

# Formularization Method for Calculating the Breakaway and Break-in Points and the Corresponding Gain of Root Locus Graphs

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**Abstract:** Break points, break-away and break-in points, are an essential part in root locus technique for single input single output linear invariant control systems. The importance of Break points comes from the fact that at the Break points at least two roots of the characteristic equation of the closed loop control system change their type from real to a complex at the break away point, and from complex to real at break-in point. This change affects the response of the system which can be crucial for some of systems' applications. The conditions for being a Break point are analyzed and a new formulated systematic method for finding the Break points and their corresponding gains is presented. An efficient algorithm was developed and can be solved analytically. There is no mathematical differentiation during calculation and the algorithm can be programmed easily. The developed algorithm is applicable for any order of transfer function of a linear invariant control system. This method is compared with other common methods to show its merits and effectiveness.

**Keywords:** formularization method for break points calculation, breakaway points' calculation, breakaway formula, gain at breakaway point, break-in points' calculation

## 1. INTRODUCTION

One of the common methods used in the analysis and design of a linear time invariant single input and single output of feedback control systems is the root locus method. The root locus technique in control systems was first introduced by Evans in the years 1948 and 1950, [1,2]. The root locus method is a technique used to determine the locus of the closed loop poles which are the roots of the closed loop characteristic equation on the complex s-plane as a function of a parameter of interest, where s is the complex variable. Usually the parameter of interest is the static gain of the open loop system. The parameter of interest value is varied theoretically from zero up to infinity.

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The root locus method developed by Evans is a powerful and efficient tool used to examine stability, to analyze and to design a single input single output linear time invariant system. In addition to examining the stability of a control system, the proper value of the gain for a specific type of response can be found.

To investigate the data obtained from the root locus plot of a single input single output linear invariant control systems neglecting noise as it is shown in Figure 1.1 , there is a need to know the system open loop transfer function;  $G(s) = G_c(s)G_p(s)$ , its feedback transfer function  $H(s)$ , and its closed loop transfer function  $T(s)$ , where the closed loop transfer function of such a system with a negative feedback is

$$T(s) = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)H(s)} = \frac{G(s)}{1 + G(s)H(s)}. \tag{1.1}$$

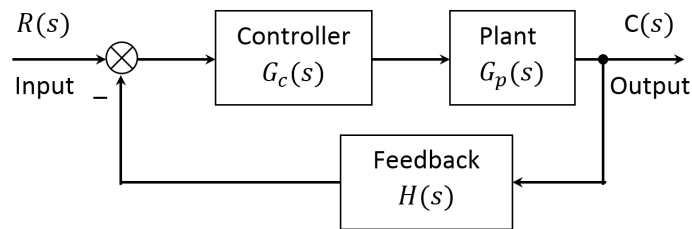


Fig. 1.1. Feedback Control System Block Diagram.

The Characteristic Equation (C.E.) of a control system is obtained by setting the denominator of the closed loop transfer function  $T(s)$  to zero, so it is:

$$1 + G(s)H(s) = 0, \tag{1.2}$$

where the mathematical expression of  $G(s)H(s)$  has a rational form of two polynomials with constant coefficients such as

$$G(s)H(s) = K \frac{N(s)}{D(s)}. \tag{1.3}$$

The degree of the numerator’s polynomial  $N(s)$  is  $m$ , and its roots are called zeros, while the degree of the denominator’s polynomial  $D(s)$  is  $n$ , and its roots are called poles. In general  $n \geq m$ .

The locus of the roots of the characteristic equation of a control system on the s-plane is obtained by varying the static gain  $K$ , theoretically  $0 \leq K < \infty$  . The path which connects the locus of the roots is known as the root locus of the system, and any point on the root locus path should satisfy two conditions: the angle condition, and the magnitude condition. It is known that the root locus begins at a pole when  $K = 0$  and ends at zero when  $K \rightarrow \infty$  , of the open loop transfer function. To satisfy the angle condition of the root locus on the real axis of the s-plane, the segments of the root loci on this axis are located to the left of an odd

number of poles and zeros.

Therefore, the static gain  $K$  value of a control system defines the roots of the characteristic equation and consequently the loci of the roots on the root locus graph on the complex  $s$ -plane. As a result, it defines the system response. From Laplace inverse transform the time domain response of any real root is exponential, while an oscillatory response occurs when there are at least one complex conjugate pair of roots of the characteristic equation [3–5]. The total response of a system is exponential with no oscillation if all its components have exponential responses. This occurs when all its roots are real. Root loci leaving the real axis have complex conjugate pair of roots and consequently have an oscillatory response.

