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On Uniform Convergence Property of Solutions for Periodic Differential Inclusions with Asymptotically Stable Sets

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Abstract: The paper considers a periodic differential inclusion with an asymptotically stable set. The uniform character of convergence of solutions to an asymptotically stable set is established. An exponential estimate is obtained for solutions of a periodic differential inclusion homogeneous in state vector. Examples of control systems leading to consideration of periodic differential inclusions are given. These results can find applications in the stability analysis of control systems with periodic parameters, in particular, servomechanisms whose elements operate on AC, control systems with pulse amplitude modulation, and systems used to solve problems related to investigating vibrations of milling machines.

Keywords: time-invariant differential inclusion, periodic differential inclusion, homogeneous differential inclusion, asymptotically stable set, control system

1. INTRODUCTION

Studying control systems has led to the use of differential inclusions theory. Consider a control system

$$\dot{x} = f(t, x, u), \tag{1.1}$$

where $x \in R^n$, $\dot{x} = dx/dt$ is the velocity vector, t is time, and $u = u(t)$ is the control on which the next constraint is imposed

$$u(t) \in U, \tag{1.2}$$

U is an arbitrary set, $U \subset R^n$.

Under fairly general assumptions control system (1.1) with constraint (1.2) is equivalent to the differential inclusion

$$\dot{x} \in F(t, x), \tag{1.3}$$

where $F(t, x)$ denotes a multivalued mapping, i.e., a function that assigns a set $F(t,x) \subset R^n$ to each time t and each point $x \in R^n$ in the state space. Differential inclusion (1.3) can be adopted not only for a control system with given control constraints but also for other objects. For example, such objects include systems of differential inequalities, implicit differential equations, control systems with state-space constraints, systems with variable structure and with sliding modes, and differential equations with discontinuous right-hand sides.

Theory of differential inclusions is well developed and presented in systematic form in [1,4,17]. The use of differential inclusions in control systems theory is covered in the monograph [6]. The examples leading to differential inclusions are given below in Section 4.

In some cases, e.g., such as the problem of absolute stability, the study of linear nonstationary systems, the matrix of the right-hand side of which satisfies interval constraints, and the stability analysis of control systems that contain elements with incomplete information linear-selectionable differential inclusions can be used. A linear-selectionable inclusion is an inclusion of the form

$$\dot{x} \in F(t,x),\ \ F(t,x) = \{y : y = A(t)x,\ A(t) \in \Psi(t)\}, \tag{1.4}$$

where $x, y \in R^n$ and $\Psi(t)$ is a set in the space of $n \times n$ matrices. An inclusion of the form (1.4) is called a linear-selectionable inclusion, because the multivalued mapping $F(t,x)$ in (1.4) is the union of linear single-valued mappings (selectors) $A(t)x,\ A(t) \in \Psi(t)$. In the case of a time-invariant linear-selectionable inclusion, the right-hand side F of this inclusion – the matrix A and the set Ψ – are time-invariant. A linear-selectionable inclusion is said to be asymptotically stable if its trivial solution $x \equiv 0$ is asymptotically stable.

Time-invariant linear-selectionable inclusions are studied in a number of publications. For time-invariant linear-selectionable inclusions for which the set Ψ is a compact or convex polyhedron, necessary and sufficient conditions of the zero solution asymptotic stability were obtained in [12,13] on the base of Lyapunov functions method. The papers [7,12,13] give various algebraic criteria for the asymptotic stability of time-invariant linear-selectionable inclusions.

The publications on time-periodic (in short, periodic) differential inclusions (e.g., [2,8,9]) were mostly devoted to the existence of periodic solutions. Few investigations were focused on the analysis of solutions of periodic differential inclusions and their properties. For example, the weak asymptotical and weak exponential stability of an equilibrium of a periodic differential inclusion were studied in [5,18]. In accordance with the definitions introduced therein, an equilibrium of a given periodic differential inclusion is weakly asymptotically (weakly exponentially) stable if there exists at least a single solution satisfying the standard definitions conditions of the asymptotical (exponential) stability for a differential inclusion equilibrium. The method consists in the design of a first-approximation inclusion and further analysis of the properties of its solutions. In addition, the theorems of the weak asymptotical (weak exponential) stability of an equilibrium of an original inclusion using the corresponding properties of an equilibrium of its first approximation-inclusion were established. It was demonstrated that the proposed method can be used to study the weak asymptotical (weak exponential) stability of the inclusions equivalent to control systems.

