q}\left(\Omega[\psi]\right)\}\right)\leq M_1\sigma 2D_1\int_0^T e^{Ls}\omega\left(s,1\right)ds\cdot\gamma\left(\Omega\right)=$$
$$2M_1\sigma D_1\int_0^T e^{Ls}\omega\left(s,1\right)ds\cdot\gamma\left(\Omega\right),$$

where $\sigma=\sup_{t\in[0,T]}\sigma(t)$.

Using Lemma 3.5, we have for each $t\in[0,T]$:

$$\chi\left(\mathcal{S}\circ BW^{-1}\big(x_1-\mathcal{G}(T)\psi(0)-\zeta\circ\mathcal{S}f(t):f\in\mathcal{Q}\left(\Omega[\psi]\right)\right)\leq$$
$$2D_1\int_0^t 2M_1\sigma D_1\Big(\int_0^T e^{Ls}\omega\left(s,1\right)ds\Big)d\tau\cdot\gamma\left(\Omega\right)\leq$$
$$4D_1M_1\sigma T\int_0^T e^{Ls}\omega\left(s,1\right)ds\cdot\gamma\left(\Omega\right).$$

Inequality (3.7) and the last inequality imply the following

$$\gamma(\Omega)\leq\sup_{t\in[0,T]}\left(2D_1\left(1+2M_1\sigma D_1T\right)\int_0^t e^{-L(t-s)}\omega\left(s,1\right)ds\right)\gamma\left(\Omega\right)=q_2\cdot\gamma\left(\Omega\right),$$

therefore

$$\gamma\left(\Omega\right)=0,$$

thus

$$\varphi\left(\Omega[\psi]_t\right)=0$$

for each $t\in[0,T]$.

Now we will show that the set Ω is equicontinuous. We take sequences $\{y_n\}_{n=1}^{\infty}\subset\Omega,\quad n\geq 1$ and $\{f_n\}_{n=1}^{\infty}$, $f_n\in\mathcal{Q}(y_n[\psi])$. From conditions $(\mathcal{Q}2)$ and $(\mathcal{Q}3)$ it follows that the sequence $\{f_n\}_{n=1}^{\infty}$ is L^p-semicompact, and therefore by Lemma 3.4 the sequence $\{\mathcal{S}f_n\}_{n=1}^{\infty}$

is relatively compact. Hence
$$mod_{\mathcal{C}}(\{\mathcal{S}f_n\}_{n=1}^{\infty}) = 0.$$

From the conditions that the operators B, W^{-1}, ζ are bounded and linear, we can conclude that
$$mod_{\mathcal{C}}\Big(S \circ BW^{-1}(x_1 - \mathcal{G}(T)\psi(0) - \zeta\{Sf_n\}_{n=1}^{\infty})\Big) = 0.$$

Thus
$$\nu\left(\{\mathcal{S} \circ \mathcal{Q}\left(\Omega[\psi]\right) + \mathcal{S} \circ BW^{-1}\big(x_1 - \mathcal{G}(T)\psi(0) - \zeta \circ S \circ \mathcal{Q}(\Omega[\psi])\big)\}\right) = (0,0),$$

but then it follows from the inequality (3.6) that
$$\nu(\Omega) = (0,0),$$

and the last expression yields that the set Ω is relatively compact. □

To prove the main theorem of the paper, we need the following statements, known as the Gronwall - Bellman Lemma and the generalized Gronwall - Bellman Lemma.

Lemma 3.6:
Let $v(t)$ and $f(t)$ be nonnegative continuous functions on the segment $[a,b]$, moreover
$$v(t) \leq c + \int_a^t f(s)v(s)ds,\ t \in [a,b],$$
where c is a positive constant. Then for each $t \in [a,b]$ the inequality
$$v(t) \leq ce^{\int_a^t f(s)ds},$$
holds.