In root locus method the root locus begins at poles and migrate toward zeros as a result of varying the static gain  $K$  value. If two root loci leave two poles on the real axis of the  $s$ -plane and move in opposite directions as shown in Figure 2.2 they will collide at a point, the point of collision is called a Breakaway point, and if a complex conjugate pair roots move in opposite directions toward the real axis they will collide at a point on the real axis is called Break-in point. At a Break point the characteristic equation has double roots for a certain value of  $K$ . Geometrically the Break point is on the root locus segments between two poles, or between two zeros. For a larger value of  $K$  than the value at the Breakaway point the characteristic equation will have at least one complex conjugate pair of roots and consequently the two loci split and leave the real axis in opposite direction and continue to move in a symmetrical way, while for larger  $K$  than its value at the Break-in point the characteristic equation will have two different real roots and consequently the root locus split and continue to move in different directions on the real axis. The process of the Break in point is an opposite manner to the Breakaway point. In this article Breakaway points and Break-in points are called just Break points

In this work a new formulated method for finding the real Breakaway points and Break-in points which are along the real axis is developed and investigated. There are a number of methods on how to find the Break points and the corresponding gains, [3, 6]. The common method is based on finding a local extremum for the parameter of interest, usually the static gain. This is done by differentiating the mathematical expression of the parameter of interest which is a function of the complex variable  $s$ , and then setting the derivative to zero. Then solving for  $s = \sigma$  to get various roots. In this method not all the roots are Break points and then there is a need to continue to figure out which point is a Break point. The roots at the Break point are real and are along the root locus segment on the real axis between two poles or between two zeros. The mathematical expression of the open loop transfer function usually has a rational form of two polynomials, numerator and denominator, and its differentiation becomes cumbersome especially for a higher degree of the numerator. A second method was presented by Remec [7], and it is based on the theory of equations. It is a tabulation method similar to Routh-Hurwitz array construction which is a long process. A third method is called the Transition method by Franklin [8]. This method is based on the fact that the natural logarithm has a zero derivative at the same point that the derivative of

the parameter of interest expression is also zero. In this method no need for differentiation but there is a need to do a number of polynomials factors multiplication to get the final form of the equation, and then to figure out what are the applicable roots out of all roots. It has the same numerator as the differentiation of the fractional form of the two polynomials. A fourth method was developed by Krishnan [9] and it is based on successive differentiation of numerator and denominator to form arrays for an algorithm. It is a long process method.

This work is divided into two parts. In the first part the development of a new method for calculating the Break points, Breakaway points and Break in points, and their corresponding gain. The second part is a comparison with the common methods and the analysis of the proposed efficiency.

**2. DERIVATION OF NEW METHOD FOR BREAK POINTS CALCULATION**

Substitute Equation 1.3 into the C.E., Equation 1.2, to get

$$1 + \frac{KN(s)}{D(s)} = \frac{D(s) + KN(s)}{D(s)} = 0. \tag{2.4}$$

It means that the numerator is zero and the new equation is

$$D(s) + KN(s) = 0. \tag{2.5}$$

Where the numerator N(s) and the denominator D(s) in Equation (1.3) are polynomials of order *m* and *n* respectively such as

$$N(s) = b_m s^m + b_{m-1} s^{m-1} + b_{m-2} s^{m-2} + \dots + b_2 s^2 + b_1 s + b_0. \tag{2.6}$$

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots + a_2 s^2 + a_1 s + a_0. \tag{2.7}$$

Substitute Equations (2.6) and (2.7) into Equation (2.5) to get the characteristic equation C.E. of the control system as

$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots + a_2 s^2 + a_1 s + a_0 + K (b_m s^m + b_{m-1} s^{m-1} + b_{m-2} s^{m-2} + \dots + b_2 s^2 + b_1 s + b_0) = 0. \tag{2.8}$$

Collect equal terms to obtain the general form of the characteristic equation, Equation (2.9). This equation is a polynomial of order *n* for *n* ≥ *m*, and constant coefficients *c<sub>i</sub>*.

$$c_n s^n + c_{n-1} s^{n-1} + c_{n-2} s^{n-2} + \dots + c_2 s^2 + c_1 s + c_0 = 0. \tag{2.9}$$

Where

$$c_q = a_q + K b_q \text{ for } q = 1, 2, \dots, n, \text{ and } b_q = 0 \text{ for } q > m. \tag{2.10}$$

It means that the numerator is zero and the new equation is

$$s_1 = s_2 = \sigma. \quad (2.11)$$

Equation (11) means that the characteristic equation at Break points has the factor

$$(s + \sigma)^2 = s^2 + 2\sigma s + \sigma^2. \quad (2.12)$$

If Equation (2.12) is a factor of the characteristic equation, Equation (2.9), the remainder of their division should be zero as shown in Example 1.

