Adv Syst Sci Appl 2021; **03**:22–30
Published online at https://ijassa.ipu.ru.

On Spectral Decomposition of Generalized Bessel Potentials

M.L. Goldman[1], N.H. Alkhalil[1*]

[1] *Mathematical Institute named S.M. Nikolskii*
Peoples' Friendship University of Russia (RUDN University)
Moscow 117198, Miklukho-Maklaya str. 6, Russia

Abstract: We establish the localization condition for the γ-means spectral decomposition by the system of fundamental functions of the Laplace operator in an arbitrary multi-dimensional domain. The result is obtained in terms of belonging of decomposing function to the spaces of the generalized Bessel potential. For conditions of localization, we apply the exact estimates for the modulus of continuity of the potential. The generalized Bessel potentials are constructed using convolutions of functions with kernels that generalize the classical Bessel–MacDonald kernels. In contrast to the classical case, non-power singularities of kernels are allowed in the vicinity of the origin. Differential properties of potentials arc described by using the k-th order modulus of continuity in the uniform norm.

Keywords: spectral decomposition, the generalized Bessel potential, the modulus of continuity of the potential, Laplace operator

1. INTRODUCTION

The paper is organized as follows. Section 1 contains basic definitions of the potential theory. The main properties of kernels are considered and basic spaces for potentials are described. Sections 3.1, 3.2 contain some auxiliary results. In Section 3.1 we give the condition of the embedding of the space $H_E^G(\mathbb{R}^n)$ of generalized Bessel potentials into the space $C(\mathbb{R}^n)$ of bounded uniformly continuous functions, and obtain the estimates for modulus of continuity of potentials, see Theorem 3.1.

In Section 3.2 we give the conditions for localization of γ-means of spectral decomposition in terms of properties of modulus of continuity for decomposing function, see Theorem 3.2.

In Section 4.1 we prove Theorem 4.1 giving the conditions for including of the space $H_E^G(\mathbb{R}^n)$ into the scheme of spectral decomposition.

Finally, Theorem 4.2 in Section 4.2 gives the conditions for localization of spectral decomposition for generalized Bessel potentials constructed over the basic weighted Lorentz spaces.

*Corresponding author: khaleel.almahamad1985@gmail.com

2. BASIC DEFINITIONS

Let $v > 0$ be measurable function on $\mathbb{R}_+$. The Lorentz space $\Lambda^p(v)$ is the space of measurable functions on $\mathbb{R}^n$ with finite (quasi) norms (see [1])

$$\|f\|_{\Lambda^p(v)} = \begin{cases} \left(\int_0^\infty f^*(t)^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}; & 0 < p < \infty; \\ \operatorname*{ess\,sup}_{t\in\mathbb{R}_+}\left\{f^*(t)v(t)\right\}; & p = \infty. \end{cases} \tag{2.1}$$

Here $f^* : \mathbb{R}_+ \to [0, \infty]$ is the decreasing rearrangement of function $f : \mathbb{R}^n \to \mathbb{R}$, i.e f^* is a nonnegative decreasing right-continuous function on $\mathbb{R}_+ = (0, \infty)$ which is equimeasurable with f:

$$\mu_n\left\{x \in \mathbb{R}^n : \ |f(x)| > y\right\} = \mu_1\left\{t \in \mathbb{R}_+ : \ |f^*(t)| > y\right\}, \quad y \in \mathbb{R}_+, \tag{2.2}$$

where μ_n is the n-dimensional Lebesgue measure. We assume that
$0 < V(t) := \int_0^t v(\tau)\mathrm{d}(\tau) < \infty,\ t \in \mathbb{R}_+$, and

$$\sup_{t\in\mathbb{R}_+}\left[\frac{V(2t)}{V(t)}\right] < \infty, \tag{2.3}$$

is the so called Δ_2-condition. Under these assumptions $E(\mathbb{R}^n) = \Lambda^p(v)$ is a (quasi) Banach space which gives an important example of a rearrangement invariant space (shortly:RIS), because of property:

$$g^* \le f^*,\ \ f \in E(\mathbb{R}^n) \ \Rightarrow\ g \in E(\mathbb{R}^n),\ \ \|g\|_E \le \|f\|_E.$$

