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The Modular Inequalities for Hardy-type Operators on Monotone Functions in Orlicz Space

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Abstract: This paper is devoted to the study of modular inequality for general Hardy–Copson type operator restricted on the cone of monotone functions from weighted Orlicz space with general weight.

Keywords: modular inequalities, norm inequalities, Orlicz space, cone of decreasing functions, positively homogeneous operators

1. INTRODUCTION

In this paper, we consider modular inequalities for Hardy-type operators on the cone Ω of positive decreasing functions from weighted Orlicz spaces. We use a general theorem (proved in [10]) on the reduction of modular inequalities for positively homogeneous operators on the cone Ω, which enables passing to modular inequalities for modified operators on the cone of all positive functions from Orlicz space. It is based on the duality theorem describing the associated norm for the cone Ω. We follow, mostly, the notation used in the book [2, Sec. 8, Chap. 4] of Bennett and Sharpley. In the paper, we concretize modular inequalities for the case in which the positive operator is a Hardy-type operator. It is shown that, in that case, the modified operator is a generalized Hardy operator in the Jim Quile Sun notation [1]. This allows us to use approaches developed in [8–10], as well as results obtained by Jim Quile Sun [1] to establish the explicit criteria for the validity of modular inequalities.

2. AUXILIARY DEFINITIONS

Definition 2.1:
(i) A Banach function space, shortly BFS, $E = E(\mathbb{R}^n)$ *is a Banach space of Lebesgue measurable functions* $f\colon \mathbb{R}^n \to \mathbb{C}$ *with monotone norm, i.e. such that*

$$|f| \le g, \quad g \in E \quad \textit{implies} \quad f \in E, \quad \|f\|_E \le \|g\|_E. \tag{2.1}$$

(ii) A BFS E *is called a rearrangement-invariant space, shortly: RIS, if its norm is monotone with respect to rearrangements,*

$$f^* \le g^*, \quad g \in E \quad \textit{implies} \quad f \in E, \quad \|f\|_E \le \|g\|_E. \tag{2.2}$$

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Here f^* is the decreasing rearrangement of the function, i.e. a positive decreasing right continuous function on $\mathbb{R}_+ = (0,\infty)$, which is equimeasurable with f:

$$\mu_n\big\{x \in \mathbb{R}^n :\ |f(x)| > y\big\} = \mu_1\big\{t \in \mathbb{R}_+ :\ |f^*(t)| > y\big\}, \quad y \in \mathbb{R}_+. \tag{2.3}$$

where μ_n is n-dimensional Lebesgue measure.

Definition 2.2:
The potential space $H^G_E(\mathbb{R}^n)$ on the n-dimensional Euclidean space $\mathbb{R}^n$ is defined by

$$H^G_E(\mathbb{R}^n) = \big\{u = G * f :\ f \in E(\mathbb{R}^n)\big\}, \tag{2.4}$$

where $E(\mathbb{R}^n)$ — is a rearrangement-invariant space (shortly: RIS), and

$$\|u\|_{H^G_E} = \inf\big\{\|f\|_E :\ f \in E(\mathbb{R}^n),\ G * f = u\big\}. \tag{2.5}$$

Here G is an admissible kernel, that is $G \in L_1(\mathbb{R}^n) + E'(\mathbb{R}^n)$, the convolution $G * f$ is defined as the integral

$$(G * f)(x) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} G(x-y) f(y)\, \mathrm{d}y. \tag{2.6}$$

Here $E' = E'(\mathbb{R}^n)$ is the associated RIS, i.e. RIS with the norm:

$$\|g\|_{E'} = \sup\left\{\int_{\mathbb{R}^n} |fg|\, \mathrm{d}\mu :\ f \in E,\ \|f\|_E \le 1\right\}. \tag{2.7}$$

Examples.