**Example 1.** Given the system open loop transfer function  $G(s) = \frac{k}{s(s+2)}$  and its negative unit feedback is  $H(s) = 1$ . The characteristic equation of this system is

$$1 + GH(s) = 1 + \frac{K}{s(s+2)} = \frac{s(s+2) + K}{s(s+2)} = 0. \quad (2.13)$$

As a consequence of being Equation (13) equal to zero, its numerator becomes equal to zero, and the new equation is

$$s(s+2) + K = s^2 + 2s + K = 0. \quad (2.14)$$

The two terms of remainder of Equation (2.14) are set equal to zero to get .

$$(1 - \sigma) = 0, \quad \text{and} \quad (K - \sigma^2) = 0. \quad (2.15)$$

Actually, Equation (2.15) contains two equations of sigma as the variable. Solving the two equations gives:  $\sigma = 1$ ,  $K = \sigma^2 = (1)^2 = 1$ . Then the Breakaway point is at point  $s = -\sigma = -1$  and the gain is  $K=1$ . These values are validated by the root locus plot shown in Figure 2.2.

The two terms of the remainder of the division result of the general form of characteristic equation of a second order system are

$$-c_1 + 2c_2\sigma = 0, \quad \text{and} \quad c_0 - c_2\sigma^2 = 0. \quad (2.16)$$

A similar results to Equation (2.16) are obtained from the division of a higher order systems. By setting the two remainders of each order equal to zero, gives the two equations that are in Table 2.1. Those two equations are the two conditions that the second order polynomial of Equation (2.12) is a factor of the characteristic equation of the system.

Substitute the coefficients of the equations of the two conditions of order n which are given in Table 2.1 as in Equation (2.10) to obtain

$$\sum_{q=1}^n (-1)^q q (a_q + Kb_q) \sigma^{q-1} = 0. \quad (2.17)$$

$$(a_0 + Kb_0) + \sum_{q=1}^n (-1)^{q-1} (q-1) (a_q + Kb_q) \sigma^q = 0. \quad (2.18)$$

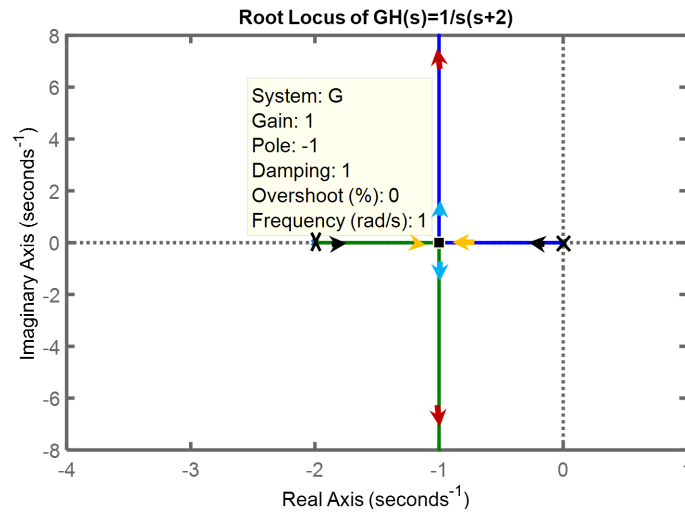


Fig. 2.2. Root Locus plot of a second order control system of Example 1, showing the direction of roots migration and the breakaway point at  $s = -1$ .

Table 2.1. The two conditions for systems of orders= 2, 3, . . . , n.

Degree of C.E.	First Condition	Second Condition
2	$-c_1 + 2c_2\sigma = 0.$	$c_0 - c_2\sigma^2 = 0.$
3	$-c_1 + 2c_2\sigma - 3c_3\sigma^2 = 0.$	$c_0 - c_2\sigma^2 + 2c_3\sigma^3 = 0.$
4	$-c_1 + 2c_2\sigma - 3c_3\sigma^2 + 4c_4\sigma^3 = 0.$	$c_0 - c_2\sigma^2 + 2c_3\sigma^3 - 3c_4\sigma^4 = 0.$
5	$-c_1 + 2c_2\sigma - 3c_3\sigma^2 + 4c_4\sigma^3 - 5c_5\sigma^4 = 0.$	$c_0 - c_2\sigma^2 + 2c_3\sigma^3 - 3c_4\sigma^4 + 4c_5\sigma^5 = 0.$
⋮	⋮	⋮
n	$\sum_{q=1}^n (-1)^q q c_q \sigma^{q-1} = 0.$	$c_0 + \sum_{q=1}^n (-1)^{q-1} (q-1) c_q \sigma^q = 0.$

