Adv Syst Sci Appl 2021; **01**:139–149
Published online at https://ijassa.ipu.ru.

Application of the Minimum Principle of a Tikhonov Smoothing Functional in the Problem of Processing Thermographic Data

Eugeniy Laneev[1]*, Natalia Chernikova[1], Obaida Baaj[1]

[1] *Peoples' Friendship University of Russia, Moscow, Russia*

Abstract: The paper considers a method for correcting thermographic images. Mathematical processing of thermograms is based on the analytical continuation of the stationary temperature distribution as a harmonic function from the surface of the object under study to the heat sources. The continuation is performed by solving an ill posed mixed problem for the Laplace equation in a cylindrical region of rectangular cross-section. The cylindrical area is bounded by an arbitrary surface and plane. The Cauchy conditions are set on the surface-the boundary values of the desired function and its normal derivative. Inhomogeneous conditions of the first kind are set on the side faces of the cylinder. The problem is the inverse of the corresponding mixed problem for the Poisson equation. In this paper, an approximate solution of the problem is obtained that is stable with respect to the error in the Cauchy data and inhomogeneity in the boundary conditions. In the course of constructing an approximate solution, the problem is reduced to the Fredholm integral equation of the first kind, which is solved using the minimum smoothing functional principle. The convergence of the approximate solution of the problem is proved when the regularization parameter is matched to the error in the data.

Keywords: termogram, ill-posed problem, inverse problem, Cauchy problem for the Laplace equation, integral equation of the first kind, Tikhonov regularization method

1. INTRODUCTION

Digital technologies have penetrated into all branches of human activity and one of the urgent problems is to improve the quality and information content of representations of research results, in particular, the quality of images obtained from measurement data, through their mathematical (digital) processing. This applies, for example, to images obtained by thermal imaging methods using a thermal imager that registers thermal electromagnetic radiation from the surface of the object under study in the infrared range. In particular, in medicine, thermal imaging has become an effective means of early diagnostics [1]. The image on the thermogram, which is a map of the temperature distribution on the surface of the patient's body, makes it possible to assess functional abnormalities in the state of his internal organs. At the same time, the image on the thermogram in some cases turns out to be somewhat distorted due to the processes of thermal conductivity and heat exchange. The paper proposes a method for correcting the image on a thermogram within a certain mathematical model. As a corrected thermogram, the image of the temperature distribution on the plane near the density of heat sources is considered as more accurately transmitting the image of heat sources. It is proposed to obtain this distribution as a result of the continuation (similar to the continuation of gravitational fields in Geophysics problems [2]) of the temperature distribution from the surface from which the original thermogram is taken. The continuation is obtained by solving

*Corresponding author: elaneev@yandex.ru

the inverse problem to a certain mixed boundary value problem for the Poisson equation. The considered inverse problem is ill-posed, since small errors in the initial data (the initial thermogram) may correspond to significant errors in the solution of the inverse problem. To construct its stable approximate solution, we use the Tikhonov regularization method [3], based on optimization methods [4].

2. STATEMENT OF THE PROBLEM

Let's consider a physical and then a mathematical model, within which we will set the inverse problem.

The physical model is a homogeneous heat-conducting body in the form of a rectangular cylinder, bounded by the surface S and containing heat sources with a time-independent density function that create a stationary temperature distribution in the body. We associate the density function of heat sources with the object under study. We assume that a given temperature distribution is maintained on the side faces of the cylinder, and on the surface S there is a convective heat exchange with the external environment of temperature U_0, described by Newton's law, according to which the heat flux density at a point on the surface is directly proportional to the temperature difference inside and outside.

Let's move on to the mathematical model. In the cylinder of rectangular cross section

$$D^\infty = \{(x,y,z):\ 0<x<l_x,\ 0<y<l_y,\ -\infty<z<\infty\} \subset \mathbb{R}^3$$

consider a cylindrical domain

$$D(F,\infty) = \{(x,y,z):\ 0<x<l_x,\ 0<y<l_y,\ F(x,y)<z<\infty\}, \tag{2.1}$$

bounded by the surface

$$S = \{(x,y,z):\ 0<x<l_x,\ 0<y<l_y,\ z=F(x,y)<H\}. \tag{2.2}$$

Let Γ be the sum of side faces of the domain $D(F,\infty)$. In the domain $D(F,\infty)$ consider the following mixed boundary value problem for the Laplace equation

$$\begin{array}{ll} \Delta u(M) = \rho(M), & M \in D(F,\infty), \\ \left.\dfrac{\partial u}{\partial n}\right|_S = h(U_0 - u)\Big|_S, & \\ u|_\Gamma = f_1, & \\ u \text{ is bounded when } z \to \infty. & \end{array} \tag{2.3}$$