$$E = L_p, \quad 1 \le p \le \infty \Rightarrow E' = L_{p'}; \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

$$L_1' = L_\infty; \quad L_\infty' = L_1.$$

For the RIS $E(\mathbb{R}^n)$, $E'(\mathbb{R}^n)$, we consider the spaces $\tilde{E} = \tilde{E}(\mathbb{R}_+)$, $\tilde{E'} = \tilde{E}'(\mathbb{R}_+)$ — their Luxemburg representations [2], i.e. RIS for which the following equalities are satisfied

$$\|f\|_E = \|f^*\|_{\tilde{E}}, \quad \|g\|_{E'} = \|g^*\|_{\tilde{E}'}.$$

We denote:

$$f^{**}(t) = \frac{1}{t} \int\limits_0^t f^*(\tau)\, \mathrm{d}\tau; \quad t \in \mathbb{R}_+. \tag{2.8}$$

We introduce the class of monotone functions $\mathfrak{I}_n(R)$, $R > 0$ as follows. The function $\theta :\ (0,R) \to \mathbb{R}_+$ belongs to the class $\mathfrak{I}_n(R)$, if θ satisfies the following conditions:
decreasing and continuous at $(0,R)$;
There is a constant $c \in \mathbb{R}_+$, such that

$$\int\limits_0^r \theta(\rho)\rho^{n-1}\, \mathrm{d}\rho \le c\theta(r) r^n, \quad r \in (0,R). \tag{2.9}$$

Now, we introduce

$$\varphi(\tau) = \theta\left(\left(\frac{\tau}{V_n}\right)^{\frac{1}{n}}\right) \in \mathfrak{I}_1(T), \quad T = V_n R^n.$$

where V_n is the volume of the unit ball in $\mathbb{R}^n$.

$$f_\varphi(t;\tau) = \min\{\varphi(t), \varphi(\tau)\} = \begin{cases} \varphi(t), & t > \tau, \\ \varphi(\tau), & \tau > t. \end{cases} \tag{2.10}$$

Definition 2.3:
Let $\theta \in \mathfrak{I}_n(\infty)$. The potentials $u \in H_E^G(\mathbb{R}^n)$ are called generalized Riesz potentials, if

$$G(x) \cong \theta(|x|), \quad x \in \mathbb{R}^n, \quad (\cong \text{ means two-sided estimate}).$$

Definition 2.4:
Let $\theta \in \mathfrak{I}_n(R)$. The potentials $u \in H_E^G(\mathbb{R}^n)$ are called generalized Bessel potentials, if

$$G(x) = G_R^0(x) + G_R^1(x);$$

$$B_R = \{x \in \mathbb{R}^n : \ |x| < R\}, \quad R \in \mathbb{R}_+,$$

$$G_R^1(x) = G(x)\chi_{B_R^c}(x), \quad G_R^0(x) = G(x)\chi_{B_R}(x),$$

$$G_R^0(x) \cong \theta(|x|), \quad x \in B_R, \quad G_R^1(x) \in (L_1 \cap E')(\mathbb{R}^n), \quad \int\limits_{\mathbb{R}^n} G\, dx \neq 0.$$

Definition 2.5:
Function Φ: $[0, +\infty) \to [0, +\infty)$ is called N-function if

$$\Phi(t) = \int\limits_0^t \phi(\tau)\, \mathrm{d}\tau; \quad \text{where } \phi \text{ is continuous}, \quad 0 < \phi \uparrow; \quad \phi(0) = 0, \quad \phi(\infty) = \infty.$$

Let ϕ^{-1} be the right continuous inverse function of ϕ, and define

$$\Psi(t) = \int\limits_0^t \phi^{-1}(\tau)\, \mathrm{d}\tau.$$

Ψ is called the complementary function of Φ.

Definition 2.6:
a) An N-function Φ is said to satisfy the Δ_2-condition (we write $\Phi \in \Delta_2$) if there is a constant $B > 0$, such that

$$\Phi(2t) \leq B\Phi(t), \quad \forall t > 0. \tag{2.11}$$

b) We write $\Phi_1 \prec\prec \Phi_2$ if there is a constant $L_0 > 0$, such that inequality

$$\sum_i \Phi_2 \circ \Phi_1^{-1}(a_i) \leq L_0 \Phi_2 \circ \Phi_1^{-1}(\sum_i a_i), \tag{2.12}$$

holds for every sequence $\{a_i\}$ with $a_i \geq 0$.

c) Let v be a positive, measurable weight function and Φ be an N-function. The Orlicz space $L_{\Phi,v}$ consists of all measurable function f (modulo equivalence almost everywhere) with

$$\|f\|_{\Phi,v} = \inf\left\{\lambda > 0, \int_0^\infty \Phi\big(\lambda^{-1}|f(x)|\big)_v(x)\, \mathrm{d}x \leq 1\right\} < \infty. \tag{2.13}$$

We call $\|\cdot\|_{\Phi,v}$ the Luxemburg norm.