Rewriting the equations of the two conditions for the existence of Break points, Equations (2.17) and (2.18), so that each equation is a sum of two polynomials. The first polynomial is constructed based only on the coefficients of the numerator  $N(s)$ , and the second polynomial is constructed based only on the coefficients of the denominator  $D(s)$  and the static gain  $K$  as a factor yields

$$\sum_{q=1}^n (-1)^q q a_q \sigma^{q-1} + K \sum_{q=1}^n (-1)^q q b_q \sigma^{q-1} = 0. \tag{2.19}$$

$$a_0 + \sum_{q=1}^n (-1)^{q-1} (q-1) a_q \sigma^q + K \left[ b_0 + \sum_{q=1}^n (-1)^{q-1} (q-1) b_q \sigma^q \right] = 0. \tag{2.20}$$

For simplicity Equation (2.19) and Equation (2.20) may written as follows

$$P_1(\sigma) + KP_2(\sigma) = 0. \quad (2.21)$$

$$P_3(\sigma) + KP_4(\sigma) = 0. \quad (2.22)$$

Where the  $P_s$  polynomials are

$$P_1(\sigma) = \sum_{q=1}^n (-1)^q q a_q \sigma^{q-1}. \quad (2.23)$$

$$P_2(\sigma) = \sum_{q=1}^n (-1)^q q b_q \sigma^{q-1}. \quad (2.24)$$

$$P_3(\sigma) = a_0 + \sum_{q=1}^n (-1)^{q-1} (q-1) a_q \sigma^q. \quad (2.25)$$

$$P_4(\sigma) = b_0 + \sum_{q=1}^n (-1)^{q-1} (q-1) b_q \sigma^q. \quad (2.26)$$

Solve for  $K$  in Equation (2.21) and in Equation (2.22) to have

$$K = -\frac{P_1(\sigma)}{P_2(\sigma)}. \quad (2.27)$$

$$K = -\frac{P_3(\sigma)}{P_4(\sigma)}. \quad (2.28)$$

Eliminate  $K$  to obtain

$$\frac{P_1(\sigma)}{P_2(\sigma)} = \frac{P_3(\sigma)}{P_4(\sigma)}. \quad (2.29)$$

Then a cross multiplication gives

$$P(\sigma) = P_1(\sigma)P_4(\sigma) - P_2(\sigma)P_3(\sigma) = 0. \quad (2.30)$$

Equation (2.30) is called the Break polynomial equation, and it is a polynomial in sigma. Part of the polynomial roots, sigma values, are the Break points.

### **Algorithm Steps**

Finding of the break points and the corresponding  $K$  gain for a given control system has the following steps:

step 1. Write the rational form of the system transfer function as shown in Equation(1.3) .

- step 2. Write the four polynomials  $P_1(\sigma)$ ,  $P_2(\sigma)$ ,  $P_3(\sigma)$ , and  $P_4(\sigma)$  using the formulas given by Equations (2.23)-(2.26) in the same polynomial's terms order that is shown in Equations (2.6) or (2.7).
- step 3. Substitute the four polynomials into Equation (2.30) to obtain the Break polynomial equation.
- step 4. Solve the Break equation (2.30) for sigma. The roots  $\sigma_i$  are the solution of the Break polynomial equation.
- step 5. Find the applicable values of sigma (Break Points) which satisfy the Root Locus technique condition.
- step 6. Evaluate the four polynomials in Equations (2.23)-(2.26) only for the applicable values of sigma.
- step 7. Calculate the gain  $K$ . If  $P_2(\sigma) \neq 0$  Equation (2.27) is used, else Equation (2.28) is used.

Note that in Equations (2.23)-(2.26) the polynomials terms' degree is from lower degree to higher degree, and when using MATLAB software the terms' order in the polynomial array have to be flipped. The flowchart of the procedure is shown in Figure 2.3.