The problem (2.3) corresponds to the steady-state temperature distribution created with heat sources of the distribution density function ρ, on the surface S – the third boundary condition is set, corresponding to convective heat exchange with the external environment of temperature U_0 with the coefficient $h,$ on the boundary Γ the temperature is set as a function f_1. We assume that the density carrier ρ is located in the domain $z > H.$

We also assume that the functions ρ, f_1 are such that the solution of the problem (2.3) exists in $C^2(D(F,\infty)) \bigcap C^1(\overline{D(F,\infty)})$. In particular, the solution of the problem (2.3) gives the boundary value $u|_S$.

Now let's set the inverse problem.

Inverse problem 1. Let within the model (2.3) be set the following functions

$$f = u|_S, \quad f_1 = u|_\Gamma. \tag{2.4}$$

We need to find a continuous function ρ.

Note that density recovery is associated with the same difficulties as solving the inverse potential problem [5], for which significant restrictions on uniqueness classes are known. Therefore, to solve the inverse problem, we apply the approach [2] used in Geophysics problems. Source of information about the density of ρ we will consider the function $u|_{z=H}$ on the plane $z = H$, that is closer to the density carrier ρ than the surface S.

Since the carrier of the function ρ by the condition of the problem (2.3) is located in the domain $z > H$, then the solution of the problem (2.3) in the domain

$$D(F, H) = \{(x, y, z) : \ 0 < x < l_x, \ 0 < y < l_y, \ F(x, y) < z < H\} \tag{2.5}$$

satisfies the Laplace equation. The sum of side faces of the domain $D(F, H)$ denote by Γ_H. Instead of the inverse problem 1, we will solve the following inverse problem

Inverse problem 2. Let within the model (2.3) be set the following functions

$$f = u|_S, \quad f_1 = u|_{\Gamma_H}. \tag{2.6}$$

We need to find a solution u to the boundary value problem in the domain $D(F, H)$

$$\begin{array}{ll} \Delta u(M) = 0, & M \in D(F, H), \\ u|_S = f, & \\ \dfrac{\partial u}{\partial n}\Big|_S = h(U_0 - f)\Big|_S, & \\ u|_{\Gamma_H} = f_1. & \end{array} \tag{2.7}$$

We will consider the function $u|_{z=H}$ as a source of information about the density ρ.

We assume that the functions f, f_1 in (2.6), (2.7) are taken from the set of solutions of the direct problem (2.3), so the solution of the inverse problem exists in $C^2(D(F, H)) \bigcap C^1(\overline{D(F, H)})$.

We note that in the problem (2.7) on the surface S of the form (2.2), Cauchy conditions are set, that is, the boundary values f of the desired function u and the values of its normal derivative are set, so the problem (2.7) has a unique solution. The boundary $z = H$ of the domain $D(F, H)$ of the form (2.5) is free and, thus, the problem (2.7) is unstable with respect to data errors, i.e. it is ill-posed.

We will construct an explicit representation of the exact solution of the problem (2.7).

3. EXACT SOLUTION OF THE PROBLEM

Let's construct an exact solution of the problem (2.7), following the scheme [6, 7]. Consider the source function $\varphi(M, P)$ of the Dirichlet problem in the cylinder D^∞:

$$\begin{array}{ll} \Delta u(P) = \rho(P), & P \in D^\infty, \\ u|_{x=0,l_x} = 0, & u|_{y=0,l_y} = 0, \\ u \to 0 & \text{when} \quad |z| \to \infty, \end{array} \tag{3.8}$$

i.e.,

$$\varphi(M, P) = \frac{1}{4\pi r_{MP}} + W(M, P), \tag{3.9}$$

where r_{MP} is the distance between points M and P and $W(M, P)$ is a harmonic function of point P.