The Orlicz norm of a function f is given by

$$\|f\|'_{\Psi,v} = \sup\left\{\int_0^\infty |fg|v\, \mathrm{d}x : \int_0^\infty \Psi(|g|)_v\, \mathrm{d}x \leq 1\right\}. \tag{2.14}$$

Remark 2.1:
$L_{\Phi,v}$ is a Banach space and the Luxemburg and Orlicz norms are equivalent . In fact,

$$\|f\|_{\Phi,v} \le \|f\|'_{\Psi,v} \le 2\|f\|_{\Phi,v}.$$

We assume $M(R_+)$ is the set of Lebesgue-measurable almost everywhere finite functions, M_+ is the cone of almost everywhere positive functions from $M = M(R_+)$;

$$M_+ = \big\{f \in M(R_+) : \; f > 0\big\}.$$

Consider the cone of positive decreasing functions from the Orlicz space:

$$\Omega = \big\{f \in L_{\Phi,v} : \; 0 \le f \downarrow\big\} \tag{2.15}$$

For $g \in M_+$, we introduce the following associated norm on the cone Ω:

$$\|g\|'_{\Omega} = \sup\bigg\{\int_0^{\infty} fg \,\mathrm{d}t : \; f \in \Omega; \; \|f\|_{\Phi,v} \le 1\bigg\}. \tag{2.16}$$

We formulate the result that generalize some previous results of papers [3], [5–7].

Proposition 2.1:
([4]). Let Φ, Ψ be the complementary N-functions, the N-function Φ satisfies Δ_2-condition, let $v \in M_+$, and let

$$0 < V(t) := \int_0^t v \,\mathrm{d}\tau < \infty, \quad t \in R_+, \quad V(+\infty) = +\infty. \tag{2.17}$$

The following two-sided estimate holds:

$$\|g\|'_{\Omega} \cong \|\mathfrak{R}_0(g)\|_{\Psi,v} = \inf\bigg\{\lambda > 0 : \; \int_0^{\infty} \Psi\big(\lambda^{-1}|\mathfrak{R}_0(g;t)|\big)v(t)\,\mathrm{d}t \le 1\bigg\}, \tag{2.18}$$

where

$$\mathfrak{R}_0(g;t) := V(t)^{-1}\int_0^t g(\tau)\,\mathrm{d}\tau, \quad t \in R_+. \tag{2.19}$$

Here and below we use the notation

$$A \cong B \Leftrightarrow \exists\, c \in [1,\infty) : \; c^{-1} \le A/B \le c. \tag{2.20}$$

In the following considerations, we will use the formula for the conjugate operator:

$$\mathfrak{R}_0^*(f;\tau) = \int_\tau^{\infty} \frac{f(t)}{V(t)}\,\mathrm{d}t, \quad \tau \in R_+. \tag{2.21}$$

Let us now state the main result of this section allowing us to reduce modular inequalities for operators on the cone Ω to modular inequalities for modified operators on the cone M_+.

Proposition 2.2:
([10]). Let T and T^* be positively homogeneous operators that map M_+ to M_+ and are

adjoint, i.e.,

$$\int\limits_{R_+} gTf\,\mathrm{d}\tau = \int\limits_{R_+} fT^*g\,\mathrm{d}\tau, \quad f,\, g \in M_+. \tag{2.22}$$