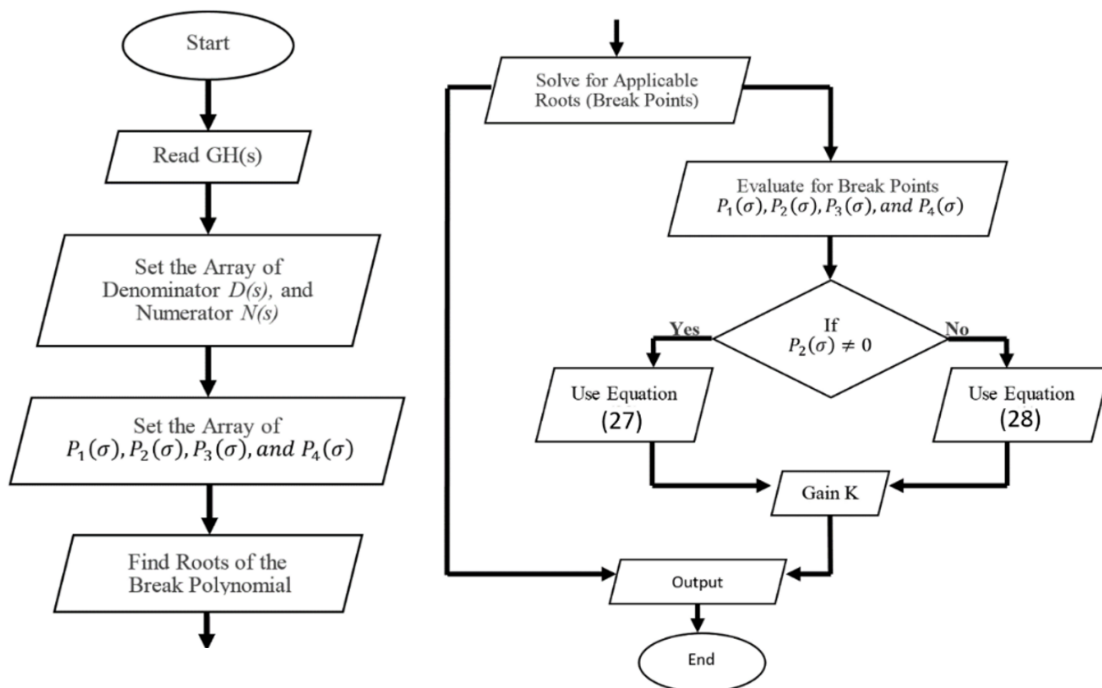


Fig. 2.3. Flowchart of the algorithm steps.



### 3. ANALYSIS AND CALCULATION PROCEDURE OF THE BREAK POINTS

The proposed Formularization method is used to obtain the Break polynomial which is based on coefficients of the open loop transfer function's polynomials. Substitution of those coefficients into Equations (2.23) – (2.26) yields the four polynomials  $p1-p4$ . Then the four polynomials are substituted into Equation (2.30) to obtain the Break polynomial.

Then solving the Break polynomial equation to obtain its roots. The Break points are some of the roots which are located on the root locus segment that satisfy the angle condition of the root locus graph on the real axis. It means that the applicable roots (sigma values) are on the real axis and are located to left of an odd number of poles and zeros of the system open loop transfer function. Specifically, they are between two poles or between two zeros on the real axis. By knowing the actual Break points the corresponding gains are obtained by substituting the actual Break points' values into Equation (2.27) or into Equation (2.28).

The calculated value of  $K$  is a "border" value where two roots at least alter their type between real and complex and vice versa for any increase in  $K$ . For Breakaway point the two real roots become complex pair roots, and for Break-in point a complex conjugate roots alter to real roots.

To show the calculation steps of this Formularization method and to compare it with other common methods it is used in the solution of two examples for a two control systems. The transfer function in the second example has two zeros and three poles, while the transfer function in the third example has two zeros and four poles. The objective of having two finite zeros is to enable the existence of both breakaway point, and break-in point.

**Example 2.** In this example the control system open loop transfer function is

$$GH(s) = K \frac{(s+3)(s+5)}{s(s+2)(s+4)}. \quad (3.31)$$

The polynomials of the numerator and denominator are

$$D(s) = s^3 + 6s^2 + 8s. \quad (3.32)$$

$$N(s) = s^2 + 8s + 15. \quad (3.33)$$

The coefficients of both polynomials are:  $a_3 = 1$ ,  $a_2 = 6$ ,  $a_1 = 8$ ,  $a_0 = 0$ ,  $b_3 = 0$ ,  $b_2 = 1$ ,  $b_1 = 8$ ,  $b_0 = 15$ .

Substitute into the P's polynomials, Equations (23-26), to have

$$P_1(\sigma) = -a_1 + 2a_2\sigma - 3a_3\sigma^2 = -8 + 12\sigma - 3\sigma^2. \quad (3.34)$$

$$P_2(\sigma) = -b_1 + 2b_2\sigma - 3b_3\sigma^2 = -8 + 2\sigma. \quad (3.35)$$

$$P_3(\sigma) = a_0 - a_2\sigma^2 + 2a_3\sigma^3 = -6\sigma^2 + 2\sigma^3. \quad (3.36)$$

$$P_4(\sigma) = b_0 - b_2\sigma^2 + 2b_3\sigma^3 = 15 - \sigma^2. \tag{3.37}$$

Or into Equation (2.30) to get the Break polynomial

$$P(\sigma) = (-8 + 12\sigma - 3\sigma^2)(15 - \sigma^2) - (-8 + 2\sigma)(-6\sigma^2 + 2\sigma^3) = 0. \tag{3.38}$$

Simplify to obtain the Break polynomial equation such as

$$P(\sigma) = \sigma^4 - 16\sigma^3 + 85\sigma^2 - 180\sigma + 120 = 0. \tag{3.39}$$

Solve to get the Break polynomial's roots as

$$\sigma_i = 1.2221, 7.8311, 3.4734 + 0.6888i, 3.4734 - 0.6888i. \tag{3.40}$$

The applicable roots are:  $[\sigma_{1,2} = 1.2221, 7.8311]$ , and substitution these two values to obtain the two corresponding gains as:  $[K = 0.393, 12791]$ . Then the Break-away point is:  $[s_1 = -\sigma_1 = -1.222]$ , and the Break-in point is:  $[s_2 = -\sigma_2 = -7.831]$ .