The source function can be obtained by the reflection method as a sum of point source functions with period $2l_x$ in the variables x and period $2l_y$ in the variable y,

$$\varphi(M, P) = \frac{1}{4\pi} \sum_{n,m=-\infty}^{\infty} \Big(\frac{1}{r_{1,nm}} - \frac{1}{r_{2,nm}} - \frac{1}{r_{3,nm}} + \frac{1}{r_{4,nm}} \Big), \tag{3.10}$$

where

$$\begin{aligned} r_{1,nm} &= [(x_M - x_P + 2l_x n)^2 + (y_M - y_P + 2l_y m)^2 + (z_M - z_P)^2]^{1/2},\\ r_{2,nm} &= [(x_M + x_P + 2l_x n)^2 + (y_M - y_P + 2l_y m)^2 + (z_M - z_P)^2]^{1/2},\\ r_{3,nm} &= [(x_M - x_P + 2l_x n)^2 + (y_M + y_P + 2l_y m)^2 + (z_M - z_P)^2]^{1/2},\\ r_{4,nm} &= [(x_M + x_P + 2l_x n)^2 + (y_M + y_P + 2l_y m)^2 + (z_M - z_P)^2]^{1/2}, \end{aligned}$$

and, in particular, $r_{1,00} = r_{MP}$.

Let $M \in D(F,H)$. Then, applying the Green formulas in the domain $D(F,H)$ to the function $u(P)$, i.e., the solution of problem (2.7), and to the functions $\frac{1}{4\pi r_{MP}}$ and $W(M,P)$ in (3.9), we obtain

$$u(M) = \int\limits_{\partial D(F,H)} \Big[\frac{\partial u}{\partial n}(P)\frac{1}{4\pi r_{MP}} - u(P)\frac{\partial}{\partial n_P}\frac{1}{4\pi r_{MP}}(M,P)\Big] d\sigma_P, \quad M \in D(F,H) \tag{3.11}$$

and

$$0 = \int\limits_{\partial D(F,H)} \Big[\frac{\partial u}{\partial n}(P)W(M,P) - u(P)\frac{\partial W}{\partial n_P}(M,P)\Big] d\sigma_P, \quad M \in D(F,H). \tag{3.12}$$

Summing (3.11) and (3.12) taking into account (3.9) we obtain

$$u(M) = \int\limits_{\partial D(F,H)} \Big[\frac{\partial u}{\partial n}(P)\varphi(M,P) - u(P)\frac{\partial \varphi}{\partial n_P}(M,P)\Big] d\sigma_P, \quad M \in D(F,H). \tag{3.13}$$

Given homogeneous boundary conditions for φ and inhomogeneous ones for u on the side faces Γ_H of the cylindrical domain $D(F,H)$, we obtain

$$\begin{aligned} u(M) = &\int\limits_S \Big[h(U_0 - f(P))\varphi(M,P) - f(P)\frac{\partial \varphi}{\partial n_P}(M,P)\Big] d\sigma_P - \\ &- \int\limits_{\Gamma_H} \Big[f_1(P)\frac{\partial \varphi}{\partial n_P}(M,P)\Big] d\sigma_P + \int\limits_{\Pi(H)} \Big[\frac{\partial u}{\partial n}(P)\varphi(M,P) - u(P)\frac{\partial \varphi}{\partial n_P}(M,P)\Big] d\sigma_P, \end{aligned}$$

where

$$\Pi(H) = \{(x,y,z):\ 0 < x < l_x,\ 0 < y < l_y,\ z = H\}. \tag{3.14}$$

In the domain $z_M < H$, we introduce the notation

$$\Phi(M) = \int\limits_S \Big[h(U_0 - f(P))\varphi(M,P) - f(P)\frac{\partial \varphi}{\partial n_P}(M,P)\Big] d\sigma_P - \int\limits_{\Gamma_H} \Big[f_1(P)\frac{\partial \varphi}{\partial n_P}(M,P)\Big] d\sigma_P, \tag{3.15}$$

$$v(M) = \int\limits_{\Pi(H)} \Big[\frac{\partial u}{\partial n}(P)\varphi(M,P) - u(P)\frac{\partial \varphi}{\partial n_P}(M,P)\Big] d\sigma_P, \quad z_M < H. \tag{3.16}$$

Then we obtain the solution of the problem (2.7) in the form

$$u(M) = v(M) + \Phi(M), \quad M \in D(F,H), \tag{3.17}$$

where the function Φ is calculated from known functions f and f_1.