Let Φ_1, Φ_2 *be* N*-functions,* u, v, $w \in M_+$, *and let condition* (2.17) *holds. Let the operator* $\mathfrak{R}_0$ *be given by formula* (2.19). *Then the following inequalities are equivalent:*

$$\exists\, c_1 \in R_+ :\ \Phi_2^{-1}\Big\{\int_{R_+} \Phi_2(wTf)u\,\mathrm{d}t\Big\} \le \Phi_1^{-1}\Big\{\int_{R_+} \Phi_1(c_1 f)v\,\mathrm{d}t\Big\}, \quad f \in \Omega; \tag{2.23}$$

$$\exists\, c_3 \in R_+ :\ \Phi_2^{-1}\Big\{\int_{R_+} \Phi_2\big(wT\mathfrak{R}_0^*(vf)\big)u\,\mathrm{d}t\Big\} \le \Phi_1^{-1}\Big\{\int_{R_+} \Phi_1(c_3 f)v\,\mathrm{d}t\Big\}, \quad f \in M_+. \tag{2.24}$$

Definition 2.7:
The generalized Hardy Operators are operators of the form

$$Kf(x) = \int\limits_0^x k(x,t)f(t)\,\mathrm{d}t, \quad K^*g(t) = \int\limits_t^{+\infty} k(x,t)g(x)\,\mathrm{d}x, \tag{2.25}$$

where

a) $k : \big\{(x,t) \in R^2 :\ 0 < t < x < +\infty\big\} \to [0,+\infty)$;

b) $k(x,t) \ge 0$ *is nondecreasing in* x, *nonincreasing in* t;

c) $k(x,y) \le D\big(k(x,t) + k(t,y)\big)$, *for some constant* D, (2.26)

whenever $0 \le y \le t < x < +\infty$

Proposition 2.3:
([1]). Let Φ_1, Φ_2 *be* N*-function and* $\Phi_1 \prec\prec \Phi_2$, *and* K *be a generalized Hardy operator* (2.25). *Let* a, b, v *and* ω *be positive weight functions. Then there exists a constant* $A > 0$ *such that*

$$\Phi_2^{-1}\Big(\int\limits_0^{+\infty} \Phi_2(aKf)\omega\,\mathrm{d}x\Big) \le \Phi_1^{-1}\Big(\int\limits_0^{+\infty} \Phi_1(Afb\,\mathrm{d}x)v\Big)$$

for all positive, measurable functions f *if and only there exists a constant* C *such that*

$$\Phi_2^{-1}\Big(\int\limits_r^{+\infty} \Phi_2\Big(\frac{a(x)}{C}\Big\|\frac{k(r;\cdot)\chi_{(0,r)}(\cdot)}{\varepsilon vb}\Big\|_{\Psi_{1(\varepsilon v)}}\Big)\omega(x)\,\mathrm{d}x\Big) \le \Phi_1^{-1}\Big(\frac{1}{\varepsilon}\Big)$$

and

$$\Phi_2^{-1}\Big(\int\limits_r^{+\infty} \Phi_2\Big(\frac{a(x)}{C}\Big\|\frac{\chi_{(0,r)}(\cdot)}{\varepsilon vb}\Big\|_{\Psi_{1(\varepsilon v)}} k(x;r)\Big)\omega(x)\,\mathrm{d}x\Big) \le \Phi_1^{-1}\Big(\frac{1}{\varepsilon}\Big)$$

holds for ε, $r > 0$.

Proposition 2.4:
([1]). Let Φ_1, Φ_2 be N-function and $\Phi_1 \prec\prec \Phi_2$, and K^* be a generalized Hardy operator (2.25).
Let a, b, v and ω be positive weight functions. Then there exists a constant $A > 0$ such that

$$\Phi_2^{-1}\Big(\int_0^{+\infty} \Phi_2(aK^*f)\omega \,\mathrm{d}t\Big) \le \Phi_1^{-1}\Big(\int_0^{+\infty} \Phi_1(Abf)v \,\mathrm{d}t\Big)$$

holds for all positive, measurable functions f if and only there exists a constant C such that

$$\Phi_2^{-1}\Big(\int_0^{r} \Phi_2\Big(\frac{a(t)}{C}\Big\|\frac{k(\cdot;r)\chi_{(r,+\infty)}(\cdot)}{\varepsilon vb}\Big\|_{\Psi_{1(\varepsilon v)}}\Big)\omega(t)\,\mathrm{d}t\Big) \le \Phi_1^{-1}\Big(\frac{1}{\varepsilon}\Big)$$

and

$$\Phi_2^{-1}\Big(\int_0^{r} \Phi_2\Big(\frac{a(t)}{C}\Big\|\frac{\chi_{(r,+\infty)}(\cdot)}{\varepsilon vb}\Big\|_{\Psi_{1(\varepsilon v)}}k(r;t)\Big)\omega(t)\,\mathrm{d}t\Big) \le \Phi_1^{-1}\Big(\frac{1}{\varepsilon}\Big)$$

holds for ε, $r > 0$.