The problems of absolute and robust stability of control systems with periodic variable parameters were solved in [10,11,14,15]. In particular, control systems with periodic parameters under consideration were proved to be equivalent to a periodic differential inclusion in the sense of the coincidence of the sets of absolutely continuous solutions. As was demonstrated in [16], in some cases solutions of periodic differential inclusions with the asymptotically stable trivial solution have the same properties as solutions of autonomous differential inclusions.

This paper continues the research of [16]. It considers periodic differential inclusions with an asymptotically stable set. The remainder of this paper is structured as follows. In Section 1 we consider periodic differential inclusion of general form and give preliminary remarks. The definition of an asymptotically stable set is also given. In Section 2 the uniform character of convergence of solutions to an asymptotically stable set is established. For solutions of periodic differential inclusion that is homogeneous in state vector we derive an exponential estimate. In Section 4 examples of control systems leading to the periodic differential inclusions are given. In the final section we offer concluding remarks.

2. STATEMENT OF THE PROBLEM

Consider the dynamic systems described by periodic differential inclusion

$$\dot{x} \in F(t,x), \quad F(t,x) \equiv F(t+T,x), \ \ t \geq 0, \ x \in R^n, \ \mathrm{T} = \mathrm{const}, \ \ \mathrm{T} > 0. \qquad (2.1)$$

Everywhere below we assume that in some domain $G = \{0 \leq t \leq T, \ x \in G_R, \ G_R = \{x_0 : \|x_0\| \leq R\}\}$, the multivalued function $F(t,x)$ satisfies the main conditions [4, p. 60]; i.e., for all $(t,x) \in G$ the set $F(t,x) \subset R^n$ is nonempty, bounded, closed, and convex, and the function $F(t,x)$ is upper semicontinuous [4, p. 52] with respect to (t,x).

A solution of inclusion (2.1) is understood as an absolutely continuous vector function $x(t)$ defined on an open or closed interval I and satisfying (2.1) almost everywhere on I.

By virtue of periodicity of the multivalued function $F(t,x)$ in t, when studying the properties of solutions $x(t,t_0,x_0)$ of inclusion (2.1), we can assume without loss of generality that $t_0 \in [0,T]$.

By the definition of solutions and the periodicity of the right-hand side of (2.1) in t the solutions of this inclusion have two properties as follows. If a function $x(t)$ is a solution of inclusion (2.1) (for $\alpha < t < \beta$), then

1) the function $x(t+kT)$, where $\alpha - kT < t < \beta - kT$ and k denotes any integer, is also a solution of inclusion (2.1); moreover, the solutions $x(t)$ and $x(t+kT)$ have the same trajectory;

2) for any $t_0 \in [0,T]$, t_1, and t such that $t_0 \leq t_1 \leq t$, the equality $x(t,t_1,x(t_1)) = x(t,t_0,x_0)$ holds, where $x(t_1) = x(t_1,t_0,x_0)$.

Let $a \in R^n$, $b \in R^n$ be points (vectors) with coordinates a_i and b_i respectively, $i = \overline{1,n}$, and also let $B \subset R^n$ be a set. The distance ρ between two points or between a point and a set is interpreted as the nonnegative values

$$\rho(a,b) = \|a-b\| = \left(\sum_{i=1}^{n} (a_i - b_i)^2 \right)^{1/2}, \qquad \rho(a,B) = \inf_{b \in B} (a,b).$$

As is well-known, the function $\rho(x,B)$ is uniformly continuous and for any points $x \in R^n$ and $y \in R^n$

$$|\rho(\mathrm{x},B) - \rho(\mathrm{y},B)| \leq \rho(x,\mathrm{y}).$$

A closed ε-neighborhood M^{ε} of a set M^{ε} is a set of such points x that $\rho(x,\mathrm{M}) \leq \varepsilon$. Let $M^{\varepsilon_0} \subset G$, $\varepsilon_0 > 0$.

Definition 2.1: *A set M is asymptotically stable for inclusion (2.1) if for any $\varepsilon > 0$ there exists a value $\delta(\varepsilon) > 0$ such that, for each x_0 satisfying the inequality $\rho(x_0,M) \leq \delta(\varepsilon)$, there exists a solution with the initial condition $x(t_0) = x_0$ and all solutions with the above-mentioned property are extendable on the interval $t_0 \leq t < \infty$ and also satisfy the conditions*

$$\rho(x(t),M) \leq \varepsilon \ \ \textit{for} \ t_0 \leq t < \infty \ \textit{and} \ \rho(x(t),M) \to 0 \ \ \textit{as} \ t \to \infty.$$

The problem is to study solutions of inclusion (2.1) with an asymptotically stable set M.