Lemma 3.7:
Let $h(t),\ u(t)$ and $v(t)$ be nonnegative functions integrable on $[a,b]$ satisfying the inequality:
$$v(t) \leq u(t) + \int_a^t h(s)v(s)ds, \quad t \in [a,b].$$
Then the following inequality holds:
$$v(t) \leq u(t) + \int_a^t e^{\int_a^t h(\theta)d\theta} h(s)u(s)ds, \quad t \in [a,b].$$

Theorem 3.3:
Let a causal multioperator $\mathcal{Q} : \mathcal{C}((-\infty,T];E) \to Cv(L^p([0,T];E)),\ 1 \leq p \leq \infty,$ satisfy conditions $(\mathcal{Q}1)$–$(\mathcal{Q}3)$ and a linear causal operator $\mathcal{S} : L^p([0,T];E) \to C([0,T];E)$ satisfy conditions $(\mathcal{S}1)$–$(\mathcal{S}3)$. Then, under conditions $(I1), (I2), (W)$ the set Σ_ψ of all solutions to problem (3.1)-(3.4) *is a non-empty compact set.*

Proof

Let us show that the set of all solutions $y \in \mathcal{D}$ of a one-parameter inclusion
$$y \in \lambda\Gamma(y), \quad \lambda \in [0,1], \tag{3.8}$$
is a priori bounded. We divide the proof into three cases: $p = 1$, $1 < p < \infty$, $p = \infty$.

Let $p = 1$, if $y \in \mathcal{D}$ satisfies condition (3.8), then for each $t \in [0, T]$, using assumptions $(\mathcal{B}0), (\mathcal{S}1), (I1), (I2)$ and (3.5), we have the following estimates:

$$\|y(t)\|_E \leq \|\mathcal{G}(t)\big(\psi(0) + \Sigma_{t_k<t} I_k(y(t_k))\big)\|_E + D\int_0^t \|f(s)\|_E ds +$$

$$DM_1M_2\int_0^t \|x_1 - \mathcal{G}(T)\psi(0) - \zeta \circ \mathcal{S}f(s)\|_E ds + DM_1M_2\int_0^t \|\Sigma_{k=1}^m \mathcal{G}(T) I_k(y(t_k))\|_E ds \leq$$

$$M(l\|\psi\|_{\mathcal{BC}} + mN) + D\int_0^t \|f(s)\|_E ds + DM_1M_2\int_0^t \|x_1 - \mathcal{G}(T)\psi(0)\|_E ds +$$

$$DM_1M_2\int_0^t \|\zeta \circ \mathcal{S}f(s)\|_E ds + DMM_1M_2mNT \leq$$

$$M(l\|\psi\|_{\mathcal{BC}} + mN) + D\int_0^t \|f(s)\|_E ds + DM_1M_2\left(\|x_1\|_E + Ml\|\psi\|_{\mathcal{BC}}\right)T +$$

$$DM_1M_2\int_0^t \left(D\int_0^T \|f(s)\|_E ds\right) d\tau + DMM_1M_2mNT \leq$$

$$M(l\|\psi\|_{\mathcal{BC}} + mN) + D\int_0^t \|f(s)\|_E ds + DM_1M_2\left(\|x_1\|_E + Ml\|\psi\|_{\mathcal{BC}}\right)T +$$

$$D^2M_1M_2T\int_0^T \|f(s)\|_E ds + DMM_1M_2mNT \leq$$

$$M(l\|\psi\|_{\mathcal{BC}} + mN) + DM_1M_2T\left(\|x_1\|_E + Ml\|\psi\|_{\mathcal{BC}} + MmN\right)$$

$$+(1 + DM_1M_2T)D\int_0^T \|f(s)\|_E ds,$$

where $f \in \mathcal{Q}(y[\psi])$ and, therefore, by condition $(\mathcal{Q}2)$:

$$\|f(s)\|_E \leq \alpha(s)(1 + \|y[\psi]\|_{\mathcal{C}}).$$

Then

$$\|y(t)\|_E \leq M(l\|\psi\|_{\mathcal{BC}} + mN) + DM_1M_2T\left(\|x_1\|_E + Ml\|\psi\|_{\mathcal{BC}} + MmN\right) +$$