(see C. Bennett and R. Sharpley [1]). Moreover, $E' = E'(\mathbb{R}^n)$ is the associated RIS for $E(\mathbb{R}^n)$, i.e. E' is RIS with the norm:

$$\|g\|_{E'} = \sup\left\{\int_{\mathbb{R}^n} |fg|\,\mathrm{d}\mu_n : \ f \in E,\ \ \|f\|_E \le 1\right\}. \tag{2.4}$$

For $1 < p < \infty$ the description of the associated space for $E(\mathbb{R}^n) = \Lambda^p(v)$ was obtained by E. Sawyer [11]. Namely,

$$\|g\|_{E'} = \sup_{0\le h\downarrow} \frac{\int_0^\infty g^*(\tau)h(\tau)\,\mathrm{d}\tau}{\left(\int_0^\infty h(\tau)^p v(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p}}} \approx \left(\int_0^\infty \left(\int_0^\xi g^*(\tau)\mathrm{d}\tau\right)^{p'} \frac{v(\xi)d\xi}{V(\xi)^{p'}}\right)^{\frac{1}{p'}}, \tag{2.5}$$

Here the symbol $\approx$ means that the ratio of left and right hand sides is bounded between positive constants depending only on p (and not on v or g). The potential space $H_E^G \equiv H_E^G(\mathbb{R}^n)$ for $E(\mathbb{R}^n) = \Lambda^p(v)$ is defined as the set of convolutions of potential kernel G with all functions belonging to the basic RIS $E(\mathbb{R}^n)$:

$$H_E^G(\mathbb{R}^n) = \left\{u = G * f : \ f \in E(\mathbb{R}^n)\right\}. \tag{2.6}$$

We define

$$\|u\|_{H_E^G} = \inf\left\{\|f\|_E : \ f \in E(\mathbb{R}^n),\ \ G * f = u\right\}. \tag{2.7}$$

We assume that the kernel G of a representation (2.7) is admissible, i.e

$$G \in L_1(\mathbb{R}^n) + E'(\mathbb{R}^n).$$

Here the convolution $G * f$ is defined as the integral

$$(G * f)(x) = \int_{\mathbb{R}^n} G(x-y) f(y) \, \mathrm{d}(y).$$

For function Φ: $\mathbb{R}_+ \to [0, \infty)$ we define

$$\varphi(\tau) = \Phi\left(\left(\frac{\tau}{V_n}\right)^{\frac{1}{n}}\right), \quad \tau \in \mathbb{R}_+. \tag{2.8}$$

Definition 2.1:
Let k, $n \in \mathbb{N}$; $R \in \mathbb{R}_+$. We say that function Φ belongs to the class $\Im_{k,n}(R)$ if it satisfies the following conditions:

1. $0 < \Phi \downarrow$ *on* $(0, R)$*;* $\exists\, c \in \mathbb{R}_+$ *such that*

$$\int_0^r \Phi(\rho)\rho^{n-1} \, \mathrm{d}\rho \le c\, \Phi(r) r^n, \quad r \in (0, R); \quad \int_R^\infty \Phi(\rho)\rho^{n-1} \, \mathrm{d}\rho < \infty. \tag{2.9}$$

2. $G(x) := \Phi(|x|) \in C^k(\mathbb{R}^n \backslash 0)$*, and for*

$$G_k(x) := \sum_{|\alpha|=k} |D^\alpha G(x)|, \quad x \in \mathbb{R}^n \backslash 0,$$

the estimate holds: for some $c_1 \in \mathbb{R}_+$

$$|G_k(x)| \le c_1 \Psi_k(|x|), \quad x \in \mathbb{R}^n \backslash 0; \tag{2.10}$$

where $\Psi_k \in C(\mathbb{R}_+)$*, and for* $T = V_n R^n$

$$\varphi_k(\tau) := \Psi_k\left(\left(\frac{\tau}{V_n}\right)^{\frac{1}{n}}\right) \le \tau^{-k/n} \varphi(\tau), \quad \tau \in (0, T]; \tag{2.11}$$

$$\int_T^\infty \varphi_k(\tau) \, \mathrm{d}\tau < \infty. \tag{2.12}$$

Definition 2.2:
For RIS $E(\mathbb{R}^n)$ the space $H_E^G(\mathbb{R}^n)$ (2.6)-(2.7) *with kernel $G(x) = \Phi(|x|)$, where $\Phi \in \Im_{k,n}(R)$, is called the space of generalized Bessel potentials.*