To validate the result the root locus plot is drawn using MATLAB software as shown in Figure 3.4. The MATLAB root locus plot shows very close results. The insignificant differenc in the results is due to the graphical plot accuracy.

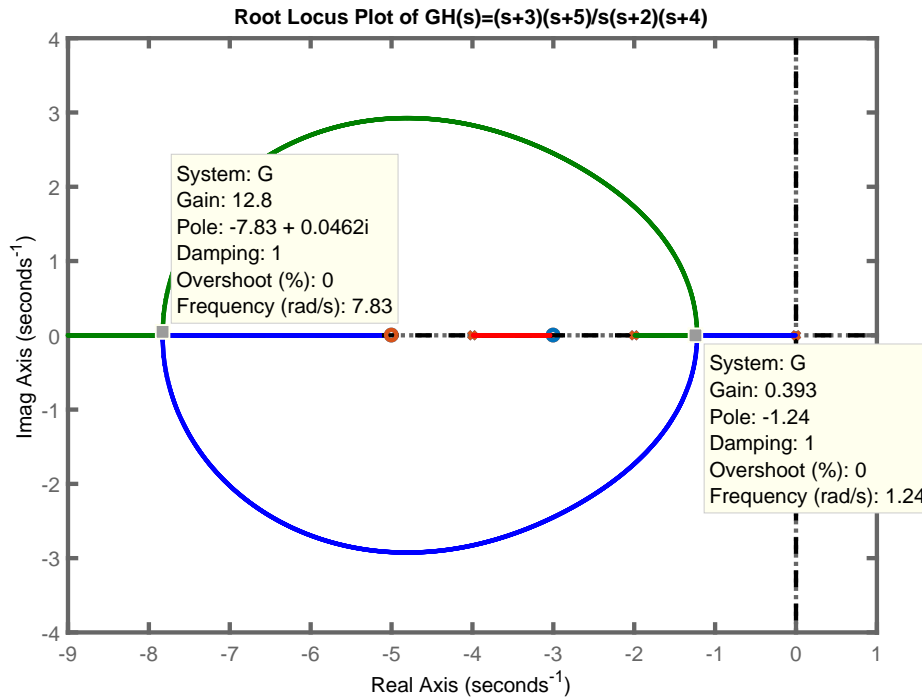


Fig. 3.4. Root Locus plot of the control system of Example 2.

**Example 3.** In this example the open loop transfer function of the control system is

$$GH(s) = K \frac{(s+3)(s+4)}{s(s+2)(s+5)(s+8)}. \quad (3.41)$$

The polynomials of the numerator and denominator are

$$D(s) = s^4 + 15s^3 + 66s^2 + 80s. \quad (3.42)$$

$$N(s) = s^2 + 7s + 12. \quad (3.43)$$

The coefficients of both polynomials are:

$$a_4 = 1, \quad a_3 = 15, \quad a_2 = 66, \quad a_1 = 80, \quad a_0 = 0, \quad b_4 = 0, \quad b_3 = 0, \quad b_2 = 1, \quad b_1 = 7, \quad b_0 = 12$$

Substitute in the Ps polynomials, Equations (23-26), to have

$$P_1(\sigma) = -a_1 + 2a_2\sigma - 3a_3\sigma^2 + 4a_4\sigma^3 = -80 + 132\sigma - 45\sigma^2 + 4\sigma^3. \quad (3.44)$$

$$P_2(\sigma) = -b_1 + 2b_2\sigma - 3b_3\sigma^2 + 4b_4\sigma^3 = -7 + 2\sigma. \quad (3.45)$$

$$P_3(\sigma) = a_0 - a_2\sigma^2 + 2a_3\sigma^3 - 3a_4\sigma^4 = -66\sigma^2 + 30\sigma^3 - 3\sigma^4. \quad (3.46)$$

$$P_4(\sigma) = b_0 - b_2\sigma^2 + 2b_3\sigma^3 - 3b_4\sigma^4 = 12 - \sigma^2. \quad (3.47)$$

Then substitute into Equation (2.30) to get the Break polynomial

$$P(\sigma) = (-80 + 132\sigma - 45\sigma^2 + 4\sigma^3)(12 - \sigma^2) +$$

$$- (-66\sigma^2 + 30\sigma^3 - 3\sigma^4)(-7 + 2\sigma) = 0. \quad (3.48)$$

Collect equal terms to obtain

$$P(\sigma) = 2\sigma^5 - 36\sigma^4 + 258\sigma^3 - 922\sigma^2 + 1584\sigma - 960 = 0. \quad (3.49)$$