$$(1 + DM_1M_2T)D\int_0^T \alpha(s)(1 + \|y[\psi]\|_{\mathcal{C}}) ds \leq$$

$$M(l\|\psi\|_{\mathcal{BC}} + mN) + DM_1M_2T\left(\|x_1\|_E + Ml\|\psi\|_{\mathcal{BC}} + MmN\right) +$$

$$(1 + DM_1M_2T)D\|\alpha\|_{L^\infty}\int_0^T \left(1 + \|y[\psi]_s\|_{\mathcal{BC}} + \sup_{s\in[0,T]} \|y(s)\|_E\right) ds.$$

From property $\mathcal{B}1$ (iii) it follows that

$$\|y[\psi]_s\|_{\mathcal{BC}} + \sup_{s\in[0,T]} \|y(s)\|_E \leq H\|\psi\|_{\mathcal{BC}} + (K + 1)\|y\|_{\mathcal{P}C}, \tag{3.9}$$

where $H(t) \leq H, K(t) \leq K, t \in [0, T]$. Then, we obtain the following estimates

$$\|y(t)\|_E \le M(l\|\psi\|_{\mathcal{BC}} + mN) + DM_1M_2T\left(\|x_1\|_E + Ml\|\psi\|_{\mathcal{BC}} + MmN\right) +$$
$$(1 + DM_1M_2T)D\|\alpha\|_{L^\infty} \int_0^T \left(1 + H\|\psi\|_{\mathcal{BC}} + (K+1)\|y\|_{\mathcal{P}C}\right) ds \le$$
$$M(l\|\psi\|_{\mathcal{BC}} + mN) + DM_1M_2T\left(\|x_1\|_E + Ml\|\psi\|_{\mathcal{BC}} + MmN\right) +$$
$$(1 + DM_1M_2T)D\|\alpha\|_{L^\infty}\left(1 + H\|\psi\|_{\mathcal{BC}}\right)T +$$
$$(1 + DM_1M_2T)D\|\alpha\|_{L^\infty} \int_0^T (K+1)\|y\|_{\mathcal{P}C} ds.$$

The last expression is a non-decreasing function of t, so we get

$$\|y\|_{\mathcal{P}C} \le M(l\|\psi\|_{\mathcal{BC}} + mN) + DM_1M_2T\left(\|x_1\|_E + Ml\|\psi\|_{\mathcal{BC}} + MmN\right) +$$
$$(1 + DM_1M_2T)D\|\alpha\|_{L^\infty}\left(1 + H\|\psi\|_{\mathcal{BC}}\right)T +$$
$$\int_0^T (1 + DM_1M_2T)D\|\alpha\|_{L^\infty}(K+1)\|y\|_{\mathcal{P}C} ds.$$

This means that the function $v(t) = \|y\|_{\mathcal{P}C}$ satisfies the estimate

$$v(t) \le M(l\|\psi\|_{\mathcal{BC}} + mN) + DM_1M_2T\left(\|x_1\|_E + Ml\|\psi\|_{\mathcal{BC}} + MmN\right) +$$
$$(1 + DM_1M_2T)D\|\alpha\|_{L^\infty}\left(1 + H\|\psi\|_{\mathcal{BC}}\right)T + \int_0^T (1 + DM_1M_2T)D\|\alpha\|_{L^\infty}(K+1)v(s)ds.$$

Applying Lemma 3.6, we obtain the required a priori boundedness:

$$v(t) = \|y\|_{\mathcal{P}C} \le Ue^{(1+DM_1M_2T)D\|\alpha\|_{L^\infty}(K+1)} = \gamma_1,$$

where

$$U = M(l\|\psi\|_{\mathcal{BC}} + mN) + DM_1M_2T\left(\|x_1\|_E + Ml\|\psi\|_{\mathcal{BC}} + MmN\right) +$$
$$(1 + DM_1M_2T)D\|\alpha\|_{L^\infty}\left(1 + H\|\psi\|_{\mathcal{BC}}\right)T.$$

