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On Some Global Properties of Multivalued Simple Waves

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Abstract: The Cauchy problem for one-dimensional quasilinear wave equation is considered. In the case that its solutions are multivalued simple waves, we derive explicit expressions in quadratures by introducing global characteristic coordinates. The main result of the paper is a theorem on some global properties of multivalued simple waves.

Keywords: hyperbolic quasilinear wave equation; multivalued solution; Cauchy problem; simple wave

INTRODUCTION

A number of models in continuum mechanics concern the wave equation

$$z_{xx} - g^2(z_y)z_{yy} = 0, \tag{0.1}$$

where $g = g(q)$ is a given positive coefficient,

$$g(q) > 0, \tag{0.2}$$

cf. [1], [2], and [3], and the Monge notations

$$p = z_x(x, y), \;\; q = z_y(x, y) \tag{0.3}$$

are used. The variable x designates time, y is the spatial coordinate, and z is the displacement of continuum. In particular, vibration of a string, cf. [3], [1]; wave propagation in a bar of elastic-plastic material, cf. [4]; and isentropic flows of a compressible gas with plane symmetry, cf. [4], [5], are described by equations of the type (0.1).

Equation (0.1) is a quasilinear second-order partial differential equation with two independent variables x and y and unknown function

$$z = z(x, y). \tag{0.4}$$

Due to condition (0.2) it is *hyperbolic*.

It is common knowledge what is a *classical* solution of equation (0.1). Classical solutions have a rather serious drawback: in finite time singularities, a so called *gradient catastrophes*, can develop in them, cf. [2] and [1]. The latter means that there is a point (x, y) such that in some its vicinity a classical solution z (0.4) itself and its first derivatives p and q (0.3) are bounded but at least one of second derivatives is unbounded. This is a motivation and reason to generalize a notion of classical solution and define the notion of a multivalued solution, which is a relatively well-known one, cf. [6], [7], [8], and [9].

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1. MULTIVALUED SOLUTIONS

The differential 2-form

$$\omega_2 = dp \wedge dy - g(q)dx \wedge dq, \tag{1.5}$$

is in an obvious way associated with the left hand side of the equation (0.1), cf. [8], and the linear differential form and its exterior derivative

$$\begin{aligned} &\omega_0 = dz - pdx - qdy, \\ &\omega_1 = d\omega_0 = dx \wedge dp + dy \wedge dq \end{aligned} \tag{1.6}$$

are associated with equalities (0.3). An immersion

$$\sigma : S \longrightarrow \mathbb{R}^5 \tag{1.7}$$

of a two-dimensional Hausdorff paracompact manifold S is a *multivalued* solution of equation (0.1) if it satisfies the following system of exterior differential equations

$$\sigma^*\omega_0 = 0, \quad \sigma^*\omega_1 = 0, \quad \sigma^*\omega_2 = 0, \tag{1.8}$$

cf. [7] and [8]. Obviously, the graphic of a classical solution is a multivalued solution but not vice versa. And it can be proven that any classical solution is a part of *maximal* multivalued solution, cf. [10]. If a gradient catastrophe takes place at a point $s \in S$, then $(d\sigma^*x \wedge d\sigma^*y)(s) = 0$. Multivalued solutions have a serious advantage over classical solutions because gradient catastrophes do not happen to them; cf. [10].

The linear differential forms

$$\begin{aligned} &\omega_{j,1} = dp - (-1)^j g(q)dq, \\ &\omega_{j,2} = dy - (-1)^j g(q)dx \end{aligned} \tag{1.9}$$

will be called *characteristic*. It is not hard to see that the equalities

$$\begin{aligned} &\omega_2 - g(p)\omega_1 = \omega_{1,1} \wedge \omega_{1,2}, \\ &\omega_2 + g(p)\omega_1 = \omega_{2,1} \wedge \omega_{2,2}, \\ &\omega_2 = \frac{1}{2}(\omega_{1,1} \wedge \omega_{1,2} + \omega_{2,1} \wedge \omega_{2,2}), \\ &\omega_1 = \frac{1}{2g(p)}(\omega_{1,1} \wedge \omega_{1,2} - \omega_{2,1} \wedge \omega_{2,2}) \end{aligned}$$