If the solution of the problem (2.7) exists, then the function v of the form (3.16), harmonic in the domain

$$D(-\infty, H) = \{(x, y, z) : \; 0 < x < l_x, \; 0 < y < l_y, \; -\infty < z < H\},$$

can be represented in $D(F, H) \subset D(-\infty, H)$ according to (3.17) in the form $v = u - \Phi$ and then it may be defined on the boundary of $\Pi(H)$ as a continuous function

$$v\mid_{z=H} = u\mid_{z=H} - \Phi\mid_{z=H} = v_H. \tag{3.18}$$

Thus, the function v can be viewed as a solution of the problem

$$\begin{array}{ll} \Delta v(M) = 0, & M \in D(-\infty, H), \\ v|_{z=H} = v_H, & \\ v|_{x=0,l_x} = 0, & v|_{y=0,l_y} = 0, \\ v \to 0 & \text{when} \quad z \to -\infty, \end{array} \tag{3.19}$$

and the function v can be expressed in terms of the function v_H by using the Green function of problem (3.19) as follows:

$$v(M) = -\int\limits_{\Pi(H)} \frac{\partial G}{\partial n_P}(M, P) v_H(P) dx_P dy_P, \quad M \in D(-\infty, H), \tag{3.20}$$

where

$$\frac{\partial G}{\partial n_P}(M, P)\Big|_{P\in\Pi(H)} = \frac{\partial G}{\partial z_P}(M, P)\Big|_{P\in\Pi(H)} =$$
$$= -\frac{4}{l_x l_y} \sum_{n,m=1}^{\infty} \exp\big\{k_{nm}(-H + z_M)\big\} \sin\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y} \sin\frac{\pi n x_P}{l_x} \sin\frac{\pi m y_P}{l_y}, \tag{3.21}$$

$$k_{nm} = \pi\left(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}\right)^{1/2}. \tag{3.22}$$

It follows that if problem (2.7) has a solution, then (3.20) implies that the function v in the domain $D(-\infty, H)$ can be represented as the Fourier series

$$v(M) = v(x, y, z) = -\sum_{n,m=1}^{\infty} (\tilde{v_H})_{nm} \exp\big\{k_{nm}(z - H)\big\} \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y}, \tag{3.23}$$

$$(\tilde{v_H})_{nm} = \frac{4}{l_x l_y} \int\limits_0^{l_x}\int\limits_0^{l_y} v_H(x', y') \sin\frac{\pi n x'}{l_x} \sin\frac{\pi m y'}{l_y} dx' dy', \tag{3.24}$$

of a complete system of functions

$$\left\{\sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y}\right\}_{n,m=1}^{\infty}. \tag{3.25}$$

The series (3.23) uniformly converges in the domain $D(-\infty, H - \varepsilon)$ for any $\varepsilon > 0$, because

$$\big|(\tilde{v_H})_{nm} \exp\big\{k_{nm}(z - H)\big\} \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y}\big| \leqslant \big|(\tilde{v_H})_{nm}\big| \exp\big\{-\varepsilon k_{nm}\big\}.$$

Thus, it follows from the representation (3.17) of the solution of problem (2.7) and from (3.23) that, to obtain an explicit expression for the exact solution of problem (2.7), it suffices to express the function v_H (3.18) in terms of the prescribed functions f and f_1.

Let us show that the function v_H satisfies a Fredholm integral equation of the first kind. Let $M \in D(-\infty, F)$, where

$$D(-\infty, F) = \{(x, y, z) : \, 0 < x < l_x, \, 0 < y < l_y, \, -\infty < z < F(x, y)\}.$$

Applying the Green formula in the domain $D(F, H)$ to the function $u(P)$, i.e., a solution of problem (2.7), and to a function $\varphi(M, P)$ of the form (3.9), we, by analogy with (3.11), (3.12), and (3.13), obtain the relation

$$0 = \int\limits_{\partial D(F,H)} \Big[\frac{\partial u}{\partial n}(P)\varphi(M,P) - u(P)\frac{\partial \varphi}{\partial n_P}(M,P)\Big] d\sigma_P, \quad M \in D(-\infty, F).$$

From this, with regard to the homogeneous boundary conditions for the function φ and to the inhomogeneous boundary conditions for u and notation (3.15) and (3.16), we obtain

$$v(M) = -\Phi(M), \quad M \in D(-\infty, F). \tag{3.26}$$

Let $a < \min\limits_{(x,y)} F(x, y)$ and $M \in \Pi(a)$, where $\Pi(a)$ is a domain of the form (3.14) for $z = a$. Then, by formulas (3.26) and (3.20), we obtain the integral equation of the first kind

$$\int\limits_{\Pi(H)} \frac{\partial G}{\partial n_P}(M,P) v_H(P) dx_P dy_P = \Phi(M), \quad M \in \Pi(a). \tag{3.27}$$