For the case $1 < p < \infty$, by using the same properties $(\mathcal{B}0), (\mathcal{S}1), (I1), (I2)$ and (3.5), we have

$$\|y(t)\|_E \le M(l\|\psi\|_{\mathcal{BC}} + mN) + \Big(D\int_0^t \|f(s)\|_E^p ds\Big)^{1/p} +$$
$$D^{1/p}M_1M_2T^{1/p}(\|x_1\|_E + Ml\|\psi\|_{\mathcal{BC}}) + D^{1/p}M_1M_2\Big(\int_0^t \|\zeta \circ (\mathcal{S}f)(t)\|_E^p ds\Big)^{1/p} +$$
$$D^{1/p}MM_1M_2mNT^{1/p} \le M(l\|\psi\|_{\mathcal{BC}} + mN) +$$
$$D^{1/p}M_1M_2T^{1/p}(\|x_1\|_E + Ml\|\psi\|_{\mathcal{BC}} + MmN) + D^{2/p}M_1M_2T^{1/p}\Big(\int_0^T \|f(s)\|_E^p ds\Big)^{1/p} +$$
$$\Big(D\int_0^t \|f(s)\|_E^p ds\Big)^{1/p} \le M\left(l\|\psi\|_{\mathcal{BC}} + mN\right) +$$
$$D^{1/p}M_1M_2T^{1/p}\left(\|x_1\|_E + Ml\|\psi\|_{\mathcal{BC}} + MmN\right) +$$

$$D^{2/p}M_1M_2T^{1/p}\left(\int_0^T \alpha^p(s)(1+\|y[\psi]\|_{\mathcal{C}})^p ds\right)^{1/p} + \left(D\int_0^t \alpha^p(s)(1+\|y[\psi]\|_{\mathcal{C}})^p ds\right)^{1/p} \leq$$
$$M\left(l\|\psi\|_{\mathcal{BC}} + mN\right) + D^{1/p}M_1M_2T^{1/p}\left(\|x_1\|_E + Ml\|\psi\|_{\mathcal{BC}} + MmN\right) +$$
$$D^{2/p}M_1M_2T^{1/p}\left(\int_0^T \alpha^p(s)(1+\|y[\psi]_s\|_{\mathcal{BC}} + \sup_{s\in[0,T]}\|y(s)\|_E)^p ds\right)^{1/p} +$$
$$\left(D\int_0^t \alpha^p(s)(1+\|y[\psi]_s\|_{\mathcal{BC}} + \sup_{s\in[0,T]}\|y(s)\|_E)^p ds\right)^{1/p}.$$

Using inequality(3.9), we obtain the following estimate

$$\|y(t)\|_E \leq M\left(l\|\psi\|_{\mathcal{BC}} + mN\right) + D^{1/p}M_1M_2T^{1/p}\left(\|x_1\|_E + Ml\|\psi\|_{\mathcal{BC}} + MmN\right) +$$
$$D^{2/p}M_1M_2T^{1/p}\left(\int_0^T \alpha^p(s)(1+H\|\psi\|_{\mathcal{BC}})^p ds + \int_0^T \alpha^p(s)(K+1)^p\|y\|^p_{\mathcal{PC}} ds\right)^{1/p} +$$
$$\left(D\int_0^t \alpha^p(s)(1+H\|\psi\|_{\mathcal{BC}})^p ds + D\int_0^t \alpha^p(s)(K+1)^p\|y\|^p_{\mathcal{PC}} ds\right)^{1/p} \leq$$
$$M\left(l\|\psi\|_{\mathcal{BC}} + mN\right) + D^{1/p}M_1M_2T^{1/p}\left(\|x_1\|_E + Ml\|\psi\|_{\mathcal{BC}} + MmN\right) +$$
$$D^{2/p}M_1M_2T^{1/p}2^{1/p}\left(\int_0^T \alpha^p(s)(1+H\|\psi\|_{\mathcal{BC}})^p ds\right)^{1/p} +$$
$$D^{2/p}M_1M_2T^{1/p}2^{1/p}\left(\int_0^T \alpha^p(s)(K+1)^p\|y\|^p_{\mathcal{PC}} ds\right)^{1/p} +$$
$$(2D)^{1/p}\left(\int_0^t \alpha^p(s)(1+H\|\psi\|_{\mathcal{BC}})^p ds\right)^{1/p} + (2D)^{1/p}\left(\int_0^t \alpha^p(s)(K+1)^p\|y\|^p_{\mathcal{PC}} ds\right)^{1/p} \leq$$
$$M\left(l\|\psi\|_{\mathcal{BC}} + mN\right) + D^{1/p}M_1M_2T^{1/p}\left(\|x_1\|_E + Ml\|\psi\|_{\mathcal{BC}} + MmN\right) +$$
$$D^{2/p}M_1M_2T^{1/p}2^{1/p}(1+H\|\psi\|_{\mathcal{BC}})\left(\int_0^T \alpha^p(s) ds\right)^{1/p} +$$
$$D^{2/p}M_1M_2T^{1/p}2^{1/p}(K+1)\left(\int_0^T \alpha^p(s)\|y\|^p_{\mathcal{PC}} ds\right)^{1/p} +$$
$$(2D)^{1/p}(1+H\|\psi\|_{\mathcal{BC}})\left(\int_0^t \alpha^p(s) ds\right)^{1/p} + (2D)^{1/p}(K+1)\left(\int_0^t \alpha^p(s)\|y\|^p_{\mathcal{PC}} ds\right)^{1/p}.$$