3. RESULTS

Theorem 3.1: *If inclusion (2.1) has an asymptotically stable set M, then there exists a value $\delta_0 > 0$ such that all solutions $x(t,t_0,x_0)$ of inclusion (2.1) satisfy the condition*

$$\rho(x(t), M) \to 0 \quad as \quad t \to \infty \tag{3.1}$$

uniformly with respect to (t_0, x_0) *for any* $t_0 \in [0, T]$ *and* $x_0 \in M^{\delta_0}$.

Proof. First, let us show that there exists a value $\delta_0 > 0$ such that all solutions of inclusion (2.1) with initial condition $x(t_0) = x_0$, $x_0 \in M^{\delta_0}$, satisfy condition (3.1) uniformly with respect to x_0 for any given $t_0 \in [0, T]$.

Suppose the contrary. Then for any $\delta > 0$, there exists $\eta(\delta) > 0$, $t_0 \in [0, T]$, a sequence of solutions $x_k(t)$ of inclusion (2.1), a numerical sequence $t_k \geq t_0 + kT$, and a sequence of vectors x_0^k, $k = 1,2,...,$ such that $x_0^k \in M^{\delta}$ and

$$\rho(x_k(t_k, t_0, x_0^k), M) > \eta(\delta) > 0, \; k = 1,2,..., \text{ as } \mathrm{t_k} \to \infty.$$

Since the set M is asymptotically stable, it follows that for any $\varepsilon > 0$ we can choose a value δ small enough to ensure that all solutions with $x(t_0) \in M^{\delta}$ satisfy the relations

$$\rho(x(t, t_0, x(t_0)), M) \leq \varepsilon \;\; (t_0 \leq t < \infty), \qquad \rho(x(t), M) \to 0 \;\; (\mathrm{t} \to \infty). \tag{3.2}$$

Consequently, there exists a number $\mu > 0$ such that all solutions of inclusion (2.1) with $x(t_0) \in M^{\mu}$ satisfy the inequality

$$\rho(x(t, t_0, x(t_0)), M) \leq \eta(\delta) \quad (t_0 \leq t < \infty). \tag{3.3}$$

Let us show that all $x_k(t, t_0, x_0^k)$ satisfy the inequalities

$$x_0^k \in M^{\delta}, \; \mu \leq \rho(x_k(t_0 + mT, t_0, \mathrm{x}_0^{\mathrm{k}}), M) \leq \varepsilon, \quad \mathrm{m} = \overline{1, \mathrm{k}}, \;\; \mathrm{k} = 1,2,... \tag{3.4}$$

Indeed, otherwise there would exist $\tilde{k}$ and $m(\tilde{k})$ $(1 \leq m(\tilde{k}) \leq \tilde{k}$) such that

$$\rho(x_{\tilde{k}}(t_0 + m(\tilde{k})T, t_0, \mathrm{x}_0^{\tilde{\mathrm{k}}}), M) < \mu, \qquad x_0^{\tilde{k}} \in M^{\delta}.$$

Then the solution

$$z(t) = z(t, t_0, x_{\tilde{k}}(t_0 + m(\tilde{k})T, t_0, x_0^{\tilde{k}})) = x_{\tilde{k}}(t + m(\tilde{k})T, t_0 + m(\tilde{k})T, x_{\tilde{k}}(t_0 + m(\tilde{k})T, t_0, x_0^{\tilde{k}})) =$$

$$= x_{\tilde{k}}(t + m(\tilde{k})T, t_0, x_0^{\tilde{k}}) \qquad (t_0 \leq t < \infty)$$

of inclusion (2.1) would satisfy the relations

$$\rho(z(t_0), M) = \rho(x_{\tilde{k}}(t_0 + m(\tilde{k})T, t_0, \mathrm{x}_0^{\tilde{\mathrm{k}}}), M) < \mu,$$

$$\rho(z(t_{\tilde{k}} - m(\tilde{\mathrm{k}})\mathrm{T}, t_0, x_{\tilde{k}}(t_0 + m(\tilde{k})T, t_0, x_0^{\tilde{k}})), M) = \rho(x_{\tilde{k}}(t_{\tilde{k}}, t_0, x_0^{\tilde{k}}), M) > \eta(\varepsilon),$$

which contradict (3.3).

The sequence of segments of solutions $x_k(t)$ contains a subsequence uniformly convergent for $t_0 \leq t \leq t_1$; in turn, the subsequence contains a new subsequence uniformly convergent for $t_0 \leq t \leq t_2$, and so on. Since $F(t, x)$ satisfies the basic assumptions, it follows that the limit function $x(t)$ is a solution of inclusion (2.1) [4, p. 60], which, together with (3.4), implies that $x(t_0) \in M^{\delta}$ and $\mu \leq \rho(x(t_0 + kT, t_0, x(t_0)), M) \leq \varepsilon$, $\mathrm{k} = 1,2,...,$ and we have arrived at a contradiction with (3.2). Therefore, the assertion started at the beginning of the proof of the theorem is valid.