Remark 2.1:
Note that the classical Bessel–McDonald kernels have the form

$$G_\alpha(x) = c(\alpha, n)\rho^{-\beta} K_\beta(\rho), \quad \rho = |x| \in \mathbb{R}_+, \quad \alpha \in (0, n), \quad \beta = \frac{n-\alpha}{2},$$

where K_β is the McDonald function, see [10]. The well-known properties of these kernels show that $\Phi(\rho) = \rho^{-\beta} K_\beta(\rho) \in \Im_{k,n}(R)$ for fixed $R \in \mathbb{R}_+$, moreover $\Phi(\rho) \cong \rho^{\alpha-n}$, $\rho \in (0, R)$; $\Phi(\rho) \cong \rho^{-\beta-\frac{1}{2}} e^{-\rho}$, $\rho > R$, and our scheme includes classical Bessel potentials.

Remark 2.2:
Note that functions $\Phi \in \Im_{k,n}(R)$ may have the property

$$\Phi(\rho) = 0, \quad \rho \in [2R, \infty). \tag{2.13}$$

Thus, the case of kernels with compact support is included in our scheme.

Definition 2.3:
Let $C(\mathbb{R}^n)$ be the space of bounded and uniformly continuous functions with the norm

$$\|u\|_C = \sup_{x\in\mathbb{R}^n} |u(x)|.$$

For $u \in C(\mathbb{R}^n)$ modulus of continuity of order $k \in \mathbb{N}$ is defined as

$$\omega_C^k(u;t) = \sup\big\{\|\Delta_h^k u\|_C :\ |h| \le t\big\}, \quad t \in \mathbb{R}_+,$$

$$\Delta_h^k(u;x) = \sum_{m=0}^{k} (-1)^{k-m} C_k^m u(x+mh),$$

is the k-th difference of function u with the step $h \in \mathbb{R}^n$ at the point $x \in \mathbb{R}^n$.
For $u \in L_2(\Omega)$, where Ω is domain in $\mathbb{R}^n$, we define

$$\omega_{2,\Omega}^k(u;t) = \sup_{|h|\le t} \|\Delta_h^k u\|_{L_2(\Omega_{kh})};$$

where

$$\Omega_{kh} = \big\{x \in \Omega :\ [x, x+kh] \subset \Omega\big\}.$$

Definition 2.4:
Let $\omega \in C(0,1]$, $0 \le \omega(t) \uparrow$; $t^{-k}\omega(t) \downarrow$ on $(0,1]$. The Nikolskii-type space with generalized smoothness $H_2^{\omega(\cdot)}(\Omega)$ is defined as

$$H_2^{\omega(\cdot)}(\Omega) = \big\{u \in L_2(\Omega) :\ \|u\|_{H_2^{\omega(\cdot)}(\Omega)} < \infty\big\}, \tag{2.14}$$

where

$$\|u\|_{H_2^{\omega(\cdot)}(\Omega)} = \|u\|_{L_2(\Omega)} + \sup_{0<t\le 1}\left[\frac{\omega_{2,\Omega}^k(u;t)}{\omega(t)}\right]. \tag{2.15}$$

3. AUXILIARY THEOREMS

3.1.

Let RIS $E(\mathbb{R}^n) = \Lambda^p(v)$ be the Lorentz space, see notations and assumptions (2.1)–(2.5), and $H_E^G(\mathbb{R}^n)$ be the related space of generalized Bessel potentials (2.6)–(2.7) with $G(x) = \Phi(|x|)$, $\Phi \in \mathfrak{I}_{k,n}(R)$, see Definitions (2.1), (2.2).