are through for characteristic forms $\omega_{j,1}$, $\omega_{j,2}$ (1.9) and 2-forms ω_2 (1.5) and ω_1 (1.6). Thence, an immersion σ (1.7) is a multivalued solution of the equation (0.1) iff

$$\begin{aligned} &\sigma^*\omega_0 = 0, \\ &\sigma^*(\omega_{1,1} \wedge \omega_{1,2}) = 0, \\ &\sigma^*(\omega_{2,1} \wedge \omega_{2,2}) = 0. \end{aligned} \tag{1.10}$$

A curve

$$\gamma : \Gamma \longrightarrow \mathbb{R}^5,$$

where Γ is a one-dimensional connected manifold, will be called a *characteristic curve* of the equation (0.1) that belongs to the j-th family, $j = 1, 2$, if

$$\gamma^*\omega_0 = 0, \quad \gamma^*\omega_{j,1} = 0, \quad \gamma^*\omega_{j,2} = 0. \tag{1.11}$$

Suppose an immersion σ (1.7) is a multivalued solution of equation (0.1). Then equations (1.10) are through for this immersion. Hence the pull-backs $\sigma^*\omega_{j,1}$ and $\sigma^*\omega_{j,2}$ of the characteristic forms $\omega_{j,1}$ and $\omega_{j,1}$ (1.9) are linearly dependent and the linear algebraic equations

$$\sigma^*\omega_{j,1} = 0, \quad \sigma^*\omega_{j,2} = 0 \tag{1.12}$$

uniquely define the one-dimensional subbundle of the tangent bundle TS for $j = 1, 2$. Therefore, by the Frobenius theorem [11], for any point $s \in S$ and number $j = 1, 2$ there exists the maximal integral manifold

$$\gamma_{j,s} : \Gamma_{j,s} \longrightarrow S, \tag{1.13}$$

of the system

$$\gamma_{j,s}^*\omega_{j,1} = 0, \quad \gamma_{j,s}^*\omega_{j,2} = 0, \tag{1.14}$$

containing the point s, where $\Gamma_{j,s}$ is a connected one-dimensional manifold.

It follows from the equations (1.11), (1.12), and (1.14) that the composition $\sigma \circ \gamma_{j,s}$ of the maps (1.7) and (1.13) is a characteristic curve of the equation (0.1), belonging to the j-th family. Of cause, for classical solutions conventional characteristic curves are obtained this way.

2. CAUCHY PROBLEM

An *initial curve* or *initial value* for equation (0.1) is a smooth curve

$$l : (-\infty, +\infty) \longrightarrow \mathbb{R}^5 \tag{2.15}$$

that satisfies the conditions

$$\begin{aligned} &\omega_0(\dot{l}(\tau)) = 0, \\ &\omega_{1,1}^2(\dot{l}(\tau)) + \omega_{1,2}^2(\dot{l}(\tau)) \neq 0, \\ &\omega_{2,1}^2(\dot{l}(\tau)) + \omega_{2,2}^2(\dot{l}(\tau)) \neq 0 \end{aligned} \tag{2.16}$$

for $-\infty < \tau < +\infty$. The last two conditions mean that the initial curve l (2.15) is not characteristic, i.e. it is *free*, see (1.11).

A classical-type initial value problem at $x = 0$ for equation (0.1) is defined by conditions

$$z(0, y) = z_0(y), \quad z_x(0, y) = q_0(y), \tag{2.17}$$

where

$$\begin{aligned} z_0 &: (-\infty, +\infty) \longrightarrow \mathbb{R}, \\ q_0 &: (-\infty, +\infty) \longrightarrow \mathbb{R} \end{aligned} \tag{2.18}$$

are given functions. This initial values determine initial curve l (2.15) with the coordinates

$$\begin{aligned} &l^*x(\tau) = 0, \\ &l^*y(\tau) = \tau, \\ &l^*p(\tau) = p_0(\tau), \\ &l^*q(\tau) = q_0(\tau), \\ &l^*z(\tau) = z_0(\tau). \end{aligned} \tag{2.19}$$

Obviously, this curve is an immersion and meets conditions (2.16) since

$$\omega_{1,1}^2(\dot{l}(\tau)) = \omega_{2,2}^2(\dot{l}(\tau)) = 1$$

according to expressions (1.9).