Let us now prove the assertion of Theorem 2.1. Suppose that is fails. Then for any $\delta > 0$, there exist a number $\eta(\delta) > 0$, a sequence $x_k(t)$ of solutions of inclusion (2.1), a sequence of numbers t_0^k ($t_0^k \in [0,T]$ and $t_k \ge t_0^k + kT$), and a sequence of vectors x_0^k, $k = 1,2,...$, such that

$$x_0^k \in M^\delta, \quad \rho(x_k(t_k, t_0^k, x_0^k), M) > \eta(\delta) > 0, \; k = 1,2,...; \; t_k \to \infty. \tag{3.5}$$

Without loss of generality, we can assume that $\delta < \delta_0$, where δ_0 is a number for which the above-proved assertion holds. Since $x_0^k \in M^\delta$, $t_0^k \in [0,T]$, $k = 1,2,...$, it follows that the sequence $k = 1,2,...$ contains a subsequence $\{k'\}$ ($k' \to \infty$), such that the limits $\lim_{k' \to \infty} x_0^{k'} = x_0'$ ($x_0' \in M^\delta$) and $\lim_{k' \to \infty} t_0^{k'} = t_0'$ ($t_0' \in [0,T]$) simultaneously exist. To simplify the notation, we assume that $t_0^{k'} = t_0^k$, $x_0^{k'} = x_0^k$ $k' = k = 1,2,...$, and

$$\lim_{k \to \infty} x_0^k = x_0, \quad \lim_{k \to \infty} t_0^k = t_0. \tag{3.6}$$

Let $\delta_0 - \rho(x_0, \mathrm{M}) = \beta > 0$. By (3.6), there exist a K_1 such that

$$\rho(x_0^k, \mathrm{x}_0) < \beta / 2 \tag{3.7}$$

for all $k \ge K_1$

Since all solutions of inclusion (2.1) are equicontinuous [2, p. 61] in any closed bounded domain lying in G, it follows that they are equicontinuous in the domain $\{x \in M^{\delta_0}, \; t \in [0,T]\}$ as well. Therefore, for the point t_0, there exist a neighbourhood $S_\gamma(t_0) = \{t : |t - t_0| \le \gamma/2, \; t \in [0,T]\}$, $S_\gamma(t_0) = [a,b]$ ($\mathrm{a} = \mathrm{t}_0$ if $t_0 = 0$ and $\mathrm{b} = \mathrm{t}_0$ if $t_0 = T$), such that

$$\rho(x(b, \tilde{t}_0, x(\tilde{t}_0)), \mathrm{x}(\tilde{t}_0)) < \beta / 2. \tag{3.8}$$

for all solutions $x(t, \tilde{t}_0, x(\tilde{t}_0))$ of inclusion (2.1) with $\tilde{t}_0 \in [a,b]$ and $x(\tilde{t}_0) \in M^\delta$.

Then, taking into account relations (3.7), (3.8) and uniform continuity of the function $\varphi(x) = \rho(x, M)$ we obtain two relations

$$\rho(x_0^k, M) - \rho(x_0, M) \le |\rho(\mathrm{x}_0^k, M) - \rho(\mathrm{x}_0, M)| \le \rho(x_0^k, \mathrm{x}_0) < \beta/2,$$

$$\rho(x_k(b, t_0^k, x_0^k), M) - \rho(x_0^k, M) \le |\rho(\mathrm{x}_k(\mathrm{b}, \mathrm{t}_0^k, \mathrm{x}_0^k), M) - \rho(\mathrm{x}_0^k, M)| \le \rho(x_k(b, t_0^k, x_0^k), \mathrm{x}_0^k) < \beta/2$$

for all solutions $x_k(t)$ with $k \ge K_1$.

Adding the last two inequalities, we obtain

$$\rho(x_k(b, t_0^k, x_0^k), M) < \beta + \rho(x_0, M) = \delta_0, \quad k \ge K_1.$$

This, together with (3.5) and (3.6), means that there exist a K_2 such that the relations

$$t_0^k \le b \le T, \;\; \rho(x_k(b, t_0^k, x_0^k), M) < \delta_0, \;\; \rho(x_k(t_k, t_0^k, x_0^k), M) > \eta(\delta) \tag{3.9}$$

are simultaneously valid for all $k \ge K_2$.