From the equation (3.27) taking into account the decomposition (3.21) for $z_M = a$ we obtain the following relations between the Fourier coefficients of the unique solution v_H of this integral equation and the Fourier coefficients of its right-hand side:

$$-\,(\tilde{v_H})_{nm} \exp\big\{-k_{nm}(H-a)\big\} = \tilde{\Phi}_{nm}(a), \tag{3.28}$$

where the $\tilde{\Phi}_{nm}(a)$ are the Fourier coefficients of the function $\Phi(M)|_{M\in\Pi(a)}$,

$$\tilde{\Phi}_{nm}(a) = \frac{4}{l_x l_y} \int\limits_{\Pi(a)} \Phi(x, y, a) \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y} dx dy. \tag{3.29}$$

Note that formula (3.28) characterizes the decrease in the Fourier coefficients $\tilde{\Phi}_{nm}(a)$ with increasing n and m if, for the functions f and f_1, there exists a solution of problem (2.7) and hence a function v_H defined by (3.18). We express the Fourier coefficients $(\tilde{v_H})_{nm}$, substitute them into the series (3.23), and obtain the function v in the domain $D(-\infty, H)$:

$$v(M) = -\sum_{n,m=1}^{\infty} \tilde{\Phi}_{nm}(a) \exp\{k_{nm}(z-a)\} \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y}, \;\; M(x, y, z) \in D(-\infty, H). \tag{3.30}$$

The series (3.30), just as the series (3.23), uniformly converges in the domain $D(-\infty, H - \varepsilon)$ for any $\varepsilon > 0$ if there exists a solution of problem (2.7) for the given functions f and f_1.

Formula (3.17), where the functions v and Φ are given by (3.30) and (3.15), respectively, gives an explicit expression for the solution of problem (2.7).

4. APPROXIMATE SOLUTION OF THE PROBLEM

Let the functions f and f_1 in problem (2.7) be given with an error; i.e., let, instead of them, functions f^δ and f_1^δ be given such that

$$\|f^\delta - f\|_{L_2(S)} \leqslant \delta, \quad \|f_1^\delta - f_1\|_{L_2(\Gamma_H)} \leqslant \delta.$$

We construct an approximate solution of problem (2.7) converging to the exact solution as $\delta \to 0$. Here the function Φ defined by formula (3.15) can be obtained approximately as

$$\Phi^\delta(M) = \int\limits_S \Big[h(U_0 - f^\delta(P))\varphi(M,P) - f^\delta(P)\frac{\partial\varphi}{\partial n_P}(M,P)\Big]d\sigma_P - \\ - \int\limits_{\Gamma_H} \Big[f_1^\delta(P)\frac{\partial\varphi}{\partial n_P}(M,P)\Big]d\sigma_P. \quad (4.31)$$

We apply the Cauchy-Schwarz inequality to the difference of functions (4.31) and (3.15) for $M \in \Pi(a)$, $a < \min\limits_{(x,y)} F(x,y)$, and obtain an estimate of the right-hand side of integral equation (3.27),

$$|\Phi^\delta(M) - \Phi(M)| \leqslant h \max_{M\in\Pi(a)} \Big(\int\limits_S \varphi^2(M,P)d\sigma_P\Big)^{1/2}\|f^\delta - f\|_{L_2(S)} + \\ + \max_{M\in\Pi(a)} \Big(\int\limits_S \Big[\frac{\partial\varphi}{\partial n_P}(M,P)\Big]^2 d\sigma_P\Big)^{1/2}\|f^\delta - f\|_{L_2(S)} + \\ + \max_{M\in\Pi(a)} \Big(\int\limits_{\Gamma_H} \Big[\frac{\partial\varphi}{\partial n_P}(M,P)\Big]^2 d\sigma_P\Big)^{1/2}\|f_1^\delta - f_1\|_{L_2(\Gamma_H)} \leqslant C\delta. \quad (4.32)$$

For an approximate solution of Eq. (3.27), we take the extremal of the Tikhonov functional [3, p. 68] with zero-order stabilizer,

$$M^\alpha[w] = \Big\|\int\limits_S \frac{\partial G}{\partial n} w d\sigma - \Phi^\delta\Big\|^2_{L_2(\Pi(a))} + \alpha\|w\|^2_{L_2(\Pi(H))},\ \alpha > 0, \quad (4.33)$$

where $\Pi(a)$ and $\Pi(H)$ are domains defined by formula (3.14).