For $0 < p < \infty$ we define function on $(0,T]$, $T = V_n R^n$,

$$A_p(t) = \sup_{\tau\in(0,t^n)}\left\{\frac{1}{V(\tau)^{\frac{1}{p}}}\int_0^\tau \varphi(\xi)\,\mathrm{d}\xi\right\}, \quad 0 < p \le 1; \tag{3.16}$$

$$A_p(t) = \left\{\int_0^{t^n}\left(\int_0^\tau \varphi(\xi)\,\mathrm{d}\xi\right)^{p'}\frac{v(\tau)\,\mathrm{d}\tau}{V(\tau)^{p'}} + \left(\int_0^{t^n}\varphi(\tau)\,\mathrm{d}\tau\right)^{p'} V(t^n)^{-\frac{p'}{p}}\right\}^{\frac{1}{p'}}, \tag{3.17}$$

for $1 < p < \infty,\ p' = \dfrac{p}{p-1}$. We require that

$$A_p(t) < \infty. \tag{3.18}$$

Theorem 3.1:
[6]. In the notations and assumptions of this section the following statements hold:

1. *$H_E^G(\mathbb{R}^n) \subset C(\mathbb{R}^n)$, and there exists constant $c_1 \in \mathbb{R}_+$ such that*

$$\|u\|_{C(\mathbb{R}^n)} \le c_1 \|u\|_{H_E^G(\mathbb{R}^n)}, \quad \forall\, u \in H_E^G(\mathbb{R}^n). \tag{3.19}$$

2. *Let additionally*

$$\int_t^T \tau^{-k/n} \varphi(\tau)\, \mathrm{d}\tau \le B_0 t^{1-k/n} \varphi(t), \quad t \in (0,T], \tag{3.20}$$

with $B_0 \in \mathbb{R}_+$ independent of $t \in (0,T]$. Then, there exists constant $c_2 \in \mathbb{R}_+$ such that for any $u \in H_E^G(\mathbb{R}^n)$

$$\omega_C^k(u;t) \le c_2 A_p(t) \|u\|_{H_E^G(\mathbb{R}^n)}, \quad t \in (0,T]. \tag{3.21}$$

3.2.

Let $F \subset \mathbb{R}^n$ be an arbitrary domain, and $(-\hat{\Delta})$ a self adjoint non-negative extension of Laplace operator in D; $y(x,t)$ an ordered spectral representation of $L_2(F)$ with respect to $(-\hat{\Delta})$, and $\mathrm{d}\rho(t)$ the related spectral measure; $y(x,t) = \{y_i(x,t)\}_{i=1}^m$ be a system of fundamental functions, that is $y_i(\cdot,t) \in C^\infty(F)$, and

$$\Delta y_i(x,t) + t^2 y_i(x,t) = 0, \quad x \in G.$$

Here $m \le \infty$ is the multiplicity of the representation. For $f \in L_2(F)$ are defined Fourier transforms

$$\hat{f} := \{\hat{f}_i(t)\}_{i=1}^m; \quad \hat{f}_i(t) = \int_F f(x) y_i(x,t)\, \mathrm{d}x$$

and spectral decomposition

$$S_\mu(f;x) = \int_0^\mu \hat{f}(y) y(x,t)\, \mathrm{d}\rho(t), \quad \mu > 0, x$$

where

$$\hat{f} y = \sum_{i=1}^m \hat{f}_i y_i.$$

Let $s > 0$ and ψ be a function on $(0,1]$ with properties $0 < \psi \uparrow$ on $(0,1]$, and $\psi(t) \cong \psi(\tau)$ if $t \cong \tau$. Moreover, for $s > 0$ and $s_0 = s$ if $s \le 1$; $s_0 = 1$ if $s > 1$ we require that

$$1)\ \psi_{s_0}(t) = \int_0^t \tau^{s_0-1} \psi(\tau)\, \mathrm{d}\tau < \infty, \quad t \in (0,1], \tag{3.22}$$

$$2)\ \psi \in C^2(0,1); \quad |\psi'(\tau)| \le c\psi(\tau)\tau^{-1}, \quad |\psi''(\tau)| \le c\psi(\tau)\tau^{-2}, \quad \tau \in (0,1], \tag{3.23}$$

$$3)\ \int_0^1 (1-\tau)^{s-1} \psi(\tau)\, \mathrm{d}\tau = \Gamma(s). \tag{3.24}$$

We define γ as s-th Riemann–Liouville integral

$$\gamma(t) = \frac{1}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} \psi(\tau)\, \mathrm{d}\tau, \quad t \in (0,1]. \tag{3.25}$$

Let us introduce the γ-means of spectral decomposition as

$$\sigma_\mu^\gamma(f;x) = \int_0^\mu \hat{f}(t) y(x,t) \gamma\left(1 - \frac{t^2}{\mu^2}\right) \mathrm{d}\rho(t), \quad \mu > 0, \tag{3.26}$$

for $f \in L_2(F)$. We see that $\gamma(1) = 1$ and if $\psi(\tau) \equiv \Gamma(s+1)$, $\tau \in (0,1]$, then $\gamma(t) = t^s$, and γ-means reduce to the classical Riesz means of orders, see [2, 3].