Taking into account relations (3.9), from the sequence of solutions $x_k(t)$ of inclusion (2.1) satisfying inequalities (3.5) for $k \ge K_2$, we pass to the sequence of solutions $y_k(t) = x_k(t, t_0^k, x_0^k)$ defined by the relations

$$y_k(b) = y_0^k = x_k(b), \quad \rho(y_0^k, M) < \delta_0, \quad \rho(y_k(t_k, b, \mathrm{y}_0^k), M) > \eta(\delta), \quad \mathrm{t}_k \to \infty. \tag{3.10}$$

The existence of a sequence $y_k(t),\ k = K_2,\ K_2+1,\ K_2+2,...\ (b \le t < \infty,\ \ 0 < b \le T),$ satisfying conditions (3.10) contradicts the already known fact that relation (3.1) is valid uniformly with respect to x_0 (for $x_0 \in M^{\delta_0}$ and $t_0 \in [0,T]$) for all solutions of inclusion (2.1). The proof of Theorem 3.1 is complete. □

In what follows, we consider differential inclusions homogeneous with respect to $x \in R^n$. If B is a set in R^n and c is a number, then cB stands for a set of points of the form cx for all $x \in B.$

Definition 3.2: *A multivalued function $F(t,x)$ is homogeneous (of degree one) in x if $F(t,cx) \equiv cF(t,x)$ for all $c > 0$.*

Definition 3.3: *A differential inclusion*

$$\dot{x} \in F(t,x) \qquad (F(t,cx) \equiv cF(t,x),\ \ c > 0) \tag{3.11}$$

is homogeneous in x.

Homogeneous differential inclusion (3.11) is preserved under the substitution $x = cx_1$ with arbitrary $c > 0.$ It means that if a function $x = \varphi(t)$ is a solution of inclusion (3.11), then the function $x = c\varphi(t)$ with arbitrary $c > 0$ is also a solution.

Let us consider the differential inclusion

$$\dot{x} \in F(t,x), \quad (t \ge 0,\ x \in R^n), \quad F(t,x) \equiv F(t+T,x),\ (\mathrm{T} = \mathrm{const},\ \ \mathrm{T} > 0),$$

$$F(t,cx) \equiv cF(t,x) \ \ (c > 0), \tag{3.12}$$

periodic in t and homogeneous in $x.$ Since inclusion (3.12) is homogeneous in $x,$ we have the following assertion.

Corollary 3.1: *If a bounded set M is asymptotically stable for inclusion (3.12), then all solutions $x(t,t_0,x(t_0))$ of inclusion (3.12) with the initial conditions $t_0 \in [0,T]$* and $x_0 \in G_R$ *(where R* is an arbitrary *positive number) satisfy condition (3.1) uniformly with respect to (t_0,x_0).*

For solutions of periodic homogeneous differential inclusion (3.12) with an asymptotically stable set M an exponential estimate is valid.

Theorem 3.2: *If a bounded set M is asymptotically stable for inclusion (3.12), then there exist numbers $c_0 > 0,\ c_1 > 0$ such that any solution $x(t,t_0,x_0)$ of inclusion (3.12) satisfies the estimate*

$$\rho(x(t,t_0,\mathrm{x}_0),M) \le c_0\|x_0\|\exp(-c_1 t) \quad (\mathrm{t}_0 \le t < \infty). \tag{3.13}$$

for any t_0 and $t \ge t_0$.

Proof. By virtue of corollary 3.1, for some $\delta > 0,$ there exists a $\tau > 0$ (independent of (t_0,x_0)) such that $\tau = \tilde{k}T$ ($\tilde{k}$ is some positive integer) and $\rho(x(t,t_0,\mathrm{x}_0),M) \le \delta/2,$ $\tau + \mathrm{t}_0 \le t < \infty$, for all solutions $x(t,t_0,x_0)$ with the initial condition $\|x_0\| \le \delta.$ By virtue of Theorem 3 in [4, p. 62] the set of solutions of inclusion (3.12) is compact on the closed interval $[t_0,\mathrm{t}_0+\tau]$ in the metric of $\mathbf{C}[t_0,\mathrm{t}_0+\tau]$, whence it follows that $\rho(x(t,t_0,\mathrm{x}_0),M) \le c_2\delta\ \ (c_2 > 0)$ for $t_0 \le t \le \mathrm{t}_0 + \tau.$

If $x(t,t_0,x_0)$ is a solution with an arbitrary $x_0 \ne 0$ and $c = \delta/\|x_0\|,$ then the function $y(t,t_0,y_0) = cx(t,t_0,x_0)$ is also a solution and $\|y_0\| = \delta,$ whence it follows that

$\rho(y(t,t_0,y_0),M)\le\delta/2 \quad (\tau+t_0\le t<\infty)$ and $\rho(y(t,t_0,y_0),M)\le c_2\delta \quad (t_0\le t<t_0+\tau).$

Returning from $y(t,t_0,y_0)$ to $x(t,t_0,x_0)$, we obtain

$\rho(x(t,t_0,x_0),M)\le c_2\|x_0\| \quad (t_0\le t<t_0+\tau), \quad \rho(x(t,t_0,x_0),M)\le\|x_0\|/2 \quad (\tau+t_0\le t<\infty).$
(3.14)

for any t_0 and x_0.