The extremal can be obtained as a solution of the Euler equation for the functional (4.33) which, in the Fourier coefficients of the function w, has the form

$$\exp\{-2k_{nm}(H-a)\}\,\tilde{w}_{nm} + \alpha\tilde{w}_{nm} = -\exp\{-k_{nm}(H-a)\}\,\tilde{\Phi}^\delta_{nm}(a),$$

where

$$\tilde{\Phi}^\delta_{nm}(a) = \frac{4}{l_x l_y}\int\limits_{\Pi(a)} \Phi^\delta(x,y,a)\sin\frac{\pi n x}{l_x}\sin\frac{\pi m y}{l_y}dxdy \quad (4.34)$$

are the Fourier coefficients of the function $\Phi^\delta(M)|_{M\in\Pi(a)}$.

Solving the equation for the Fourier coefficients of the extremal and substituting the extremal w_α^δ for v_H into representation (3.23), we obtain an approximation v_α^δ to the function

v in the domain $D(-\infty, H)$,

$$v_\alpha^\delta(M) = -\sum_{n,m=1}^{\infty} \frac{\tilde{\Phi}_{nm}^\delta(a)\exp\{k_{nm}(z_M - a)\}}{1+\alpha\exp\{2k_{nm}(H-a)\}} \sin\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y}. \tag{4.35}$$

Note that coefficients of the series (4.35) differ from corresponding coefficients of the series (3.30) in the factor $(1+\alpha\exp\{2k_{nm}(H-a)\})^{-1}$, and the series (4.35) converges uniformly.

According to the representation (3.17), we obtain an approximate solution of problem (2.7) in the form

$$u_\alpha^\delta(M) = v_\alpha^\delta(M) + \Phi^\delta(M), \quad M \in D(F, H), \tag{4.36}$$

where v_α^δ and Φ^δ are the functions defined by formulas (4.35) and (4.31) respectively.

Theorem. Assume that there exists a solution of problem (2.7). Then, for any $\alpha = \alpha(\delta) > 0$ such that $\alpha(\delta) \to 0$ and $\delta/\sqrt{\alpha(\delta)} \to 0$ as $\delta \to 0$, the function $u_{\alpha(\delta)}$ of the form (4.36) uniformly converges as $\delta \to 0$ to the exact solution of problem (2.7) on any compact $K \subset D(F, H)$.

Proof. On any compact $K \subset D(F, H)$, according to representations (4.36) and (3.17), we estimate the difference

$$|u_\alpha^\delta - u| \leqslant |v_\alpha^\delta - v| + |\Phi^\delta - \Phi|. \tag{4.37}$$

Obviously, there is $\varepsilon > 0$ such that $K \subset D(-\infty, H-\varepsilon)$. For the modul of the difference $v_\alpha^\delta - v$ in the domain $D(-\infty, H-\varepsilon)$ we obtain

$$|v_\alpha^\delta - v| \leqslant |v_\alpha^\delta - v_\alpha| + |v_\alpha - v|, \tag{4.38}$$

where v_α is a function of the form (4.35) for exact functions f and f_1,

$$v_\alpha(M) = -\sum_{n,m=1}^{\infty} \frac{\tilde{\Phi}_{nm}(a)\exp\{k_{nm}(z_M - a)\}}{1+\alpha\exp\{2k_{nm}(H-a)\}} \sin\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y}. \tag{4.39}$$

To estimate the difference $v_\alpha^\delta - v_\alpha$ on the right-hand side in inequality (4.38) for $z_M < H - \varepsilon$, we use inequality (4.32)

$$|v_\alpha^\delta(M) - v_\alpha(M)| \leqslant \left| \sum_{n,m=1}^{\infty} \frac{\exp\{k_{nm}(z_M - a)\}}{1+\alpha\exp\{2k_{nm}(H-a)\}} \right| \cdot 4 \max_{P\in\Pi(a)} \left|\Phi^\delta(P) - \Phi(P)\right| \leqslant$$
$$\leqslant C_1\delta \sum_{n,m=1}^{\infty} \frac{\exp\{k_{nm}(H-\varepsilon-a)\}}{1+\alpha\exp\{2k_{nm}(H-a)\}} \leqslant$$
$$\leqslant C_1\delta \max_x \left[\frac{e^x}{1+\alpha e^{2x}}\right] \sum_{n,m=1}^{\infty} \exp\{-k_{nm}\varepsilon\} \leqslant C_2\frac{\delta}{\sqrt{\alpha}}. \tag{4.40}$$