Let us introduce the following notation:

$$c_0 = M\left(l\|\psi\|_{\mathcal{BC}} + mN\right) + D^{1/p}M_1M_2T^{1/p}\left(\|x_1\|_E + Ml\|\psi\|_{\mathcal{BC}} + MmN\right) +$$
$$(D^{2/p}M_1M_2T^{1/p}2^{1/p} + (2D)^{1/p})(1+H\|\psi\|_{\mathcal{BC}})\|\alpha\|_{L^p},$$
$$h(s) = \left(D^{2/p}M_1M_2T^{1/p}2^{1/p}(K+1) + (2D)^{1/p}(K+1)\right)^{1/p}\alpha(s).$$

Then we get:

$$\|y\|_{\mathcal{P}C} \leq c_0 + \left(\int_0^T h^p(s) \|y\|^p_{\mathcal{P}C} ds \right)^{1/p}.$$

Let $v(t) = \|y\|^p_{\mathcal{P}C}$, then from the last inequality we obtain the estimate:

$$v(t) \leq 2^p c_0^p + 2^p \int_0^T h^p(s) v(s) ds.$$

Now applying Lemma 3.7 to the last inequality, we get

$$v(t) = \|y\|^p_{\mathcal{P}C} \leq 2^p c_0^p \left(1 + \int_0^T e^{2^p \int_0^T h^p(\theta) d\theta} h^p(s) ds \right).$$

Then we have the final estimate for $1 < p < \infty$:

$$\|y\|_{\mathcal{P}C} \leq 2q_0 \sqrt[p]{1 + \int_0^T e^{2^p \int_0^T h^p(\theta) d\theta} h^p(s) ds} = \gamma_2.$$

For the case $p = \infty$, in the same way as in the case $p = 1$, the following estimate holds:

$$\|y\|_{\mathcal{P}C} \leq U_1 e^{(1 + D_1 M_1 M_2 T) D_1 \|\alpha\|_{L^\infty} (K+1)} = \gamma_3,$$

where

$$U_1 = M(l\|\psi\|_{\mathcal{BC}} + mN) + D_1 M_1 M_2 T \left(\|x_1\|_E + Ml\|\psi\|_{\mathcal{BC}} + MmN \right) +$$
$$(1 + D_1 M_1 M_2 T) D_1 \|\alpha\|_{L^\infty} \left(1 + H\|\psi\|_{\mathcal{BC}} \right) T.$$