Since after replacement of t by $t+kT$ (k is an arbitrary integer), a solution of inclusion (3.12) remains a solution, it follows from (3.14) that a solution $x(t,t_0,x_0)$ with an arbitrary x_0 satisfies the relations

$$\rho(x(t,t_0,x_0),M)\le 2^{-i}\|x_0\| \ (t_i\le t<\infty), \tag{3.15}$$

where $t_i=t_0+i\tau,\ i=1,2,...$

For any $t\ge t_0$, we choose an i such that $t_{i-1}\le t<t_i$. Then $t<t_0+i\tau$ and $i>(t-t_0)/\tau$. Therefore, the inequality

$$2^{-i}=\exp(-i\ln 2)<\exp(-\ln 2(t-t_0)/\tau)=\exp(t_0\ln 2/\tau)\exp(-t\ln 2/\tau)$$

is fulfilled. The last inequality, together with (3.15), implies (3.13) with $c_0=\exp(t_0\ln 2/\tau)$ and $c_1=\ln 2/\tau$. The proof of Theorem 3.2 is complete. □

4. EXAPMLES

Example 4.1:

In the paper [13] the nonlinear control system is considered

$$\dot{x}=Ax+\sum_{j=1}^{m}b^j\varphi_j(\sigma_j,t),\ \ \sigma_j=\langle c^j,x\rangle=\sum_{i=1}^{n}c_i^jx_i,\ \ \varphi_j(0,t),\equiv 0,\ \ j=\overline{1,m}, \tag{4.1}$$

where $x=(x_i)_{i=1}^n, x\in R^n$ is n-dimensional vector, characterizing the deviation of the system from the mode prescribed by control objective (the zero solution $x(t)\equiv o$ of system (4.1) corresponds to this mode), A is a constant square matrix of order n, and b^j and c^j $(j=\overline{1,m})$ are constant n-dimensional vectors. The brackets $\langle\cdot,\cdot\rangle$ denote the scalar product. It is also assumed that nonlinear functions $\varphi_j(\sigma_j,t),\ j=\overline{1,m}$, that define the characteristics of nonlinear elements, satisfy conditions for the existence of an absolutely continuous solution of system (4.1) for any initial conditions and inequalities

$$k_{1j}\sigma_j^2\le\varphi_j(\sigma_j,t)\sigma_j\le k_{2j}\sigma_j^2 \quad \left(-\infty<k_{1j}\le k_{2j}<+\infty,\ \ j=\overline{1,m}\right)$$

for all σ_j and $t\ge 0$. System (4.1) is equivalent, see the paper [13], to time-invariant linear-selectionable inclusion (1.4), where Ψ is a compact set in the n^2-dimensional matrix space. Here and in the following examples, equivalence is understood in the sense of solutions sets identity for the system under consideration and the corresponding inclusion under the same initial conditions.

Example 4.2:

In the paper [10] the linear nonstationary control systems

$$\dot{x}=\sum_{k=1}^{m}\lambda_k(t)A_k(t)x, \quad \lambda_k(t)\ge 0, \quad \sum_{k=1}^{m}\lambda_k(t)=1 \tag{4.2}$$

are considered, where $x=(x_i)_{i=1}^m \in R^n$ is n -dimensional state vector of the system, $A_k(t)=(\mathrm{a}_{ij}^k(t))_{i,j=1}^n$ are given continuous periodic matrices of the period $T>0$, $A_k(t+T)=A_k(t)$, $k=\overline{1,m}$, and $\lambda_k(t)$ ($k=\overline{1,m}$) are bounded measurable functions.