We estimate the difference $v_\alpha - v$ in inequality (4.38) for $z_M < H - \varepsilon$,

$$|v_\alpha - v| \leqslant \sum_{n,m=1}^{\infty} \frac{\alpha\exp\{2k_{nm}(H-a)\}\exp\{k_{nm}(H-\varepsilon-a)\}}{1+\alpha\exp\{2k_{nm}(H-a)\}} \left|\tilde{\Phi}_{nm}(a)\right|.$$

From this, using (3.28) and applying the CauchySchwarz inequality, we obtain

$$|v_\alpha - v| = \sum_{n,m=1}^{\infty} \frac{\alpha \exp\{2k_{nm}(H-a)\} \exp\{-k_{nm}\varepsilon\}}{1+\alpha \exp\{2k_{nm}(H-a)\}} |(\tilde{v_H})_{nm}| \leqslant$$

$$\leqslant \Big[\sum_{n,m=1}^{\infty} \left(\frac{\alpha \exp\{2k_{nm}(H-a)\}}{1+\alpha \exp\{2k_{nm}(H-a)\}} \right)^2 \exp\{-2k_{nm}\varepsilon\} \Big]^{1/2} \cdot \frac{2}{\sqrt{l_x l_y}} ||v_H||_{L_2}.$$

Since the series depending on the parameter α is majorized by the converging numerical series with coefficients $\exp\{-2\varepsilon k_{nm}\}$, it is possible to pass to the limit in α, and hence

$$|v_\alpha - v| \to 0 \quad \text{when} \quad \alpha \to 0. \tag{4.41}$$

It follows from (4.38), (4.40), (4.41), and the assumptions of the theorem that

$$|v^\delta_{\alpha(\delta)} - v| \to 0 \quad \text{when} \quad \delta \to 0. \tag{4.42}$$

The second difference on the right-hand side in inequality (4.37) can be estimated by analogy with (4.32). We apply the Cauchy-Schwarz inequality to this difference for $M \in K$, and obtain

$$|\Phi^\delta(M) - \Phi(M)| \leqslant h \max_{M \in K} \Big(\int_S \varphi^2(M,P) d\sigma_P \Big)^{1/2} \|f^\delta - f\|_{L_2(S)} +$$

$$+ \max_{M \in K} \Big(\int_S \big[\frac{\partial \varphi}{\partial n_P}(M,P)\big]^2 d\sigma_P \Big)^{1/2} \|f^\delta - f\|_{L_2(S)} +$$

$$+ \max_{M \in K} \Big(\int_{\Gamma_H} \big[\frac{\partial \varphi}{\partial n_P}(M,P)\big]^2 d\sigma_P \Big)^{1/2} \|f_1^\delta - f_1\|_{L_2(\Gamma_H)} \leqslant C_3 \delta.$$

From this relation, inequality (4.37), and formula (4.42) the assertion of the theorem follows.

5. NUMERICAL SOLUTION OF THE PROBLEM

The effectiveness of the proposed method for solving the problem (2.7) is shown in the following model example.

In the problem (2.3), let the surface S be the plane $\Pi(0)$, $f_1 = U_0 = 24, \quad h = 0.5, l_x = 30, l_y = 30, H = 1.4$, and the function ρ corresponds to three point sources at points in the plane $\Pi(H): (x_1, y_1) = (8.8)$, $(x_2, y_2) = (10.8)$, $(x_3, y_3) = (10.10)$. The boundary value of the solution of the model problem (2.3) in this case has the form

$$f(x,y) = U_0 + \sum_{n,m=1}^{\infty} \sum_{i=1}^{3} \frac{e^{-k_{nm}H}}{k_{nm}+h} \sin\frac{\pi n x_i}{l_x} \sin\frac{\pi m y_i}{l_y} \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y}, \tag{5.43}$$

where k_{nm} is calculated using the formula (3.22).

To set the inverse problem (2.7), we consider that the function f, calculated by the formula (5.43), a known function. Also $f_1 = U_0 = 24, \quad h = 0.5, \ l_x = 30, \ l_y = 30, \ H = 1.4$ are known.