We define

$$\omega_0(t) = \frac{t^{\frac{n-1}{2}} + s_0 - s}{\psi_{s_0}(t)}, \quad t \in (0,1]. \tag{3.27}$$

Let α, $\beta \geq 0$ be such that

$$\frac{n-2}{2} - s < \alpha \leq \beta < \min\left\{\alpha + \frac{3}{2}, \frac{n}{2} + 1\right\}. \tag{3.28}$$

Let function ω satisfy the conditions

$$\omega(t)t^{-\alpha} \uparrow, \quad \omega(t)t^{-\beta_0} \downarrow \text{ on } (0,1]; \quad \beta_0 = \min\{\beta, k\}, \tag{3.29}$$

and

$$\lim_{t \to +0} \frac{\omega(t)}{\omega_0(t)} = 0. \tag{3.30}$$

Let $\Omega \subset\subset F$, that is Ω is bounded domain, and $\bar{\Omega} \subset F$,

$$f \in H_2^{\omega(\cdot)}(\Omega) \cap L_2(F). \tag{3.31}$$

Theorem 3.2:
[4, 5]. In the notations and assumptions of Section 3.2 (3.22)–(3.31)*, let $D \subset \Omega$ and function f satisfies the condition $f(x) \equiv 0$, $x \in D$. Then, for each compact $K \subset D$ uniformly in $x \in K$ the relation holds:*

$$\lim_{\mu \to \infty} \sigma_\mu^\gamma(f;x) = 0. \tag{3.32}$$

Remark 3.1:
Theorem 3.2 gives sharp conditions for localization of γ-means of spectral decomposition. In typical situations we have $s < \frac{n-1}{2}$,

$$\psi_{s_0}(t) \cong t^{s_0}\psi(t), \quad \omega_0(t) \cong \frac{t^{\frac{n-1}{2}-s}}{\psi(t)}, \quad t \in (0,1].$$

In particular, for Riesz means, $\psi(t) \equiv \Gamma(s+1)$ and (3.30) *gives the condition*

$$\omega(t) = \bar{\bar{0}}\left(t^{\frac{n-1}{2}-s}\right).$$

4. LOCALIZATION FOR γ-MEANS OF SPECTRAL DECOMPOSITION FOR GENERALISED BESSEL POTENTIALS

4.1.

Theorem 4.1:
Let the notation and assumptions (3.16)–(3.18) *and* (3.20) *be satisfied,* $V(+\infty) = \infty$*, and moreover*

$$B_p := \sup\left[t^{\frac{1}{2}} V(t)^{-\frac{1}{p}} :\ t \in \mathbb{R}_+\right] < \infty, \quad 0 < p \le 2; \tag{4.33}$$

$$B_p := \left(\int_0^\infty \left[t^{\frac{1}{2}} V(t)^{-\frac{1}{p}}\right]^s \frac{v(t)\,\mathrm{d}t}{V(t)}\right)^{\frac{1}{s}} < \infty, \quad 2 < p < \infty, \quad s = \frac{2p}{p-2}. \tag{4.34}$$

Then, for $E(\mathbb{R}^n) = \Lambda^p(v)$ *there is the embedding of the space of generalized Bessel potentials*

$$H_E^G(\mathbb{R}^n) \subset L_2(\mathbb{R}^n). \tag{4.35}$$

Proof
Note that the conditions (4.33), (4.34) give the criterion of embedding

$$E = \Lambda^p(v) \subset L_2(\mathbb{R}^n). \tag{4.36}$$

It follows from the assertions (see [7, 8]): let

$$C_p := \sup_{f \in \Lambda^p(v)} \left[\left(\int_0^\infty f^*(\tau)^2\,\mathrm{d}\tau\right)^{\frac{1}{2}} \left(\int_0^\infty f^*(\tau)^p\, v(\tau)\,\mathrm{d}\tau\right)^{-\frac{1}{p}}\right].$$