The Break polynomial's roots are

$$\sigma_i = 1.2436, \quad 3.4909, \quad 3.5875 + j2.2987, \quad 3.5875 - j2.2987, \quad 6.0906. \quad (3.50)$$

The applicable roots are:  $[\sigma_{1,2,3} = 1.2436, \quad 6.0906, \quad 3.4909]$ , and the gains for these three values are:  $[K = 4.9312, \quad 8.0296, \quad 141.7091]$ . Then the Break away points are:  $[s_{1,2} = -\sigma_{1,2} = -1.2436, \quad -6.0906]$ , and the Break-in point is:  $[s_3 = -\sigma_3 = -3.4909]$ . It

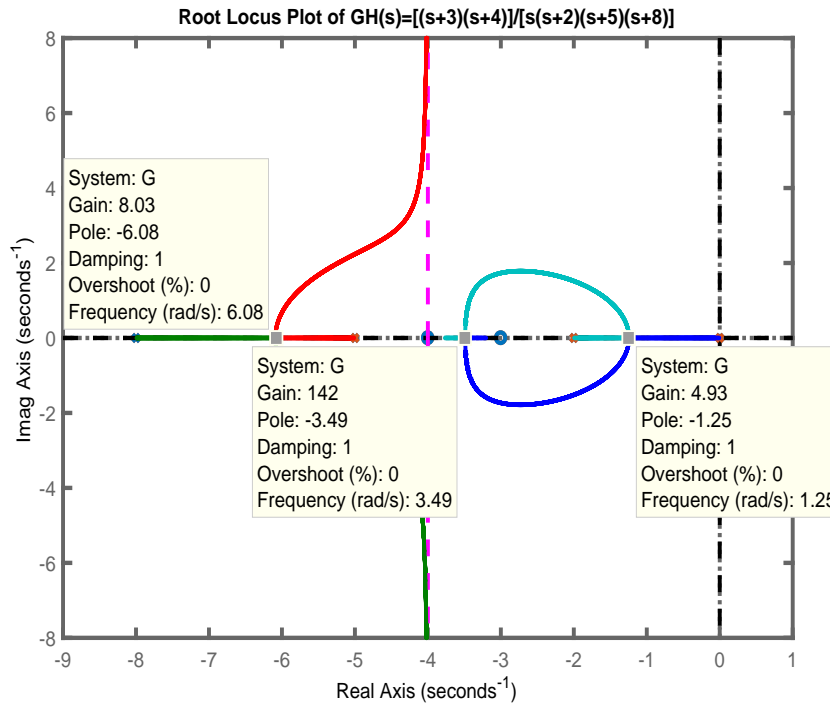


Fig. 3.5. Root Locus plot of the control system of Example 3.

can be seen that the result obtained from MATLAB root locus plot is too close to the calculated result.

#### 4. COMPARISON WITH OTHER COMMON METHODS

To examine the methods accuracy and calculation's efficiency of finding the Break points of the considered control system, the Formulated method, and other common methods are used in solving two examples, Examples 2 and 3. The accuracy, simplicity, and the amount of mathematical operations to obtain the final results are considered.

##### 4.1. Comparison with the Root Locus Sketching Method

Comparison between the Break points' values and the corresponding gains' values of the two Examples 2 and 3 in the previous section using Root Locus sketching method as shown in Figures 3.4 and 3.5 and using the Formulated method shows very small difference in the values. The difference is a result of the root locus sketching method being a semi-graphical method

##### 4.2. Comparison with the Via Differentiation Method

The characteristic equation of the control system is

$$1 + GH(s) = 1 + K \frac{N(s)}{D(s)} = 1 + K \frac{(s+3)(s+5)}{s(s+2)(s+4)} = 0. \quad (4.51)$$

Solve for  $K$  expression to have

$$K = -\frac{D(s)}{N(s)} = -\frac{s(s+2)(s+4)}{(s+3)(s+5)} = -\frac{s^3 + 6s^2 + 8s}{s^2 + 8s + 15}. \quad (4.52)$$

Via Differentiation method is to differentiate  $K$  with respect to  $s$  to obtain the local extremums of  $K$ .

$$\frac{dK}{ds} = -\frac{D'N - DN'}{N^2} = 0. \quad (4.53)$$

Or

$$D'N - DN' = 0. \quad (4.54)$$

Differentiate both polynomials  $D(s)$  and  $N(s)$  and the substitute into Equation (4.54) to get

$$(3s^2 + 12s + 8)(s^2 + 8s + 15) - (s^3 + 6s^2 + 8s)(2s + 8) = 0. \quad (4.55)$$