If for a multivalued solution (1.7) of equation (0.1) there exists an imbedding

$$L : (-\infty, +\infty) \longrightarrow S \tag{2.20}$$

such that

$$l = \sigma \circ L, \tag{2.21}$$

then σ is called a *solution of Cauchy problem* (0.1), (2.15) and L – an *initial embedding*. Thus a solution of the Cauchy problem (0.1), (2.15) is a pair (σ, L) of an immersion σ (1.7), satisfying the system (1.10), and imbedding L (2.20), satisfying (2.21). A multivalued solution (σ, L) of the initial value problem (0.1), (2.15) is said to be *definite* if for any point $s \in S$ and number $j = 1, 2$ the intersection $\gamma_{j,s}(\Gamma_{j,s}) \cap L(-\infty, +\infty)$ of the image of initial imbedding L (2.20) and characteristic curve $\gamma_{j,s}$ (1.13) consists of exactly one point.

Theorem 2.1:
(Characteristic uniformization, [10]) Let (σ, L) be a definite solution (1.7), (2.20) *of the Cauchy problem* (0.1), (2.15)–(2.16). *Then there exists a unique diffeomorphism*

$$\Phi : S \longrightarrow \Phi(S) \tag{2.22}$$

such that the following three properties hold.

(a) The image $\Phi(S)$ is a subset of $\mathbb{R}^2$ and contains its diagonal

$$\delta = \{(\tau, \tau) \in \mathbb{R}^2 | -\infty < \tau < +\infty\}.$$

(b) For $-\infty < \tau < +\infty$ the initial imbedding L (2.20) *satisfies the equality*

$$\Phi \circ L(\tau) = (\tau, \tau).$$

(c) Images $\Phi \circ \gamma_{j,s}(\tau)$ of characteristic curves $\gamma_{j,s}$ (1.13)*, where $s \in S$, lie in coordinate straight lines $u = const$ if $j = 1$ and $v = const$ if $j = 2$ for $\tau \in \Gamma_{j,s}$.*

A diffeomorphism Φ (2.22), which is uniquely defined by Theorem 2.1, is called a *characteristic uniformization* of the definite solution (σ, L). The coordinate plane of the parameters u, v and the same coordinates in it are called *characteristic* as well. The image $\Phi(S)$ of the uniformization Φ (2.22) is a *uniformized domain* and the composition

$$\psi = \sigma \circ \Phi^{-1} \tag{2.23}$$

is a *uniformized solution*.

3. SIMPLE WAVES

A solution σ (1.7) is said to be a *simple wave*, if the equality

$$d\sigma^* p \wedge d\sigma^* q = 0 \tag{3.24}$$

holds for it, cf. [4]. It is possible to show that a definite solution (σ, L) of the Cauchy problem (0.1), (2.17) is a simple wave iff either

$$p_0(y) = -G(z_0'(y)), \tag{3.25}$$

or

$$p_0(y) = G(z_0'(y)), \tag{3.26}$$

where $G = G(q)$ is a primitive function of $g = g(q)$.