Now, if we take $R \geq \max\{\gamma_1, \gamma_2, \gamma_3\}$, then we can guarantee that the set $V \subset \mathcal{D}$, given as

$$V = \{ y \in \mathcal{D} : \|y\|_{\mathcal{P}C} < R \},$$

contains all solutions of inclusion (3.8). Thus, the multioperator Γ satisfies on ∂V the condition of Theorem 2.2 with $a = 0$, hence the set of its fixed points is non-empty and compact. □

4. CONTROLLABILITY PROBLEMS FOR SEMILINEAR DIFFERENTIAL INCLUSIONS WITH A DELAY AND IMPULSE EFFECTS

4.1. Controllability problems for a first order semilinear functional differential inclusions

Consider the following system governed by a differential inclusion in a separable Banach space E :

$$y'(t) \in Ay(t) + F(t, y_t) + Bu(t), \quad t \in [0,T] \setminus \{t_1, t_2, \cdots, t_m\}, \tag{4.10}$$

$$y(\theta) = \psi(\theta), \quad \theta \in (-\infty, 0], \tag{4.11}$$

$$I_k y(t_k) = y(t_k^+) - y(t_k), \; k = 1, ..., m, \tag{4.12}$$

where $F : [0,T] \times \mathcal{BC} \to Kv(E)$ is a multivalued map, $\psi \in \mathcal{BC}$ is a given function. Suppose that

(A) $A : D(A) \subset E \to E$ is a linear closed operator, generating a C_0-semigroup $e^{At}, t \geq 0$;

a multimap $F : [0,T] \times \mathcal{BC} \to Kv(E)$ is such that:

$(F1)$ for each $\psi \in \mathcal{BC}$ the multifunction $F(\cdot, \psi) : [0,T] \to Kv(E)$ admits a measurable selection;

$(F2)$ for a.e. $t \in [0,T]$ the multimap $F(t, \cdot) : \mathcal{BC} \to Kv(E)$ is u.s.c.;

$(F3)$ there exists a function $\alpha \in L^{\infty}_{+}[0,T]$ such that

$$\|F(t,\psi)\|_E := \sup\{\|z\|_E : z \in F(t,\psi)\} \leq \alpha(t)(1 + \|\psi\|_{\mathcal{BC}})$$

for a.e. $t \in [0,T]$ and for all $\psi \in \mathcal{BC}$;

$(F4)$ there exists a function $\omega_F : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the conditions $(\omega 1)$-$(\omega 3)$ such that for each bounded set $\Omega \subset \mathcal{BC}$ we have

$$\chi(F(t,\Omega)) \leq \omega_F(t, \varphi(\Omega)) \text{ for a.e. } t \in [0,T].$$

We impose the following conditions on the impulse functions I_k:

$(I1)$ the functions $I_k : E \to E, \quad 1 \leq k \leq m$, are completely continuous.

$(I2)$ the functions $I_k : E \to E, \quad 1 \leq k \leq m$, are globally bounded, i.e., there exists $N > 0$ such that $\|I_k x\| \leq N$ for all $x \in E$.

Notice that when $q = 1$:

$$\mathcal{G}(t) = e^{At}, \quad \mathcal{T}(t) = e^{At},$$

therefore, in accordance with [24], a function $y \in \mathcal{C}((-\infty, T]; E)$, is a mild solution of problem (4.10)-(4.12), if it can be represented in the form:

$$y(t) = \begin{cases} e^{At}\big(\psi(0) + \Sigma_{t_k<t} I_k(y(t_k))\big) + \int_0^t e^{A(t-s)} f(s)ds + \int_0^t e^{A(t-s)} Bu(s)ds, & t \in [0,T], \\ \psi(t), \quad t \in (-\infty, 0], \end{cases}$$

where $f \in \mathcal{P}_F(y[\psi])$, $\mathcal{P}_F$ is a superposition multioperator (see Example 2.1).