System (4.2) is equivalent to the periodic differential inclusion

$$\dot{x}\in F(t,x), \qquad F(t,x)\equiv F(t+T,x),$$

$$F(t,x)=\{y: y=\sum_{k=1}^{m}\lambda_k(t)A_k(t),\ \lambda_k(t)\geq 0,\ \sum_{k=1}^{m}\lambda_k(t)=1\}. \tag{4.3}$$

Example 4.3: In the paper [14] the linear nonstationary control system

$$\dot{x}=A(t)x, \quad x\in R^n \tag{4.4}$$

is considered, where $A(t)=(a_{ij}(t))_{i,j=1}^n$ is an arbitrary matrix ($a_{ij}(t)$, $i,j=\overline{1,n}$, are measurable functions, generally speaking, not periodic). System (4.4) almost everywhere satisfies the inequalities

$$\underline{A}(\mathrm{t})\leq A(\mathrm{t})\leq \overline{A}(\mathrm{t}), \quad \underline{A}(\mathrm{t})=(\underline{a}_{ij}(t))_{n,\mathrm{j}=1}^n, \quad \overline{A}(t)=(\overline{a}_{ij}(t))_{i,j=1}^n \tag{4.5}$$

on any finite interval of the semi-axis $[0,\infty)$. Matrix inequalities (4.5) are understood elementwise, that is $\underline{a}_{\mathrm{ij}}(\mathrm{t})\leq a_{ij}(t)\leq \overline{a}_{\mathrm{ij}}(\mathrm{t}),\ i,j=\overline{1,n},$, where $\mathrm{a}_{ij}(t), \underline{a}_{\mathrm{ij}}(\mathrm{t}), \overline{a}_{\mathrm{ij}}(\mathrm{t}), i,j=\overline{1,n}$, are arbitrary measurable functions. It is assumed that the given bounded "extreme" matrices $\underline{A}(\mathrm{t})$ and $\overline{A}(\mathrm{t})$ are periodic with a period $\mathrm{T}>0$, i.e. conditions are valid

$$\underline{A}(t+T)\equiv \underline{A}(\mathrm{t}), \quad \overline{A}(t+T)\equiv \overline{A}(t).$$

Thus, by virtue of (4.5), not one fixed system (4.4) is considered, but a set of linear nonstationary systems (4.4) with periodic interval constraints (4.5).

A series of examples leading to systems (4.2) and (4.4) with constraints (4.5) were considered in [19]. Among such systems, note tracking systems with AC elements, control systems with pulse amplitude modulation and also the systems arising in the vibration analysis of milling machines.

If the condition $A(t+T)=A(t)$ is additionally satisfied, the set of linear nonstationary systems (4.4) with periodic interval constraints (4.5) is equivalent to the linear-selectionable periodic inclusion

$$\dot{x}\in F(t,x), \quad F(t,x)=\{y: y=A(t)x,\ A(t)\in\Omega(t)\},$$

$\Omega(t)=\{A(t):\ A(t)=\lambda_1(t)\underline{A}(t)+\lambda_2(t)\overline{A}(t)\}$, $\Omega(t+T)=\Omega(t)$, where arbitrary bounded and measurable functions $\lambda_k(t),\ k=\overline{1,2}$ satisfy the conditions $\lambda_k(t)\geq 0,\ \sum_{k=1}^{2}\lambda_k(t)=1,$ $\lambda_k(t+T)=\lambda_k(t),\ t\geq t_0.$

5. CONCLUSION

The paper considers solutions of periodic differential inclusions with asymptotically stable sets. Theorem 3.1 establishes the uniform character of convergence to an asymptotically stable set. Homogeneous in state vector periodic differential inclusions are also considered. Corollary 3.1 is obtained for such inclusions. Theorem 3.2 gives the exponential estimate for solutions. Theorem 3.1 is a generalization of lemma 2, proved for autonomous differential inclusions [3]. Theorem 3.2 generalizes the estimate well known for solutions of an autonomous homogeneous (first degree) differential inclusion [4]. The examples of control systems leading to consideration of periodic differential inclusions are given.

The results obtained can find applications in the stability analysis of control systems with periodic parameters, in particular, servomechanisms whose elements operate on AC, control systems with pulse amplitude modulation, and systems used to solve problems related to investigating vibrations of milling machines.

Further research into the inclusions considered in the present paper can be related to producing weak asymptotic and weak exponential stability conditions. In addition, it seems interesting to distinguish the classes of Lyapunov functions establishing necessary and sufficient conditions for the asymptotic stability of periodic differential and difference inclusions.