3. APPLICATIONS FOR HARDY-TYPE OPERATORS

Let us now state the main result of this section allowing us to reduce modular inequalities for operators on the cone Ω to modular inequalities for modified operators on the cone M_+.

$$\mathfrak{I}(f;t) = \int_0^{+\infty} f_\varphi(t;\tau)f(\tau)\,\mathrm{d}\tau, \quad \tau \in R_+, \tag{3.27}$$

where $f_\varphi(t;\tau) = \min\{\varphi(t), \varphi(\tau)\}$.

Theorem 3.1:
Let Φ_1, Φ_2 be N-function and $\Phi_1 \prec\prec \Phi_2$, w, u, v be positive weight functions, $\mathfrak{I}$ be Hardy-type operators (3.27)*. Let the condition be satisfied*

$$A_\varphi = \sup_{t\in R_+} \frac{1}{t\varphi(t)}\Big(\int_0^t \varphi\,\mathrm{d}\tau\Big) < \infty. \tag{3.28}$$

Then there exists a constant $C > 0$ such that inequality

$$\Phi_2^{-1}\Big\{\int_{R_+} \Phi_2\big(w(t)\mathfrak{I}f\big)u(t)\,\mathrm{d}t\Big\} \le \Phi_1^{-1}\Big\{\int_{R_+} \Phi_1(Cf)v\,\mathrm{d}t\Big\}, \quad f \in \Omega, \tag{3.29}$$

holds for all positive, nonincreasing functions f if and only if there is a constant B such that the following inequalities hold for all ε, $r > 0$:

$$\Phi_2^{-1}\Big\{\int_0^{\infty} \Phi_2\Big(\frac{w(t)}{B}\cdot\frac{f_\varphi(t,r)}{\varphi(r)}\Big\|\frac{f_\varphi(\cdot,r)(\cdot)}{\varepsilon V}\Big\|_{\Psi_{1(\varepsilon v)}}\Big)u(t)\,\mathrm{d}t\Big\} \le \Phi_1^{-1}\Big(\frac{1}{\varepsilon}\Big). \tag{3.30}$$

Proof
The purpose of the first step is to reduce estimate(2.24) to the estimate for the Hardy-type operator given in [11]. For the Hardy-type operator (3.27), by using (2.21), we obtain

$$\mathfrak{I}\big(\mathfrak{R}_0^*(vf;t)\big) = \int_0^\infty f_\varphi(t;\tau)\mathfrak{R}_0^*(vf;\tau)\,d\tau = \int_0^\infty f_\varphi(t;\tau)\bigg(\int_\tau^\infty \frac{f(\xi)v(\xi)}{V(\xi)}\,\mathrm{d}\xi\bigg)\,\mathrm{d}\tau, \quad \tau \in R_+\,.$$

By changing the order of integration, we have

$$\mathfrak{I}\big(\mathfrak{R}_0^*(vf;t)\big) = \int_0^\infty \frac{f(\xi)v(\xi)}{V(\xi)}\bigg(\int_0^\xi f_\varphi(t;\tau)\,\mathrm{d}\tau\bigg)\,\mathrm{d}\xi. \tag{3.31}$$

Let us show that

$$\int_0^\xi f_\varphi(t;\tau)\,\mathrm{d}\tau \cong f_\varphi(t;\xi)\xi; \quad t,\,\xi \in R_+ = (0,+\infty). \tag{3.32}$$

From the decrease of φ and from the condition $A_\varphi < \infty$ it follows that

$$\varphi(t)t \le \int_0^t \varphi\,\mathrm{d}\tau \le A_\varphi\varphi(t)t, \quad t \in R_+. \tag{3.33}$$