Multiply and collect equal terms to obtain the polynomial

$$s^4 + 16s^3 + 85s^2 + 180s + 120 = 0. \quad (4.56)$$

The roots of this polynomial are

$$s_i = -1.2221, -7.8311, -3.4734 + 0.6888i, -3.4734 - 0.6888i. \quad (4.57)$$

As discussed before the applicable roots are  $[s_{1,2} = -1.2221, -7.8311]$ , and the corresponding Gain  $K$  is calculated by using Equation (4.52) as follows

$$K_i = -\frac{D(s_i)}{N(s_i)}. \quad (4.58)$$

$$K_1 = -\frac{(-1.2221)^3 + 6(-1.2221)^2 + 8(-1.2221)}{(-1.2221)^2 + 8(-1.2221) + 15} = 0.393. \quad (4.59)$$

$$K_2 = -\frac{(-7.8311)^3 + 6(-7.8311)^2 + 8(-7.8311)}{(-7.8311)^2 + 8(-7.8311) + 15} = 12.791. \quad (4.60)$$

**4.3. Comparison with Franklin's Via Transition Method**

In this method the Break points are determined by using:

$$\sum_{i=1}^n \frac{1}{\sigma + p_i} = \sum_{i=1}^m \frac{1}{\sigma + z_i}. \tag{4.61}$$

Applying Equation (4.61) to the open loop transfer function of the same example gives an equation which a summation of a number of fractions such as

$$\frac{1}{\sigma} + \frac{1}{\sigma + 2} + \frac{1}{\sigma + 4} = \frac{1}{\sigma + 3} + \frac{1}{\sigma + 5}. \tag{4.62}$$

Algebraic simplification gives

$$\frac{(\sigma + 2)(\sigma + 4) + \sigma(\sigma + 4) + \sigma(\sigma + 2)}{\sigma(\sigma + 2)(\sigma + 4)} = \frac{(\sigma + 5) + (\sigma + 3)}{(\sigma + 3)(\sigma + 5)}. \tag{4.63}$$

Then a cross multiplication gives

$$[(\sigma + 2)(\sigma + 4) + \sigma(\sigma + 4) + \sigma(\sigma + 2)][(\sigma + 3)(\sigma + 5)] + \\ - [(\sigma + 5) + (\sigma + 3)][\sigma(\sigma + 2)(\sigma + 4)] = 0. \tag{4.64}$$

This result is the same as in the via differentiation method as given in Equation (4.55). It can be noticed that this result, Equation (4.64), is actually the differentiation result of the factored form of the denominator and numerator of the transfer function of the control system.

After mathematical operations and algebraic simplification we get the same polynomial as in Equation (4.56)

$$\sigma^4 + 16\sigma^3 + 85\sigma^2 + 180\sigma + 120 = 0. \tag{4.65}$$

And the roots are

$$\sigma_i = -1.2221, -7.8311, -3.4734 + 0.6888i, -3.4734 - 0.6888i. \tag{4.66}$$

The applicable roots are  $[s_{1,2} = -1.2221, -7.8311]$ , and the corresponding Gain **K** is calculated the same way as in the via differentiation method, but there is no need for differentiation along the calculation

**5. CONCLUSION**

The result of the comparison via solution of example 2 shows that the new method is accurate and efficient compared to the very common root locus sketching method. The results of the Via Differentiation method are the same as in the Formulated method where s is equal negative sigma, however there is a need for polynomials differentiation, and it requires more mathematical operations.

In principle the two most used methods in class rooms are the Via Differentiation method, and the Transition method. They are very much similar. The difference is that in the Via Differentiation method the differentiation part is done by the user, and in the Transition method the differentiation part was done by the author to obtain the used formula. In both methods the Break polynomial can be found by using the following equation:

$$[All \text{ combinations of the product of } (n - 1) \text{ factors of } D(s)] \cdot N(s) + \\ - [All \text{ combinations of the product of } (m - 1) \text{ factors of } N(s)] \cdot D(s) = 0. \quad (5.67)$$

As it can be noticed that Equation (5.67) is based on the factored form of the numerator  $N(s)$ , and the denominator  $D(s)$ .

The proposed Formularization method is used to find the Break polynomial based on the substitution of the coefficients of the two polynomials of the fractional form of the open loop transfer function of a control system. This method gives the breakaway points, break-in points, and the corresponding gains.

The results of this proposed Formularization method are the same as results of the other methods which shows that it is an accurate method and requires less mathematical operations. The calculation procedure shows its merits as a systematic method. There is no mathematical differentiation, and its algorithm can be programmed easily. The Formulated method is applicable to transfer functions of any degree, and in addition to that, a constant numerator where all its coefficients are zeros simplifies and shorten the process of obtaining Break polynomial. These features simplify the calculations for finding the points at which the characteristic equation roots change type. The use of this method in plotting the root locus of a control system for a design purpose becomes more accurate especially for a larger step size of the static gain.

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