Then,

$$C_p = B_p, \quad 0 < p \le 2; \quad C_p \cong B_p, \quad 2 < p < \infty.$$

The embedding (4.36) implies the embedding

$$H_E^G(\mathbb{R}^n) \subset H_{L_2}^G(\mathbb{R}^n).$$

But, in the assumptions of Section 3.1 the kernel

$$G(x) = \Phi(|x|) \in L_1(\mathbb{R}^n),$$

and by the generalized Minkovkii inequality for convolutions

$$f \in L_2(\mathbb{R}^n) \Rightarrow u = G * f \in L_2(\mathbb{R}^n), \quad \|u\|_{L_2(\mathbb{R}^n)} \le \|G\|_{L_1(\mathbb{R}^n)} \|f\|_{L_2(\mathbb{R}^n)}.$$

This gives embedding (4.35). □

4.2.

Now, let us recall the notations and assumptions of Section 3.2.

Let the conditions of Theorem 4.1 be satisfied, and $H_E^G(\mathbb{R}^n)$ be the space of generalized Bessel potentials with $E(\mathbb{R}^n) = \Lambda^p(v)$. We assume that the notations and assumptions of Section 3.2 hold, see (3.22)–(3.28). Under assertions (3.16)–(3.18) and (3.20), we define $\omega(t) := A_p(t)$ in (3.29), (3.30).

Theorem 4.2:
Under assumptions of this Section, let D, Ω, F *be domains in* $\mathbb{R}^n$*, and* $D \subset \Omega \subset\subset F$*. If*

$$u \in H_E^G(\mathbb{R}^n); \quad u(x) \equiv 0, \quad x \in D, \tag{4.37}$$

then for each compact $K \subset D$ *uniformly in* $x \in K$ *the relation holds:*

$$\lim_{\mu \to \infty} \sigma_\mu^\gamma(u; x) = 0. \tag{4.38}$$

Proof
By Theorem 4.1 we have the embedding (4.35). It means that

$$u \in H^G_E(\mathbb{R}^n) \Rightarrow u \in L_2(\mathbb{R}^n) \Rightarrow u \in L_2(F),$$

and γ-means of spectral decomposition $\sigma^\gamma_\mu(u;x)$ are correctly defined by formula

$$\sigma^\gamma_\mu(u;x) = \int\limits_0^\mu \hat{u}(t) y(x,t) \gamma\left(1 - \frac{t^2}{\mu^2}\right) \mathrm{d}\rho(t), \tag{4.39}$$

where $y(x,t)$ is the system of fundamental functions

$$\hat{u}y = \sum_{i=1}^m \hat{u}_i y_i; \quad \hat{u}_i(t) = \int\limits_F u(x) y_i(x,t)\,\mathrm{d}x,$$

see notations in Section 3.2.

Moreover, for $u \in H^G_E(\mathbb{R}^n)$ we have

$$\omega_{2,\Omega}(u;t) \leq \omega_C(u;t)(\operatorname{mes}\Omega)^{\frac{1}{2}}. \tag{4.40}$$

Indeed,

$$\|\Delta^k_h u\|_{L_2(\Omega_{kh})} = \left(\int_{\Omega_{kh}} |\Delta^k_h u|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \leq \sup_{x\in\Omega_{kh}} |\Delta^k_h u(x)| (\operatorname{mes}\Omega_{kh})^{\frac{1}{2}},$$

and for

$$\omega^k_{2,\Omega}(u;t) = \sup_{|h|\leq t} \|\Delta^k_h u\|_{L_2(\Omega_{kh})}$$

we have the estimates

$$\omega^k_{2,\Omega}(u;t) \leq \sup_{|h|\leq t} \sup_{x\in\mathbb{R}^n} |\Delta^k_h u(x)| (\operatorname{mes}\Omega)^{\frac{1}{2}} = \\ = \sup_{|h|\leq t} \|\Delta^k_h u\|_{C(\mathbb{R}^n)} (\operatorname{mes}\Omega)^{\frac{1}{2}} = \omega^k_C(u;t)(\operatorname{mes}\Omega)^{\frac{1}{2}},$$

and (4.40) follows. Thus, by Theorem 3.1, see (3.21), we have

$$\omega^k_{2,\Omega}(u;t) \leq c_2 (\operatorname{mes}\Omega)^{\frac{1}{2}} \|u\|_{H^G_E(\mathbb{R}^n)} A_p(t), \quad t \in (0,T]. \tag{4.41}$$

Now, we apply Theorem 3.2.