1) For $\xi \le 1$ we have $f_\varphi(t;\tau) = \varphi(t)$, $\tau \in (0,\xi)$, so that

$$\int_0^\xi f_\varphi(t;\tau)\,\mathrm{d}\tau = \varphi(t)\int_0^\xi \mathrm{d}\tau = \varphi(t)\xi = f_\varphi(t;\xi)\xi. \tag{3.34}$$

2) For $\xi > t$ we have

$$\int_0^\xi f_\varphi(t;\tau)\,\mathrm{d}\tau = \int_0^t \varphi(t)\,d\tau + \int_t^\xi \varphi(\tau)\,\mathrm{d}\tau =$$

$$= \varphi(t)t + \int_t^\xi \varphi(\tau)\,\mathrm{d}\tau \underset{(3.34)}{\cong} \int_0^t \varphi(\tau)\,\mathrm{d}\tau + \int_t^\xi \varphi(\tau)\,\mathrm{d}\tau = \int_0^\xi \varphi(\tau)\,\mathrm{d}\tau \underset{(3.34)}{\cong} \varphi(\xi)\xi,$$

that is, for $\xi > t$

$$\int_0^\xi f_\varphi(t;\tau)\,\mathrm{d}\tau \cong \varphi(\xi)\xi = f_\varphi(t,\xi)\xi. \tag{3.35}$$

From (3.34), (3.35) it follows (3.32).
Substitute (3.32) in (3.31):

$$\mathfrak{I}\big(\mathfrak{R}_0^*(vf;t)\big) = \int_0^\infty \frac{f(\xi)v(\xi)}{V(\xi)} f_\varphi(t;\xi)\xi\,\mathrm{d}\xi, \tag{3.36}$$

so

$$\mathfrak{I}\big(\mathfrak{R}_0^*(vf;t)\big) = \int\limits_0^\infty f_\varphi(t;\xi)g(\xi)\,\mathrm{d}\xi, \tag{3.37}$$

where

$$g(\xi) = \frac{f(\xi)v(\xi)\xi}{V(\xi)}. \tag{3.38}$$

We obtain the equivalence of (2.24) and (3.39), where (3.39) is of the form

$$\exists\, c_3 \in R_+ :$$
$$\Phi_2^{-1}\Big\{\int_{R_+} \Phi_2\Big(w(t)\int_0^\infty f_\varphi(t;\xi)g(\xi)\,\mathrm{d}\xi\Big)u(t)\,\mathrm{d}t\Big\} \le \Phi_1^{-1}\Big\{\int_{R_+} \Phi_1(c_3\sigma g)v\,\mathrm{d}t\Big\}, \quad g \in M_+ \tag{3.39}$$

and $\sigma(t) = V(t)v^{-1}(t)t^{-1}$. As a result, introducing the operator

$$\mathfrak{I}_0(g;t) = \int\limits_0^\infty f_\varphi(t;\xi)g(\xi)\,\mathrm{d}\xi, \quad t \in R_+ \tag{3.40}$$

where the kernel

$$f_\varphi(t;\tau) = \min\big\{\varphi(t),\varphi(\tau)\big\} = \begin{cases}\varphi(t), & t > \tau,\\ \varphi(\tau), & \tau > t\end{cases}$$

(operator $\mathfrak{I}_0$ in the notations of [11]) we obtain the equivalence of the modular inequalities (2.24) and (3.39).

2. We now pass to the proof of the equivalence of inequality (3.39) and the set of conditions (3.29). To this end, we use a known result due to Jim Quile Sun which was given in our paper [11] combined with the generalizations given in [4]. Denote

$$\Phi_2^{-1}\Big\{\int_0^r \Phi_2\Big(\frac{w(t)}{B}\cdot\frac{tf_\varphi(t,r)}{\varphi(r)}\Big\|\frac{f_\varphi(\cdot,r)}{\varepsilon V}\Big\|_{\Psi_1(\varepsilon v)}\Big)u(t)\,\mathrm{d}t\Big\} \le \Phi_1^{-1}\Big(\frac{1}{\varepsilon}\Big). \tag{3.41}$$

Thus, we have shown that (2.24) ⇔ (3.39) ⇔ (3.29). Theorem 3.1 is proved. □

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