By definition (3.24), in case of simple waves linearization of equation (0.1) by means of hodograph transformation is not applicable. But it is possible to apply Theorem 2.1 and use characteristic uniformization Φ (2.22) in this case instead. Indeed, by Theorem 2.1, definition (1.13)–(1.14) of characteristic curve, and initial values (3.25) and (3.26) we get for coordinates x, y, p, q, and z of uniformized solution (2.23) the following equations

$$\begin{aligned}
&\frac{\partial}{\partial v} \lrcorner\, \psi^*\omega_{1,1} = p_v + g(p)q_v = 0,\\
&\frac{\partial}{\partial v} \lrcorner\, \psi^*\omega_{1,2} = y_v + g(p)x_v = 0,\\
&\frac{\partial}{\partial v} \lrcorner\, \psi^*\omega_0 = z_v - px_v - qy_v,\\
&\frac{\partial}{\partial u} \lrcorner\, \psi^*\omega_{2,1} = p_u - g(q)q_u = 0,\\
&\frac{\partial}{\partial u} \lrcorner\, \psi^*\omega_{2,2} = y_u - g(p)x_u = 0,\\
&\frac{\partial}{\partial u} \lrcorner\, \psi^*\omega_0 = z_u - px_u - qy_u
\end{aligned} \tag{3.27}$$

where the symbol $\lrcorner$ denotes the interior multiplication, see [11] (section 2.11), and the initial values

$$\begin{aligned}
&z_0(\tau,\tau) = z_0(\tau),\\
&q_0(\tau,\tau) = z_0'(\tau),\\
&p_0(\tau,\tau) = -G(z_0'(\tau))
\end{aligned} \tag{3.28}$$

in the case of (3.25) and

$$\begin{aligned}
&z_0(\tau,\tau) = z_0(\tau)),\\
&q_0(\tau,\tau) = z_0'(\tau)),\\
&p_0(\tau,\tau) = G(z_0'(\tau))
\end{aligned} \tag{3.29}$$

in the case of (3.26).

Integration of Cauchy problems (3.27), (3.28) and (3.27), (3.29) allows to represent characteristic uniformization (2.23) of a simple wave in quadratures.

Proposition 3.1:
In characteristic coordinates u and v the coordinates x, y, p, q, and z of uniformization (2.23) *of any definite multivalued solution (σ, L) of the Cauchy problem* (0.1)*,* (2.17) *that is a maximal simple wave are representated in the following way:*

$$\begin{aligned}
&x(u,v) = \frac{h(u,v)}{2\sqrt{g_0(v)}},\\
&y(u,v) = v + \frac{\sqrt{g_0(v)}}{2}h(u,v),\\
&z(u,v) = z_0(v) + \frac{p_0(v) + g_0(v)z_0'(v)}{2\sqrt{g_0(v)}}h(u,v),\\
&p(u,v) = p_0(v),\\
&q(u,v) = z_0'(v)
\end{aligned} \tag{3.30}$$

in case (3.25) *and*

$$\begin{aligned} x(u,v) &= \frac{h(u,v)}{2\sqrt{g_0(u)}},\\ y(u,v) &= u - \frac{\sqrt{g_0(u)}}{2}h(u,v),\\ z(u,v) &= z_0(u) + \frac{p_0(u)+g_0(u)z_0'(u)}{2\sqrt{g_0(u)}}h(u,v),\\ p(u,v) &= p_0(u),\\ q(u,v) &= z_0'(u) \end{aligned} \tag{3.31}$$

in case (3.26). *Here* $(u,v) \in \mathbb{R}^2$*,*

$$g_0(\tau) = g(z_0'(\tau)), \quad h(u,v) = \int\limits_v^u \frac{d\tau}{\sqrt{g_0(\tau)}}.$$

Put

$$\pi : \mathbb{R}^5 \ni (x,y,p,q,z) \longmapsto (x,y) \in \mathbb{R}^2.$$

The following statement is an immediate corollary of representations (3.30) and (3.31) from proposition 3.1. It gives sufficient conditions for projection $\pi \circ \sigma$ of a maximal simple wave (σ, L) to be a proper map of degree 1.

Theorem 3.1:
If

$$\lim_{\tau\to\infty} \frac{g_0(\tau)}{\tau} = 0, \quad \lim_{\tau\to\infty} |h(\tau,0)| = +\infty,$$

then the projection $\pi \circ \sigma$ *of the maximal simple wave* (σ, L) *is a proper map of degree* 1 *and, therefore,*

$$\pi \circ \sigma(\mathbb{R}^2) = \mathbb{R}^2.$$

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