This estimates shows that

$$u \in H^G_E(\mathbb{R}^n) \Rightarrow u \in H^{\omega(\cdot)}_2(\Omega), \quad \omega(t) = A_p(t). \tag{4.42}$$

Finally, $u \in H^{\omega(\cdot)}_2(\Omega) \cap L_2(F)$, and we obtain the assertion (4.38) by applying of the Theorem 3.2. □

Remark 4.1:
Note that if kernel G of generalized Bessel potentials is compactly supported on $B_{2R} = \{x \in \mathbb{R}^n : |x| < 2R\}$, see the condition (2.13)*, we have for $u \in H^G_E(\mathbb{R}^n)$*

$$u(x) = (G * f)(x) \equiv 0, \quad x \in D,$$

function $f \in E(\mathbb{R}^n)$ *satisfies the condition*

$$f(x) \equiv 0, \quad x \in D^{2R} = \big\{x \in \mathbb{R}^n : \ \rho(x, D) < 2R\big\},$$

where

$$\rho(x, D) = \inf\big\{|x - y| : \ y \in D\big\}$$

is the distance from x *to* D.

Remark 4.2:
The posing of the problem and the results of Sections 3.2 and 4.1 belong to M.L. Goldman. The other results of paper obtained by N.H. Alkhalil.

Remark 4.3:
In this paper we consider the differential properties of generalized Bessel potentials. Their integral properties are connected with estimates of Hardy-type operators on the cones of monotone functions considered in [12].

ACKNOWLEDGEMENTS

The research of M.L. Goldman is supported by the Ministry of Science and Higher Education of the Russian Federation: agreement No. 075-03-2020-223/3 (FSSF-2020-0018).

REFERENCES

1. Bennett, C., & Sharpley, R. (1988) *Interpolation of operators*. Boston: Acad. Press, 1988. (Pure and Appl. Math.; V. 129).
2. Il'in, V.A., & Alimov, Sh.A. (1971) Conditions for the convergence of spectral decompositions that correspond to self-adjoint extensions of elliptic operators. I, *Differential Equations* 1971, **7**(4), 670–710.
3. Il'in, V.A., & Alimov, Sh.A. (1971) Conditions for the convergence of spectral decompositions that correspond to self-adjoint extensions of elliptic operators. II, *Differential Equations* 1971, **7**(5), 851–882.
4. Ayele, T.G., & Goldman, M.L. (2014) Spaces of generalized smoothness in summability problems for Φ-means of spectral decomposition. *Eurasian Mathem. Journal* 2014, **5**(1), 61–81.
5. Goldman, M.L., & Tsegaye, G.A. (2015) Spaces with Generalized Smoothness in Summability Problems for Φ-means of Spectral Decompositions. *Springer International Publishing, Switzerland* 2015, *Current Trends in Mathematics*, 163–169.
6. Alkhalil, N.Kh. Modulus of continuity for Bessel type potentials over Lorentz space. *Eurasian Mathem. Journal*, (to appear).
7. Burenkov, V.I., & Goldman, M.L. (1995) Calculation of the norm of a positive operator on the cone of monotone functions. *Proc. of the Steklov Inst. Math.*, 1995, **210**, 65–89.
8. Goldman, M.L. (1998) Hardy-type inequalities on the cone of quasimonotone functions. *Russian Academy of Sciences, Far-Eastern Branch, Research Report 98/31, Khabarovsk*, 1998, 1–70.
9. Goldman, M.L., & Malysheva, A. (2013) Estimates of the uniform modulus of continuity for Bessel potentials. *Proc. of the Steklov Inst. Math.* 2013, **283**, 1–12.
10. Nikol'skii, S.M. *Approximation of Functions of Several Variables and Imbedding Theorems*. Springer, Berlin-Heidelberg-New York, 1975.
11. Sawyer, E. (1990) Boundedness of classical operators on classical Lorentz spaces. *Studia Math.*, **96**, 145–158.
12. Almohammad, Kh. (2021). The Modular Inequalities for Hardy-type Operators on Monotone Functions in Orlicz Space. *Advances in Systems Science and Applications*, **21(2)**, 133–141.