The fact that the superposition multioperator $\mathcal{P}_F : \mathcal{C}((-\infty,T];E) \multimap L^1([0,T];E)$ satisfies condition $(\mathcal{Q}1)$ can be verified by Lemma 5.1.1 from [24]. Conditions $(\mathcal{Q}2)$ and $(\mathcal{Q}3)$ for $\mathcal{P}_F$ follow from $(F3)$ and $(F4)$, respectively. Taking into account Lemma 3.1, we can consider relation (4.10) as a special case of functional inclusion (3.1) with $\mathcal{Q} = \mathcal{P}_F$, and $\mathcal{S} = \mathcal{L}$ is the Cauchy operator.

As a direct consequence of Theorem 3.3, we obtain the following result.

Theorem 4.1:
Suppose that conditions (A), $(F1)$–$(F4)$, $(I1)$, $(I2)$, (W) hold. Then the set of solutions to problem (4.10)-(4.12)*,* (3.4) *is a non-empty compact subset of the space* $\mathcal{C}((-\infty, T]; E)$.

4.2. Controllability problems for fractional semilinear functional differential inclusions

Let us recall the notion of the Caputo fractional derivative.

Definition 4.1:
The Caputo fractional derivative of an order $q \in (0,1)$ of a function $g \in C^1([0,T];E)$ is the function ${}^C D_0^q g$ of the following form:

$${}^C D_0^q g(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} g(s)\, ds,$$

where Γ is the Euler's gamma-function

$$\Gamma(q) = \int_0^{\infty} x^{q-1} e^{-x} dx.$$

Consider the following control system governed by a functional differential inclusion in a separable Banach space E :

$$ {}^{C}D_0^q y\,(t) \in Ay\,(t) + F\,(t, y_t) + Bu(t),\;\; t \in [0,T] \setminus \{t_1, t_2, \cdots, t_m\}, \tag{4.13} $$

$$ y\,(\theta) = \psi(\theta), \quad \theta \in (-\infty, 0], \tag{4.14} $$

$$ I_k y(t_k) = y(t_k^+) - y(t_k),\; k = 1, ..., m, \tag{4.15} $$

where $\psi \in \mathcal{BC}$ is a given function. Suppose that

(A) $A : D\,(A) \subset E \to E$ is a linear closed operator, generating a C_0-semigroup $\{U(t)\}_{t\geq 0}$.

Assume also that a multimap $F : [0,T] \times \mathcal{BC} \to Kv(E)$ satisfies conditions $(F1)$–$(F4)$ from Example 2.1, impulse functions obey the conditions $(I1), (I2)$.

A function $y \in \mathcal{C}((-\infty, T]; E)$, is a mild solution to problem (4.13)–(4.14), if it can be presented in the form (see [12]):

$$ y(t) = \begin{cases} \mathcal{G}(t)\big(\psi(0) + \Sigma_{t_k<t} I_k(y(t_k)) + \int_0^t (t-s)^{q-1}\mathcal{T}(t-s)f(s)ds + \\ \int_0^t (t-s)^{q-1}\mathcal{T}(t-s)Bu(s)ds, \quad t \in [0,T], \\ \psi(t), \quad t \in (-\infty, 0], \end{cases} $$

where $f \in \mathcal{P}_F(y[\psi])$.

The fact that the superposition multioperator $\mathcal{P}_F : \mathcal{C}((-\infty, T]; E) \multimap L^p([0,T]; E)$, $p > 1/q$ satisfies condition $(\mathcal{Q}1)$ can be verified as in the paper [28]. Conditions $(\mathcal{Q}2)$ and $(\mathcal{Q}3)$ for $\mathcal{P}_F$ follow from $(F3)$ and $(F4)$, respectively. Taking into account Lemma 3.1, we can consider the relation (4.13) as a special case of functional inclusion (3.1) with $\mathcal{Q} = \mathcal{P}_F$, and $\mathcal{S} = G$ is the Cauchy type operator.

As a direct consequence of Theorem 3.3, we obtain the following result.

Theorem 4.2:
Suppose that conditions (A), $(F1)$–$(F4)$, $(I1)$, $(I2)$, (W) *hold. Then the set of solutions to problem* (4.13)-(4.15)*,* (3.4) *is a non-empty compact subset of the space* $\mathcal{C}((-\infty, T]; E)$.

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