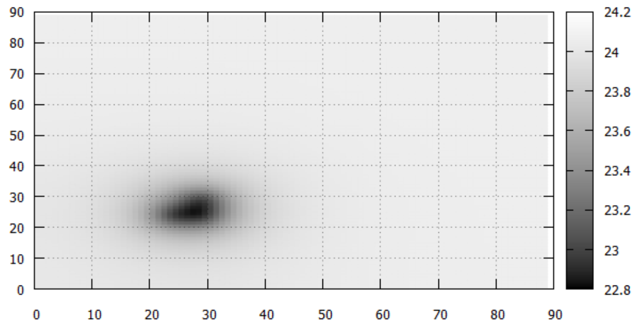

Fig. 5.1. The initial data of the inverse problem (initial thermogram)

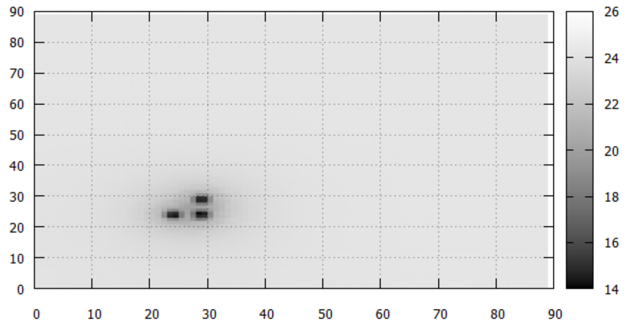

Fig. 5.2. The result of restoring the thermogram $u|_{z=H}$

To solve the inverse problem (2.7), we use the formulas (4.36), (4.35), (4.34), (4.31). In the formula (4.31) we use the representation for the fundamental solution

$$\varphi(M,P) = \frac{2}{l_x l_y} \sum_{n,m=1}^{\infty} \frac{e^{-k_{nm}|z_M - z_P|}}{k_{nm}} \sin \frac{\pi n x_M}{l_x} \sin \frac{\pi m y_M}{l_y} \sin \frac{\pi n x_P}{l_x} \sin \frac{\pi m y_P}{l_y} \tag{5.44}$$

when $z_M = a$, $z_P = 0$. The Fourier coefficients in the formula (4.34) are calculated without calculating the function Φ, similarly to [8]. When using the formula (4.34), integration is performed under the sign of the integral in (4.31) and under the sign of the sum in (5.44). Taking into account the orthogonality of the system of functions (3.25), the calculation formulas for calculating the Fourier coefficients Φ_{nm} are significantly simplified.

To obtain a numerical result, the problems (2.3), (2.7) are discretized. A uniform grid of 91x91 points is introduced on the rectangles $\Pi(a), a = -0.5$ and $\Pi(H)$. The Hamming algorithm [9, p.83] is used to sum discrete Fourier series.

The calculation results are shown in Fig.5.1 and Fig.5.2. Fig.5.1 shows the initial data of the inverse problem – the function f calculated from the discrete analog of the formula (5.43). The relative magnitude of the added error is 0.28%. The three sources are perceived as a single whole. Fig.5.2 shows the result of restoring the $u|_{z=H}$ function using the formulas (4.36), (4.35), (4.34), (4.31). Three sources are clearly visible. Regularization parameter $\alpha = 10^{-8}$. With the regularization parameter $\alpha = 0$, the solution is destroyed.

6. CONCLUSION

The inverse problem (2.7) and its stable solution can be used for mathematical processing of thermograms, in particular, in medicine [1], in order to correct the image. As already mentioned, a thermogram obtained using a thermal imager transmits an image of the structure of heat sources inside the body approximately. Refinement of the image on the thermogram can be performed within the framework of the problem (2.7). In this case, the f function will be associated with the original thermogram, and the u_H function will be considered as the result of processing the thermogram. Since the function $u|_{z=H}$ represents the temperature distribution on a plane closer to the heat sources under study than the original surface S, we can expect a more accurate reproduction of the source image on the calculated thermogram $u|_{z=H}$. The results of calculations, performed on the model example, show the effectiveness of the proposed method and algorithm based on the formulas (4.36), (4.35), (4.34), (4.31), which can be used for processing thermographic images.

ACKNOWLEDGEMENTS

The research is supported by the Russian Science Foundation, project N 21-11-00064.

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