REFERENCES

1. Aubin, J.-P., & Cellina, A. (1983). *Differential Inclusions.* Berlin Heidelberg: Springer-Verlag.
2. Filippakis, M., Michael E., & Papageorgiou, Nikolaos S. (2006). Periodic solutions for differential inclusions in R^N, *Archivum Mathematicum.*, 42(2), 115–123.
3. Filippov, A. F. (1979). Ustoychivost' dlia differentsial'nyh uravneniy c razryvnymi i mnogoznachnymi pravymi chastiami [Stability for differential equations with discontinuous and multivalued right-hand sides], *Differets. Uravn.,* 15(6), 1018-1027, [in Russian].
4. Filippov, A. F. (1985). *Differentsial'nye uravneniya s razryvnoi pravoi chast'yu* [Differential equations with discontinuous right-hand side]. Moscow, USSR: Nauka, [in Russian].
5. Gama, R. & Smirnov, G.V. (2013). Weak exponential stability for time-periodic differential inclusions via first approximation averaging, *Set-valued and variational analysis*, 21(2), 191-200, https://doi.org/10.1007/s11228-012-0216-1
6. Han, Z., Cai, X., Huang, J. (2016). *Theory of Control Systems Described by Differential Inclusions.* Berlin Heidelberg: Springer.
7. Ivanov, G.G., Alferov, G.V., & Efimova, P.A. (2017). Ustoychivost' selektorno-lineynyh differentsial'nyh vklucheniy [Stability of selector-linear differential inclusions]. *Vestnik Permskogo Universiteta, Matematika, Mekhanika, Informatika.* 2(37), 25-30, [in Russian], https://doi.org/10.17072/1993-0550-2017-2-25-30
8. Jack W., Macki, Paolo, Nistri. & Pietro, Zecca (1988) The existence of periodic solutions to non-autonomous differential inclusions, *Proc. of the Amer. Math. Soc.,* 104(3), 840-844, https://doi.org/10.2307/2046803
9. Li, G.&Xue, X. (2002). On the existence of periodic solutions for differential inclusions, *J. Math. Anal. Appl.,* 276(1), 168–183.
10. Molchanov, A. P. & Morozov, M. V. (1997). Algoritmy analiza robastnoy ustoychivosti lineynih nestatsionarnyh system upravleniya c periodicheskimi ogranicheniyami [Algorithms for robust stability analysis of linear time-varying control systems with periodic constraints]. *Automation and Remote Control,* 58(5(2)), 795-804, [in Russian].
11. Molchanov, A. P. & Morozov, M. V. (1992). Absolutnaya ustoychivost' nelineynih nestatsionarnyh system upravleniya c periodicheskoy lineynoy chast'u [Absolute stability of nonlinear nonstationary control systems with periodic linear sections]. *Automation and Remote Control,* 53(2(1)), 189-198, [in Russian].
12. Molchanov, A. P. & Pyatnitskii, E.S. (1989). Criteria of asymptotic stability of differential and difference inclusions encountered in control theory, *Systems & Control Letters.,* 13, 59–64.
13. Molchanov, A. P. & Pyatnitskii, E.S. (1986). Funktsii Lyapunova opredelyaushie neobhodimye I dostatochnie usloniya absolutnoy ustoychivosti nelineynyh nestatsionarnyh system upravlenia. [Lyapunov functions defining the necessary and sufficient conditions for absolute stability of the nonlinear nonstationary control systems.]. *Avtom. & Telemekh.,* 3, 63–73, [in Russian].

14. Morozov, M. V. (2016). Kriterii robastnoy ustoychivosti nestatsionarnih system interval'nimi ogranicheniyami [Robust Stability Criteria for Nonstationary Systems with Interval Constraints]. *Proceedings of the Institute for Systems Analysis*, 66, 4, 4-9 [in Russian].
15. Morozov, M.V. (2014). Kriterii robastnoy absolutnoy ustoychivosti diskretnyh system upravleniya s periodicheskimi ogranicheniyami [Criteria of robust absolute stability for discrete control systems with periodic constraints]. *Proceedings of Institute for Systems Analysis*, 64, 2, 13-18. [in Russian].
16. Morozov, M. V. (2000). Svoystva resheniy periodicheskih differentsial'nyh vklucheniy [Properies of solutions of periodic differential inclusions]. *Differets. Uravn.*, 36(5), 677-682. [in Russian], https://doi.org/10.1007/BF02754225
17. Smirnov, G. V. (2002). *Introduction to the Theory of differential Inclusions.* Providence, Rhode Island, USA: Amer. Math. Soc. Graduate Studies in Mathematics, 41.
18. Smirnov, G. V. (1995). Weak asymptotic stability at first approximation for periodic differential inclusions, *Nonlinear differential equations and applications.*, 2(4), 445-461, https://doi.org/10.1007/BF01210619
19. Shil'man, S.V. (1978). *Metod proizvodiatshikh funktsiy v teorii dinamitcheskikh sistem.* [The method of generating functions in the theory of dynamic systems]. Moscow, USSR: Nauka, [in Russian].